Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 3, No. 2, (2012), 185-193
ISSN : 1906-9685
http://www.math.sci.nu.ac.th

ON SOME PROPERTIES OF p-WAVELET PACKETS VIA THE WALSH-FOURIER TRANSFORM

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ABSTRACT. A novel method for the construction of orthogonal p-wavelet packets on a positive half-line $\mathbb{R}^+$ was given by the author in [*Construction of wavelet packets on p-adic field*, Int. J. Wavelets Multiresolut. Inf. Process., 7(5) (2009), pp. 553-565]. In this paper, we investigate their properties by means of the Walsh-Fourier transform. Three orthogonal formulas regarding these p-wavelet packets are derived.

KEYWORDS : p-Multiresolution analysis; p-Wavelet packets; Riesz basis; Walsh functions; Walsh-Fourier transform.
AMS Subject Classification: 42C40 42C15 42C10

1. INTRODUCTION

In the early nineties a general scheme for the construction of wavelets was defined. This scheme is based on the notion of multiresolution analysis (MRA) introduced by Mallat [13]. Immediately specialists started to implement new wavelet systems and in recent years, the concept MRA of $\mathbb{R}^n$ has been extended to many different setups, for example, Dahlke introduced multiresolution analysis and wavelets on locally compact Abelian groups [5], Lang [11] and [12] constructed compactly supported orthogonal wavelets on the locally compact Cantor dyadic group $\mathcal{C}$ by following the procedure of Daubechies [6] via scaling filters and these wavelets turn out to be certain lacunary Walsh series on the real line. Later on, Farkov [7] extended the results of Lang [11] and [12] on the wavelet analysis on the Cantor dyadic group $\mathcal{C}$ to the locally compact Abelian group G which is defined for an integer $p \geq 2$ and coincides with $\mathcal{C}$ when $p = 2$. The construction of dyadic compactly supported wavelets for $L^2(\mathbb{R}^+)$ have been given by Protasov and Farkov in [14] where the latter author has given the general construction of all compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}^+)$ arising from scaling filters with p^n many terms in [8].

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Article history : Received 20 April 2012. Accepted 20 August 2012.

It is well-known that the classical orthonormal wavelet bases have poor frequency localization. For example, if the wavelet ψ is band limited, then the measure of the supp of $(\psi_{j,k})^\wedge$ is 2^j-times that of supp $\hat{\psi}$. To overcome this disadvantage, Coifman *et al.* [4] constructed univariate orthogonal wavelet packets. The fundamental idea of wavelet packet analysis is to construct a library of orthonormal bases for $L^2(\mathbb{R})$, which can be searched in real time for the best expansion with respect to a given application. The standard construction is to start from a multiresolution analysis (MRA) and generate the library using the associated quadrature mirror filters (QMFs). The internal structure of the MRA and the speed of the decomposition schemes make this an efficient adaptive method for simultaneous time and frequency analysis of signals. The concept of the wavelet packet was subsequently generalized to $\mathbb{R}^d$ by taking tensor products, whereas Shen [18] formulated non-tensor product wavelets in $L^2(\mathbb{R}^s)$. Other notable generalizations are the non-orthogonal version of wavelet packets [2], biorthogonal wavelet packets [3], vector-valued wavelet packets [1] and higher dimensional wavelet packets with arbitrary dilation matrix [10].

Recently, Shah [16] has constructed p-wavelet packets associated with the p-MRA on the positive half-line $\mathbb{R}^+$. He proved lemmas on the so-called splitting trick and several theorems concerning the Walsh-Fourier transform of the p-wavelet packets and the construction of p-wavelet packets to show that their translates form an orthonormal basis of $L^2(\mathbb{R}^+)$. Very recently, Shah and Debnath [17], have constructed the corresponding p-wavelet frame packets on the positive half-line $\mathbb{R}^+$ by using the Walsh-Fourier transform. As one of a series of works on positive half-line $\mathbb{R}^+$, the objective of this paper is to investigate certain properties of orthogonal p-wavelet packets on the positive half-line $\mathbb{R}^+$ by virtue of the Walsh-Fourier transform.

In order to make the paper self-contained, we state some basic preliminaries, notations and definitions including the Walsh-Fourier transform, Walsh functions and p-MRA in Section 2. In Section 3, we study certain properties of orthogonal p-wavelet packets on a half-line $\mathbb{R}^+$.

2. PRELIMINARIES AND p-WAVELET PACKETS

Let p be a fixed natural number greater than 1. As usual, let $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}^+ = \{0, 1, 2, ...\}$ and $\mathbb{N} = \mathbb{Z}^+ - \{0\}$. Set $\Omega_0 = \{0, 1, 2, ..., p-1\}$ and $\Omega = \Omega_0 - \{0\}$. Denote by $[x]$ the integer part of x. For $x \in \mathbb{R}^+$ and any positive integer j, we set

$$x_j = [p^j x](\mathrm{mod}\, p), \; x_{-j} = [p^{1-j} x](\mathrm{mod}\, p). \tag{2.1}$$

We consider on $\mathbb{R}^+$ the addition defined as follows: if $z = x \oplus y$, then

$$z = \sum_{j<0} \zeta_j p^{-j-1} + \sum_{j>0} \zeta_j p^{-j}$$

with $\zeta_j = x_j + y_j (\mathrm{mod}\ p)$ $(j \in \mathbb{Z} \setminus \{0\})$, where $\zeta_j \in \Omega_0$ and x_j, y_j are calculated by (2.1). Moreover, we note that $z = x \ominus y$ if $z \oplus y = x$, where $\ominus$ denotes subtraction modulo p in $\mathbb{R}^+$.

For $x \in [0, 1)$, let $r_0(x)$ be given by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/p); \\ \varepsilon_p^{\ell}, & \text{if } x \in [\ell p^{-1}, (\ell+1)p^{-1}),\ \ell \in \Omega, \end{cases}$$

where $\varepsilon_p = \exp(2\pi i/p)$. The extension of the function r_0 to $\mathbb{R}^+$ is denoted by the equality $r_0(x+1) = r_0(x),\ x \in \mathbb{R}^+$. Then, the generalized Walsh functions $\{w_m(x) : m \in \mathbb{Z}^+\}$ are defined by

$$w_0(x) \equiv 1,\ w_m(x) = \prod_{j=0}^{k} \left(r_0(p^j x)\right)^{\mu_j}$$

where $m = \sum_{j=0}^{k} \mu_j p^j\ ,\ \mu_j \in \Omega_0,\ \mu_k \neq 0.$

For $x, w \in \mathbb{R}^+$, let

$$\chi(x, w) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j w_{-j} + x_{-j} w_j)\right), \tag{2.2}$$

where x_j, w_j are given by (2.1). Note that $\chi(x, m/p^{n-1}) = \chi(x/p^{n-1}, m) = w_m(x/p^{n-1})$ for all $x \in [0, p^{n-1}),\ m \in \mathbb{Z}^+$.

The Walsh-Fourier transform of a function $f \in L^1(\mathbb{R}^+)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^+} f(x)\overline{\chi(x, \xi)}\, dx, \tag{2.3}$$

where $\chi(x, \xi)$ is given by (2.2). The properties of the Walsh-Fourier transform are quite similar to those of the classic Fourier transform (see [15]). In particular, if $f \in L^2(\mathbb{R}^+)$, then $\hat{f} \in L^2(\mathbb{R}^+)$ and

$$\|\hat{f}\|_{L^2(\mathbb{R}^+)} = \|f\|_{L^2(\mathbb{R}^+)}.$$

If $x, y, \xi \in \mathbb{R}^+$ and $x \oplus y$ is p-adic irrational, then

$$\chi(x \oplus y, \xi) = \chi(x, \xi)\, \chi(y, \xi). \tag{2.4}$$

It is shown by Golubov *et al.* [9] that both the systems $\{\chi(\alpha, .)\}_{\alpha=0}^{\infty}$ and $\{\chi(., \alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2[0,1]$.

In the following subsection, we introduce the notion of p-multiresolution analysis on $\mathbb{R}^+$ and give the definition of orthogonal wavelets of space $L^2(\mathbb{R}^+)$.

Definition 2.1. *A p-multiresolution analysis of $L^2(\mathbb{R}^+)$ is a nested sequence of closed subspaces $\{V_j\}_{j \in \mathbb{Z}}$ such that*

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,

(ii) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^+)$ and $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$,

(iii) $f \in V_j$ if and only if $f(p.) \in V_{j+1}$ for all $j \in \mathbb{Z}$,

(iv) there exists a function φ in V_0, called the scaling function, such that $\{\varphi(. \ominus k) : k \in \mathbb{Z}^+\}$ forms an orthonormal basis for V_0.

Since $\varphi(x) \in V_0 \subset V_1$, by Definition 2.1, there exists a finitely supported sequence $\{a_k\}_{k\in\mathbb{Z}^+} \in l^2(\mathbb{Z}^+)$ such that

$$\varphi(x) = \sum_{k\in\mathbb{Z}^+} a_k\, \varphi(px \ominus k). \tag{2.5}$$

The Walsh-Fourier transform of (2.5) is given by

$$\hat{\varphi}\,(\xi) = m_0(p^{-1}\xi)\,\hat{\varphi}(p^{-1}\xi), \tag{2.6}$$

where $m_0(\xi) = \sum_{k\in\mathbb{Z}^+} a_k\, \overline{\chi(k,\xi)}$, is a Walsh polynomial called the *symbol* of $\varphi(x)$.

Let $W_j, j \in \mathbb{Z}$ be the direct complementary subspaces of V_j in V_{j+1}. Assume that there exist a set of $p-1$ functions $\{\psi_1, \psi_2, ..., \psi_{p-1}\}$ in $L^2(\mathbb{R}^+)$ such that their translates and dilations form an orthonormal basis of W_j, i.e.,

$$W_j = \overline{\text{span}}\left\{\psi_\ell(p^j. \ominus k) : k \in \mathbb{Z}^+, \ell \in \Omega\right\},\ j \in \mathbb{Z}. \tag{2.7}$$

Since $\psi_\ell(x) \in W_0 \subset V_1, \ell \in \Omega$, there exists a sequence $\left\{a_k^\ell\right\}_{k\in\mathbb{Z}^+}$ in $\Upsilon^2(\mathbb{Z}^+)$ such that

$$\psi_\ell\,(x) = \sum_{k\in\mathbb{Z}^+} a_k^\ell\, \varphi(px \ominus k),\ \ell \in \Omega. \tag{2.8}$$

Implementing the Walsh- Fourier transform for both sides of (2.8) gives

$$\hat{\psi}_\ell\,(\xi p) = m_\ell(\xi)\,\hat{\varphi}(\xi), \tag{2.9}$$

where

$$m_\ell(\xi) = \sum_{k\in\mathbb{Z}^+} a_k^\ell\, \overline{\chi(k,\xi)}. \tag{2.10}$$

Moreover, we say that $\psi_\ell, \in L^2(\mathbb{R}^+), \ell \in \Omega$ are orthogonal wavelets associated with the orthogonal scaling function $\varphi(x)$, if $\{\psi_\ell(x \ominus k) : k \in \mathbb{Z}^+, \ell \in \Omega\}$ is a basis of W_0 and

$$\langle \varphi(.), \varphi(. \ominus k)\rangle = \delta_{0,k},\ k \in \mathbb{Z}^+, \tag{2.11}$$

$$\langle \varphi(.), \psi_\ell(. \ominus k)\rangle = 0,\ \ell \in \Omega, k \in \mathbb{Z}^+, \tag{2.12}$$

$$\langle \psi_\ell(.), \psi_{\ell'}(. \ominus k)\rangle = \delta_{\ell,\ell'}\delta_{0,k},\ \ell, \ell' \in \Omega, k \in \mathbb{Z}^+. \tag{2.13}$$

We now introduce the definition of p-wavelet packets (as defined in [11]) associated with the scaling function $\varphi(x)$.

Definition 2.2. *Let the orthonormal scaling function $\varphi(x)$ and $\psi_\ell(x), \ell \in \Omega$ satisfy refinement equation (2.5) and wavelet equation (2.8), respectively. Then, for all $n \in \mathbb{Z}^+$, define the functions $\omega_n(x)$ recursively by*

$$\omega_n(x) = \omega_{pr+s}(x) = \sum_{k\in\mathbb{Z}^+} p\, a_k^s\, \omega_r(px \ominus k),\ s \in \Omega_0 \tag{2.14}$$

where $r \in \mathbb{Z}^+$ is the unique element such that $n = pr + s, s \in \Omega_0$ holds.

Applying the Walsh-Fourier transform for the both sides of (2.14) yields,

$$\hat{\omega}_{pr+s}(\xi) = m_s(p^{-1}\xi)\hat{\omega}_r(p^{-1}\xi),\ s \in \Omega_0. \tag{2.15}$$

When $r = 0$ and $s \in \Omega$, we obtain

$$\hat{\omega}_s(p\xi) = m_s(\xi)\hat{\omega}_0(\xi)$$

which shows that $\omega_0(x) = \varphi(x)$ and $\omega_s(x) = \psi_\ell(x)$.

Lemma 2.3[8]. *Let $\omega_0(x) \in L^2(\mathbb{R}^+)$. The system $\{\omega_0(. \ominus k) : k \in \mathbb{Z}^+\}$ is orthogonal in $L^2(\mathbb{R}^+)$ if and only if*

$$\sum_{k\in\mathbb{Z}^+} \hat{\omega}_0(\xi \oplus k)\overline{\hat{\omega}_0(\xi \oplus k)} = 1 \text{ for a.e. } \xi \in \mathbb{R}^+. \tag{2.16}$$

3. THE PROPERTIES OF ORTHOGONAL p-WAVELET PACKETS

In this section, we investigate the orthogonality property of the p-wavelet packets on $\mathbb{R}^+$ by virtue of the Walsh-Fourier transform.

Lemma 3.1. *Let $\{\omega_n : n \in \mathbb{Z}^+\}$ be the p-wavelet packets associated with the p-MRA $\{V_j\}_{j\in\mathbb{Z}}$. Then, we have*

$$\sum_{\ell\in\Omega_0} m_r\big(p^{-1}(\xi \oplus \ell)\big)\overline{m_s\big(p^{-1}(\xi \oplus \ell)\big)} = \delta_{r,s},\ r, s \in \Omega_0. \tag{3.1}$$

Proof. Using (2.11)–(2.13), (2.15) and Lemma 2.3, we have

$$\begin{aligned}
\delta_{r,s} &= \sum_{k\in\mathbb{Z}^+} \omega_r(\xi \oplus k)\overline{\omega_s(\xi \oplus k)} \\
&= \sum_{k\in\mathbb{Z}^+} m_r\big(p^{-1}(\xi \oplus k)\big)\hat{\omega}_0\big(p^{-1}(\xi \oplus k)\big)\overline{\hat{\omega}_0\big(p^{-1}(\xi \oplus k)\big)}\ \overline{m_s\big(p^{-1}(\xi \oplus k)\big)} \\
&= \sum_{\ell\in\Omega_0} m_r\big(p^{-1}(\xi \oplus \ell)\big) \\
&\quad \left\{\sum_{k\in\mathbb{Z}^+} \hat{\omega}_0\big(p^{-1}(\xi \oplus k) \oplus \ell\big)\overline{\hat{\omega}_0\big(p^{-1}(\xi \oplus k) \oplus \ell\big)}\right\}\overline{m_s\big(p^{-1}(\xi \oplus \ell)\big)} \\
&= \sum_{\ell\in\Omega_0} m_r\big(p^{-1}(\xi \oplus \ell)\big)\overline{m_s\big(p^{-1}(\xi \oplus \ell)\big)}.
\end{aligned}$$

Theorem 3.2. *If $\{\omega_n : n \in \mathbb{Z}^+\}$ are the p-wavelet packets with respect to the scaling function $\varphi(x)$. Then, for $n \in \mathbb{Z}^+$, we have*

$$\langle \omega_n(.), \omega_n(. \ominus k)\rangle = \delta_{0,k},\ k \in \mathbb{Z}^+. \tag{3.2}$$

Proof. We prove this result by using induction on n. It follows from (2.11) and (2.13) that the claim is true for $n = 0$ and $n = 1, 2, ..., p - 1$. Assume that (3.2) holds when $n < q$, where $q \in \mathbb{N}$. Then, we prove the result (3.2) for $n = q$. Let

$n = pr + s$, where $r \in \mathbb{Z}^+, s \in \Omega_0$ and $r < n$. Therefore, by induction assumption, we have

$$\langle \omega_r(.), \omega_r(. \ominus k)\rangle = \delta_{0,k} \iff \sum_{k\in\mathbb{Z}^+} \hat{\omega}_r(\xi \oplus k)\overline{\hat{\omega}_r(\xi \oplus k)} = 1,\ \xi \in \mathbb{R}^+.$$

Using Lemma 2.3, Lemma 3.1, and (2.15), we get

$$\begin{aligned}
\langle \omega_n(.), \omega_n(. \ominus k)\rangle &= \langle \hat{\omega}_n(.), \hat{\omega}_n(. \ominus k)\rangle \\
&= \int_{\mathbb{R}^+} \hat{\omega}_{pr+s}(\xi)\,\overline{\hat{\omega}_{pr+s}(\xi)}\chi(k,\xi)d\xi \\
&= \int_{\mathbb{R}^+} m_s(p^{-1}\xi)\hat{\omega}_r(p^{-1}\xi)\,\overline{m_s(p^{-1}\xi)}\,\overline{\hat{\omega}_r(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \sum_{k\in\mathbb{Z}^+}\int_{p([0,1]+k)} m_s(p^{-1}\xi)\hat{\omega}_r(p^{-1}\xi)\,\overline{m_s(p^{-1}\xi)}\,\overline{\hat{\omega}_r(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \int_{p[0,1]} m_s(p^{-1}\xi)\left\{\sum_{k\in\mathbb{Z}^+}\hat{\omega}_r\big(p^{-1}(\xi\oplus k)\big)\,\overline{\hat{\omega}_r\big(p^{-1}(\xi\oplus k)\big)}\right\} \\
&\quad \times \overline{m_s(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \int_{p[0,1]} m_s(p^{-1}\xi)\overline{m_s(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \int_{[0,1]}\sum_{\ell\in\Omega_0} m_s\big(p^{-1}(\xi\oplus\ell)\big)\overline{m_s\big(p^{-1}(\xi\oplus\ell)\big)}\chi(k,\xi)d\xi \\
&= \int_{[0,1]}\chi(k,\xi)d\xi \\
&= \delta_{0,k}.
\end{aligned}$$

Theorem 3.3. *Suppose $\{\omega_n : n \in \mathbb{Z}^+\}$ are the p-wavelet packets associated with the scaling function $\varphi(x)$. Then, for $r \in \mathbb{Z}^+$, we have*

$$\langle \omega_{pr+s_1}(.), \omega_{pr+s_2}(. \ominus k)\rangle = \delta_{0,k}\delta_{s_1,s_2},\ s_1, s_2 \in \Omega_0, k \in \mathbb{Z}^+. \tag{3.3}$$

Proof. By Lemma 2.3, we have

$$\begin{aligned}
\langle \omega_{pr+s_1}, \omega_{pr+s_2}(. \ominus k)\rangle &= \langle \hat{\omega}_{pr+s_1}, \hat{\omega}_{pr+s_2}(. \ominus k)\rangle \\
&= \int_{\mathbb{R}^+}\hat{\omega}_{pr+s_1}(\xi)\,\overline{\hat{\omega}_{pr+s_2}(\xi)}\chi(k,\xi)d\xi \\
&= \int_{\mathbb{R}^+} m_{s_1}(p^{-1}\xi)\hat{\omega}_r(p^{-1}\xi)\,\overline{m_{s_2}(p^{-1}\xi)}\,\overline{\hat{\omega}_r(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= p\sum_{k\in\mathbb{Z}^+}\int_{([0,1]+k)} m_{s_1}(\xi)\hat{\omega}_r(\xi)\,\overline{m_{s_2}(\xi)}\,\overline{\hat{\omega}_r(\xi)}\chi(k,p\xi)d\xi \\
&= p\int_{[0,1]} m_{s_1}(\xi)\left\{\sum_{k\in\mathbb{Z}^+}\hat{\omega}_r(\xi\oplus k)\overline{\hat{\omega}_r(\xi\oplus k)}\right\}\overline{m_{s_2}(\xi)}\chi(k,p\xi)d\xi \\
&= \int_{p[0,1]} m_{s_1}(p^{-1}\xi)\,\overline{m_{s_2}(p^{-1}\xi)}\chi(k,\xi)d\xi
\end{aligned}$$

$$
\begin{aligned}
&= \int_{[0,1]} \sum_{\ell\in\Omega_0} m_{s_1}\left(p^{-1}(\xi\oplus\ell)\right)\overline{m_{s_2}\left(p^{-1}(\xi\oplus\ell)\right)}\chi(k,\xi)d\xi \\
&= \int_{[0,1]} \delta_{s_1,s_2}\chi(k,\xi)d\xi \\
&= \delta_{0,k}\delta_{s_1,s_2}.
\end{aligned}
$$

Theorem 3.4. *Let $\{\omega_n : n\in\mathbb{Z}^+\}$ be the p-wavelet packets associated with the scaling function $\varphi(x)$. Then, for $\ell, n\in\mathbb{Z}^+$, we have*

$$
\langle \omega_\ell(.), \omega_n(.\ominus k)\rangle = \delta_{\ell,n}\delta_{0,k},\ k\in\mathbb{Z}^+. \tag{3.4}
$$

Proof. For $\ell = n$, the result (3.4) follows by Theorem 3.2. When $\ell\neq n$, and $\ell, n\in\Omega_0$, the result (3.4) can be established from Theorem 3.3. Assuming that ℓ is not equal to n, and atleast one of $\{\ell, n\}$ does not belong to Ω_0, then we can rewrite ℓ, n as $\ell = pr_1 + s_1, n = pu_1 + v_1$, where $r_1, u_1\in\mathbb{Z}^+, s_1, v_1\in\Omega_0$.

Case 1. If $r_1 = u_1$, then $s_1\neq v_1$. Therefore, (3.4) follows by virtue of (2.15), Lemma 2.3 and (3.1), i.e.,

$$
\begin{aligned}
\langle \omega_\ell(.), \omega_n(.\ominus k)\rangle &= \langle \omega_{pr_1+s_1}, \omega_{pu_1+v_1}(.\ominus k)\rangle \\
&= \langle \hat\omega_{pr_1+s_1}, \hat\omega_{pu_1+v_1}(.\ominus k)\rangle \\
&= \int_{\mathbb{R}^+} \hat\omega_{pr_1+s_1}(\xi)\,\overline{\hat\omega_{pu_1+v_1}(\xi)}\chi(k,\xi)d\xi \\
&= \int_{\mathbb{R}^+} m_{s_1}(p^{-1}\xi)\hat\omega_{r_1}(p^{-1}\xi)\,\overline{\hat\omega_{u_1}(p^{-1}\xi)}\,\overline{m_{v_1}(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \sum_{k\in\mathbb{Z}^+}\int_{p([0,1]+k)} m_{s_1}(p^{-1}\xi)\hat\omega_{r_1}(p^{-1}\xi)\,\overline{\hat\omega_{u_1}(p^{-1}\xi)}\,\overline{m_{v_1}(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \int_{p([0,1])} m_{s_1}(p^{-1}\xi)\left\{\sum_{k\in\mathbb{Z}^+}\hat\omega_{r_1}\left(p^{-1}(\xi\oplus k)\right)\overline{\hat\omega_{u_1}\left(p^{-1}(\xi\oplus k)\right)}\right\} \\
&\quad\times \overline{m_{v_1}(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \int_{[0,1]}\sum_{\ell\in\Omega_0} m_{s_1}\left(p^{-1}(\xi\oplus\ell)\right)\overline{m_{v_1}\left(p^{-1}(\xi\oplus\ell)\right)}\chi(k,\xi)d\xi \\
&= \int_{[0,1]}\delta_{s_1,v_1}\chi(k,\xi)d\xi \\
&= \delta_{0,k} = 0.
\end{aligned}
$$

Case 2. If $r_1\neq u_1$, order $r_1 = pr_2 + s_2, u_1 = pu_2 + v_2$, where $r_2, u_2\in\mathbb{Z}^+$ and $s_2, v_2\in\Omega_0$. If $r_2 = u_2$, then $s_2\neq v_2$. Similar to Case 1, (3.4) can be established. When $r_2\neq u_2$, we order $r_2 = pr_3 + s_3, u_2 = pu_3 + v_3$, where $r_3, u_3\in\mathbb{Z}^+$ and $s_3, v_3\in\Omega_0$. Thus, after taking finite steps (denoted by κ), we obtain $r_\kappa, u_\kappa\in\Omega_0$ and $s_\kappa, v_\kappa\in\Omega_0$. If $r_\kappa = u_\kappa$, then $s_\kappa\neq v_\kappa$. Similar to the Case 1, (3.4) follows. If $r_\kappa\neq u_\kappa$, then it follows from (2.11)–(2.13) that

$$
\langle \omega_{r_\kappa}(.), \omega_{u_\kappa}(.\ominus k)\rangle = 0,\ k\in\mathbb{Z}^+ \iff \sum_{k\in\mathbb{Z}^+}\hat\omega_{r_\kappa}(\xi\oplus k)\overline{\hat\omega_{u_\kappa}(\xi\oplus k)} = 0,\ \xi\in\mathbb{R}^+.
$$

Furthermore, we obtain

$$
\begin{aligned}
\langle \omega_r(.), \omega_u(.\ominus k)\rangle &= \langle \hat{\omega}_r(.), \hat{\omega}_u(.\ominus k)\rangle \\
&= \langle \hat{\omega}_{pr_1+s_1}, \hat{\omega}_{pu_1+v_1}(.\ominus k)\rangle \\
&= \int_{\mathbb{R}^+} \hat{\omega}_{pr_1+s_1}(\xi)\, \overline{\hat{\omega}_{pu_1+v_1}(\xi)} \chi(k,\xi) d\xi \\
&= \int_{\mathbb{R}^+} m_{s_1}(p^{-1}\xi) m_{s_2}(p^{-2}\xi) \hat{\omega}_{r_2}(p^{-2}\xi)\, \overline{\hat{\omega}_{u_2}(p^{-2}\xi)}\, \overline{m_{v_1}(p^{-1}\xi)} \\
&\quad \times \overline{m_{v_2}(p^{-2}\xi)} \chi(k,\xi) d\xi \\
&= \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= \int_{\mathbb{R}^+} \left\{ \prod_{\ell=1}^{\kappa} m_{s_\ell}(p^{-\ell}\xi) \right\} \hat{\omega}_{r_\kappa}(p^{-\kappa}\xi)\, \overline{\hat{\omega}_{u_\kappa}(p^{-\kappa}\xi)} \left\{ \prod_{\ell=1}^{\kappa} \overline{m_{v_\ell}(p^{-\ell}\xi)} \right\} \chi(k,\xi) d\xi \\
&= \sum_{k\in\mathbb{Z}^+} \int_{p^\kappa([0,1]+k)} \left\{ \prod_{\ell=1}^{\kappa} m_{s_\ell}(p^{-\ell}\xi) \right\} \left\{ \hat{\omega}_{r_\kappa}\left(p^{-\kappa}\xi\right) \overline{\hat{\omega}_{u_\kappa}\left(p^{-\kappa}\xi\right)} \right\} \\
&\quad \times \left\{ \prod_{\ell=1}^{\kappa} \overline{m_{v_\ell}(p^{-\ell}\xi)} \right\} \chi(k,\xi) d\xi \\
&= \int_{p^\kappa[0,1]} \left\{ \prod_{\ell=1}^{\kappa} m_{s_\ell}(p^{-\ell}\xi) \right\} \left\{ \sum_{k\in\mathbb{Z}^+} \hat{\omega}_{r_\kappa}\left(p^{-\kappa}(\xi\oplus k)\right) \overline{\hat{\omega}_{u_\kappa}\left(p^{-\kappa}(\xi\oplus k)\right)} \right\} \\
&\quad \times \left\{ \prod_{\ell=1}^{\kappa} \overline{m_{v_\ell}(p^{-\ell}\xi)} \right\} \chi(k,\xi) d\xi \\
&= \int_{p^\kappa[0,1]} \left\{ \prod_{\ell=1}^{\kappa} m_{s_\ell}(p^{-\ell}\xi) \right\} .0. \left\{ \prod_{\ell=1}^{\kappa} \overline{m_{v_\ell}(p^{-\ell}\xi)} \right\} \chi(k,\xi) d\xi = 0.
\end{aligned}
$$

Therefore, for any $\ell, n \in \mathbb{Z}^+$, result (3.4) is established.

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