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ON SOME I-CONVERGENT SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULI

VAKEEL.A. KHAN[1,*], SUTHEP SUANTAI[2] AND KHALID EBADULLAH[1]

[1]Department of Mathematics A.M.U, Aligarh-202002, India

[2] Department of Mathematics, Chiang Mai University, Chiang Mai 50200, Thailand

ABSTRACT. In this article we introduce the sequence spaces $c_0^I(F)$, $c^I(F)$ and $l_\infty^I(F)$ for the sequence of modulii $F = (f_k)$ and study some of the properties of these spaces. The results here in proved are analogus to those by Vakeel.A.Khan and Khalid Ebadullah [Theory and Applications of Mathematics and Computer Science,1(2)(2011): 22-30].

KEYWORDS : Ideal; Filter; Sequence of moduli; Lipschitz function; I-convergence field; I-convergent; Monotone; Solid spaces

1. INTRODUCTION

Throughout the article $\mathbb{N}, \mathbb{R}, \mathbb{C}$and ω denotes the set of natural,real,complex numbers and the class of all sequences respectively.

The notion of the statistical convergence was introduced by H.Fast [5].Later on it was studied by J.A.Fridy[6, 7] from the sequence space point of view and linked it with the summability theory.

The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko,Šalát and Wilezyński [18]. Later on it was studied by Šalát[26], Tripathy and Ziman [32] and Demirci[3], Das, Kostyrko, Wilczynski, and Malik [2], Mursaleen and Alotaibi [23], Mursaleen, Mohiuddine, and Edely [24], Mursaleen, and Mohiuddine [25], Mursaleen and Mohiuddine [26], Sahiner, Gurdal, Saltan and Gunawan [33] and Kumar [19]. Here we give some preliminaries about the notion of I-convergence.

Let X be a non empty set. Then a family of sets I$\subseteq 2^X$(power set of X)is said to be an ideal if I is additive i.e, A,B∈I ⇒A∪ B∈I and hereditary i.e., A∈I, B⊆A⇒B∈I.

A non-empty family of sets $\pounds(I) \subseteq 2^X$ is said to be filter on X if and only if $\Phi \notin \pounds$(I),for A,B∈ $\pounds$(I) we have A∩B∈ $\pounds$(I) and for each A∈ $\pounds$(I)and A⊆B implies B∈ $\pounds$(I).

An Ideal I$\subseteq 2^X$ is called non-trivial if I$\neq 2^X$.

* *Corresponding author.*
Email address : vakhan@math.com (V.A. Khan), suantai@yahoo.com (S. Suantai) and khalidebadullah@gmail.com (K. Ebadullah).
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A non-trivial ideal I$\subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\}$ $\subseteq$I.

A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal J$\neq$I containing I as a subset.

For each ideal I, there is a filter $\pounds$(I) corresponding to I.
i.e., $\pounds$(I) = $\{K \subseteq N : K^c \in I\}$,where K^c = N-K.

The idea of modulus was structured in 1953 by Nakano.(See[27]). A function f : [0,∞)$\longrightarrow$[0,∞) is called a modulus if

(1)f(t) = 0 if and only if t = 0,

(2) f(t+u)$\leq$ f(t)+ f(u) for all t,u$\geq$0,

(3) f is increasing, and

(4) f is continuous from the right at zero.

Ruckle [28, 29, 30] used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : (f(|x_k|)) \in X\}.$$

This space is an FK space ,and Ruckle[28] proved that that the intersecton of all such $X(f)$ spaces is ϕ, the space of all finite sequences.
The space $X(f)$ is closely related to the space l_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$.Thus Ruckle[29] proved that,for any modulus f,

$$X(f) \subset l_1 \text{ and } X(f)^\alpha = l_\infty$$

Where

$$X(f)^\alpha = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}.$$

The space $X(f)$ is a Banach space with respect to the norm

$$||x|| = \sum_{k=1}^{\infty} f(|x_k|) < \infty.\text{(See[29])}.$$

Spaces of the type $X(f)$ are a special case of the spaces structured by B. Gramsch [10]. From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Therefore symmetric sequence spaces, which are locally convex have been frequently studied by D.J.H Garling[8, 9], G.Köthe[17] and W.H.Ruckle[28, 29, 30].

After then E.Kolk gave an extension of $X(f)$ by cosidering a sequence of modulii $F = (f_k)$ and defined the sequence space

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}.\text{(See[15, 16])}.$$

Definition 1.1. A sequence space E is said to be solid or normal if $(x_k) \in$ E implies $(\alpha_k x_k) \in$ E for all sequence of scalars (α_k) with $|\alpha_k| < 1$ for all k $\in$ N.

Definition 1.2. A sequence space E is said to be monotone if it contains the cannonical preimages of all its stepspaces.

Definition 1.3. A sequence space E is said to be covergencefree if $(y_k)$$\in$ E whenever $(x_k)$$\in$ E and $x_k = 0$ implies $y_k = 0$.

Definition 1.4. A sequence space E is said to be a sequencealgebra if $(x_k y_k) \in$ E whenever $(x_k) \in$ E, $(y_k) \in$ E.

Definition 1.5. A sequence space E is said to be symmetric if $(x_{\pi(k)}) \in$ E whenever $(x_k) \in$ E where π(k) is a permutation on $\mathbb{N}$.

Definition 1.6. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$. $\{k \in N : |x_k - L| \geq \epsilon\} \in$ I. In this case we write $I - \lim x_k = L$.
The space c^I of all I-convergent sequences to L is given by

$$c^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I, \text{for some L} \in \mathbb{C}\}.$$

Definition 1.7. A sequence $(x_k) \in \omega$ is said to be I-null if L = 0 .In this case we write $I - \lim x_k$ = 0.

Definition 1.8. A sequence $(x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number m=m(ϵ) such that $\{k \in N : |x_k - x_m| \geq \epsilon\} \in$ I.

Definition 1.9. A sequence $(x_k) \in \omega$ is said to be I-bounded if there exists M >0 such that $\{k \in N : |x_k| > M\} \in I$.

Definition 1.10. Take for I the class I_f of all finite subsets of $\mathbb{N}$.Then I_f is a non-trivial admissible ideal and I_f convergence coincides with the usual convergence with respect to the metric in X.(see[18]).

Definition 1.11. For I= I_δ the class of all $A \subset \mathbb{N}$ with $\delta(A) = 0$ respectively. I_δ is a non-trivial admissible ideal,I_δ-convergence is said to be logarithmic statistical covergence.(see[18]).

Definition 1.12. A map $\hbar$ defined on a domain $D \subset X$ i.e., $\hbar : D \subset X \to \boldsymbol{R}$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where $K$is known as the Lipschitz constant. The class of K-Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$(see[32]).

Definition 1.13. A convergence field of I-covergence is a set

$$F(I) = \{x = (x_k) \in l_\infty : \text{there exists } I - \lim x \in \boldsymbol{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of l_∞ with respect to the supremum norm, $F(I) = l_\infty \cap c^I$(See[31]).
Define a function $\hbar : F(I) \to \boldsymbol{R}$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \to \boldsymbol{R}$ is a Lipschitz function.(see [1, 4, 11, 12, 13, 14, 20, 21, 22, 26, 31, 34, 35]).

Throughout the article $l_\infty, c^I, c_0^I, m^I$ and m_0^I represent the bounded , I-convergent, I-null, bounded I-convergent and bounded I-null sequence spaces respectively.

In this article we introduce the following classes of sequence spaces.

$$c^I(F) = \{(x_k) \in \omega : I - \lim f_k(|x_k|) = L \text{ for some L}\} \in I$$
$$c_0^I(F) = \{(x_k) \in \omega : I - \lim f_k(|x_k|) = 0\} \in I$$
$$l_\infty^I(F) = \{(x_k) \in \omega : \sup_k f_k(|x_k|) < \infty\} \in I$$

We also denote by

$$m^I(F) = c^I(F) \cap l_\infty(F)$$

and

$$m_0^I(F) = c_0^I(F) \cap l_\infty(F)$$

The following Lemmas will be used for establishing some results of this article.

Lemma 1.14. *Let E be a sequence space.If E is solid then E is monotone.*

Lemma 1.15. *Let K∈ £(I) and M⊆N. If M∉I, then M∩N ∉I.*

Lemma 1.16. *If I $\subset 2^N$ and M⊆N. If M ∉I, then M∩N ∉I.*

2. MAIN RESULTS

Theorem 2.1. *For any sequence of moduli $F = (f_k)$,the classes of sequences $c^I(F), c_0^I(F), m^I(F)$ and $m_0^I(F)$ are linear spaces.*

Proof: We shall prove the result for the space $c^I(F)$.
The proof for the other spaces will follow similarly.
Let $(x_k), (y_k) \in c^I(F)$ and let α, β be the scalars. Then

$$I - \lim f_k(|x_k - L_1|) = 0, \text{ for some} L_1 \in c;$$

$$I - \lim f_k(|y_k - L_2|) = 0, \text{ for some} L_2 \in c.$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{k \in N : f_k(|x_k - L_1|) > \frac{\epsilon}{2}\} \in I, \quad (1)$$

$$A_2 = \{k \in N : f_k(|y_k - L_2|) > \frac{\epsilon}{2}\} \in I. \quad (2)$$

Since f_k is a modulus function, we have

$$f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2) \leq f_k(|\alpha||x_k - L_1|) + f_k(|\beta||y_k - L_2|)$$

$$\leq f_k(|x_k - L_1|) + f_k(|y_k - L_2|).$$

Now, by (1) and (2), $\{k \in$ N: $f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|) > \epsilon\} \subset A_1 \cup A_2$.
Therefore $(\alpha x_k + \beta y_k) \in c^I(F)$.
Hence $c^I(F)$ is a linear space.

Theorem 2.2. *A sequence $x = (x_k) \in m^I(F)$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that*

$$\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I(F). \quad (3)$$

Proof: Suppose that $L = I - \lim x$. Then

$$B_\epsilon = \{k \in \mathbb{N} : |x_k - L| < \frac{\epsilon}{2}\} \in m^I(F). \text{ For all } \epsilon > 0.$$

Fix an $N_\epsilon \in B_\epsilon$.Then we have

$$|x_{N_\epsilon} - x_k| \leq |x_{N_\epsilon} - L| + |L - x_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $k \in B_\epsilon$.
Hence $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I(F)$.

Conversly, suppose that $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I(F)$.
That is $\{k \in \mathbb{N} : (|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I(F)$ for all $\epsilon > 0$. Then the set

$$C_\epsilon = \{k \in \mathbb{N} : x_k \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in m^I(F) \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I(F)$ as well as $C_{\frac{\epsilon}{2}} \in m^I(F)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m^I(F)$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi,$$

that is

$$\{k \in \mathbb{N} : x_k \in J\} \in m^I(F),$$

that is

$$diam J \leq diam J_\epsilon,$$

where the diam of J denotes the length of interval J.
In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq ....$$

with the property that $diam I_k \leq \frac{1}{2} diam I_{k-1}$ for (k=2,3,4,.....) and $\{k \in \mathbb{N} : x_k \in I_k\} \in m^I(F)$ for (k=1,2,3,4,......).
Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim x$. So that $f_k(\xi) = I - \lim f_k(x)$, that is $L = I - \lim f_k(x)$.

Result 2.3. The spaces $c_0^I(F)$ and $m_0^I(F)$ are solid and monotone .

Proof:We shall prove the result for $c_0^I(F)$. Let $x_k \in c_0^I$(F).Then

$$I - \lim_k f_k(|x_k|) = 0. \tag{4}$$

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all k $\in$ N. Then the result follows from (4) and the following inequality
$f_k(|\alpha_k x_k|) \leq |\alpha_k| f_k(|x_k|) \leq f_k|x_k|)$ for all k $\in$ N.
That the space c_0^I(F) is monotone follows from the Lemma 1.14.
For $m_0^I(F)$ the result can be proved similarly.

Result 2.4. The spaces $c^I(F)$ and $m^I(F)$ are neither solid nor monotone in general .

Proof:Here we give a counter example.
Let I = I_δ and f(x) = x^2 for all x$\in$ $[0, \infty)$. Consider the K-step space $X_K(f)$ of X defined as follows,

Let $(x_k) \in X$and let $(y_k) \in X_K$ be such that

$$(y_k) = \begin{cases} (x_k), \text{if k is even,} \\ 0, otherwise. \end{cases}$$

Consider the sequence (x_k) defined by $(x_k) = 1$ for all k $\in$ N.
Then $(x_k) \in c^I(F)$ but its K-stepspace preimage does not belong to $c^I(F)$. Thus $c^I(F)$ is not monotone.Hence $c^I(F)$ is not solid.

Result 2.5. The spaces $c^I(F)$ and $c_0^I(F)$ are sequence algebras.

Proof: We prove that $c_0^I(F)$ is a sequence algebra.
Let $(x_k), (y_k) \in c_0^I(F)$. Then

$$I - \lim f_k(|x_k|) = 0$$

and

$$I - \lim f_k(|y_k|) = 0.$$

Then we have

$$I - \lim f_k(|(x_k.y_k)|) = 0.$$

Thus $(x_k.y_k) \in c_0^I(F)$ is a sequence algebra.
For the space $c^I(F)$, the result can be proved similarly.

Result 2.6. The spaces $c^I(F)$ and $c_0^I(F)$ are not convergence free in general.

Proof:Here we give a counter example.
Let $I = I_f$ and $f(x) = x^3$ for all $x \in [0, \infty)$. Consider the sequence (x_k) and (y_k) defined by

$$x_k = \frac{1}{k} \quad and\ y_k = k \text{ for all } k \in \mathrm{N}.$$

Then $(x_k) \in c^I(F)$ and $c_0^I(F)$, but $(y_k) \notin c^I(F)$ and $c_0^I(F)$.
Hence the spaces $c^I(F)$ and $c_0^I(F)$ are not convergence free.

Result 2.7. If I is not maximal and $I \neq I_f$,then the spaces $c^I(F)$ and $c_0^I(F)$ are not symmetric.

Proof: Let $A \in I$ be infinite and $f(x) = x$ for all x$\in [0, \infty)$.
If

$$x_k = \begin{cases} 1, \text{for } k \in A, \\ 0, otherwise. \end{cases}$$

Then by lemma 1.16 $x_k \in c_0^I(F) \subset c^I(F)$.
Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$.Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), \text{for } k \in K, \\ \psi(k), otherwise, \end{cases}$$

is a permutation on $\mathbb{N}$, but $x_{\pi(k)} \notin c^I(F)$ and $x_{\pi(k)} \notin c_0^I(F)$.
Hence $c_0^I(F)$ and $c^I(F)$are not symmetric.

Theorem 2.3. *Let $F = (f_k)$ be the sequence of modulii. Then $c_0^I(F) \subset c^I(F) \subset l_\infty^I(F)$ and the inclusions are proper.*

Proof: Let $x_k \in c^I(F)$. Then there exists L $\in$ C such that

$$I - \lim f_k(|x_k - L|) = 0.$$

We have $f_k(|x_k|) \leq \frac{1}{2} f_k(|x_k - L|) + f_k \frac{1}{2}(|L|)$.
Taking the supremum over k on both sides we get $x_k \in l_\infty(F)$.
The inclusion $c_0^I(F) \subset c^I(F)$ is obvious.

Theorem 2.4. *The function $\hbar : m^I(F) \to \mathbb{R}$ is the Lipschitz function,where $m^I(F) = c^I(F) \cap l_\infty(F)$, and hence uniformly cotinuous.*

Proof:Let $x, y \in m^I(F)$,$x \neq y$.Then the sets

$$A_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| \geq ||x - y||\} \in I,$$
$$A_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| \geq ||x - y||\} \in I.$$

Thus the sets,

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < ||x - y||\} \in m^I(F),$$
$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| < ||x - y||\} \in m^I(F).$$

Hence also $B = B_x \cap B_y \in m^I(F)$, so that $B \neq \phi$.
Now taking k in B,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_k| + |x_k - y_k| + |y_k - \hbar(y)| \leq 3||x - y||.$$

Thus $\hbar$ is a Lipschitz function. For $m_0^I(F)$ the result can be proved similarly.

Result 2.10. If $x, y \in m^I(F)$,then $(x.y) \in m^I(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

Proof: For $\epsilon > 0$

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < \epsilon\} \in m^I(F),$$
$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| < \epsilon\} \in m^I(F).$$

Now,

$$\begin{aligned} |x_k y_k - \hbar(x)\hbar(y)| &= |x_k y_k - x_k \hbar(y) + x_k \hbar(y) - \hbar(x)\hbar(y)| \\ &\leq |x_k||y_k - \hbar(y)| + |\hbar(y)||x_k - \hbar(x)|. \end{aligned} \tag{8}$$

As $m^I(F) \subseteq l_\infty(F)$,there exists an $M \in \boldsymbol{R}$ such that $|x_k| < M$ and $|\hbar(y)| < M$.
Using (8) we get

$$|x_k y_k - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon.$$

For all $k \in B_x \cap B_y \in m^I(F)$. Hence $(x.y) \in m^I(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.
For $m_0^I(F)$ the result can be proved similarly.

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