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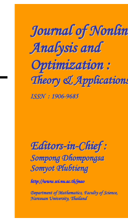

ON A HALF-DISCRETE REVERSE MULHOLLAND'S INEQUALITY

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ABSTRACT. By using the way of weight functions and the technique of real analysis, a half-discrete reverse Mulholland's Inequality with a best constant factor is given. The extension with multi-parameters and the equivalent forms are also considered.

KEYWORDS : Mulholland's inequality; Weight function; Equivalent form; Reverse.
AMS Subject Classification: 26D15.

1. INTRODUCTION

Assuming that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(\geq 0) \in L^p(0,\infty), g(\geq 0) \in L^q(0,\infty), ||f||_p = \{\int_0^\infty f^p(x)dx\}^{\frac{1}{p}} > 0, ||g||_q > 0$, we have the following Hardy-Hilbert's integral inequality (cf. [3]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} ||f||_p ||g||_q, \tag{1.1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. If $a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l^p, b = \{b_n\}_{n=1}^\infty \in l^q, ||a||_p = \{\sum_{m=1}^\infty a_m^p\}^{\frac{1}{p}} > 0, ||b||_q > 0$, then we still have the following discrete Hardy-Hilbert's inequality with the same best constant factor $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} ||a||_p ||b||_q. \tag{1.2}$$

Inequalities (1.1) and (1.2) are important in analysis and its applications (cf. [10], [13], [18]). Also we have the following Mulholland's inequality with the same best constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [3], [15]):

$$\sum_{m=2}^\infty \sum_{n=2}^\infty \frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \sum_{m=2}^\infty m^{p-1} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^\infty n^{q-1} b_n^q \right\}^{\frac{1}{q}}. \tag{1.3}$$

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In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [11] gave an extension of (1.1) (for $p=q=2$). Refinement the results of [11], Yang [14] gave some best extensions of (1.1) and (1.2) as follows: If $\lambda, \lambda_1, \lambda_2 \in \mathbf{R}, \lambda_1+\lambda_2=\lambda, k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$, $k(\lambda_1)=\int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}dt \in R_+$, $\phi(x)=x^{p(1-\lambda_1)-1}$, $\psi(x)=x^{q(1-\lambda_2)-1}$, $f(\geq 0) \in L_{p,\phi}(0,\infty)=\{f|||f||_{p,\phi}:=\{\int_0^\infty \phi(x)|f(x)|^p dx\}^{\frac{1}{p}}<\infty\}, g(\geq 0) \in L_{q,\psi}(0,\infty), ||f||_{p,\phi}, ||g||_{q,\psi}>0$, then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x,y)f(x)g(y)dxdy < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{1.4}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover if $k_\lambda(x,y)$ is also finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing for $x>0(y>0)$, then for $a_m,b_n \geq 0$, $a=\{a_m\}_{m=1}^\infty \in l_{p,\phi}=\{a|||a||_{p,\phi}:=\{\sum_{n=1}^\infty \phi(n)|a_n|^p\}^{\frac{1}{p}}<\infty\}, b=\{b_n\}_{n=1}^\infty \in l_{q,\psi}, ||a||_{p,\phi}, ||b||_{q,\psi}>0$, we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m,n)a_m b_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{1.5}$$

where the constant factor $k(\lambda_1)$ is the best possible. For $\lambda=1, k_1(x,y)=\frac{1}{x+y}$, $\lambda_1=\frac{1}{q}, \lambda_2=\frac{1}{p}$, (1.4) reduces to (1.1), and (1.5) reduces to (1.2). Some other results including the reverse Hilbert-type inequalities are provided by [19]-[9].

On half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [3]. But they did not prove that the the constant factors in the inequalities are the best possible. And Yang [12] gave a result by introducing an interval variable and proved that the constant factor is the best possible. Recently, Yang [17] gave a half-discrete Hilbert's inequality and [16] gave the following half-discrete reverse Hilbert-type inequality with the best constant factor 4:

$$\int_0^\infty f(x)\sum_{n=1}^\infty \min\{x,n\}a_n dx > 4\left\{\int_0^\infty (1-\theta_1(x))x^{\frac{3p}{2}-1}f^p(x)dx\right\}^{\frac{1}{p}}$$
$$\times \left\{\sum_{n=1}^\infty n^{\frac{3q}{2}-1}a_n^q\right\}^{\frac{1}{q}} \quad (\theta_1(x)\in(0,1)). \tag{1.6}$$

In this paper, by using the way of weight functions and the technique of real analysis, a half-discrete reverse Mulholland's inequality with a best constant factor is given as follows:

$$\int_1^\infty f(x)\sum_{n=2}^\infty \frac{a_n}{\ln xn}dx > \pi\left\{\int_1^\infty \frac{1-\theta_1(x)}{(\ln x)^{1-\frac{p}{2}}}x^{p-1}f^p(x)dx\right\}^{\frac{1}{p}}$$
$$\times \left\{\sum_{n=2}^\infty \frac{n^{q-1}}{(\ln n)^{1-\frac{q}{2}}}a_n^q\right\}^{\frac{1}{q}} \quad (\theta_1(x)\in(0,1)). \tag{1.7}$$

A best extension of (1.7) with multi-parameters, some equivalent forms are considered.

2. SOME LEMMAS

Lemma 2.1. *If $\lambda, \lambda_1 > 0, 0<\lambda_2 \leq 1, \lambda_1+\lambda_2=\lambda$, setting weight functions $\omega(n)$ and $\varpi(x)$ as follows:*

$$\omega(n) \quad : \quad = (\ln n)^{\lambda_2}\int_1^\infty \frac{1}{x(\ln xn)^\lambda}(\ln x)^{\lambda_1-1}dx, n \in \mathbf{N}\backslash\{1\}, \tag{2.1}$$

$$\varpi(x) \quad : \quad = (\ln x)^{\lambda_1} \sum_{n=2}^{\infty} \frac{1}{n(\ln xn)^{\lambda}} (\ln n)^{\lambda_2 - 1}, x \in (1, \infty), \tag{2.2}$$

then we have

$$B(\lambda_1, \lambda_2)(1 - \theta_\lambda(x)) < \varpi(x) < \omega(n) = B(\lambda_1, \lambda_2), \tag{2.3}$$

where $\theta_\lambda(x) := \frac{1}{B(\lambda_1,\lambda_2)} \int_0^{\frac{1}{\ln x}} \frac{t^{\lambda_2-1}}{(1+t)^{\lambda}} dt \in (0, 1)$ *and* $\theta_\lambda(x) = O(\frac{1}{(\ln x)^{\lambda_2}})(x \in (1, \infty))$.

Proof. Setting $t = \frac{\ln x}{\ln n}$ in (2.1), by calculation, we have

$$\omega(n) = \int_0^{\infty} \frac{1}{(1+t)^{\lambda}} t^{\lambda_1 - 1} dt = B(\lambda_1, \lambda_2).$$

Since for fixed $x > 1$,

$$h(x, y) := \frac{1}{y(\ln xy)^{\lambda}} (\ln y)^{\lambda_2 - 1} = \frac{1}{y(\ln x + \ln y)^{\lambda} (\ln y)^{1-\lambda_2}}$$

is strictly decreasing for $y \in (1, \infty)$, then we find

$$\varpi(x) < (\ln x)^{\lambda_1} \int_1^{\infty} \frac{1}{y(\ln xy)^{\lambda}} (\ln y)^{\lambda_2 - 1} dy$$

$$\overset{t=(\ln y)/(\ln x)}{=} \int_0^{\infty} \frac{t^{\lambda_2 - 1}}{(1+t)^{\lambda}} dt = B(\lambda_2, \lambda_1) = B(\lambda_1, \lambda_2),$$

$$\begin{aligned} \varpi(x) \quad &> \quad (\ln x)^{\lambda_1} \int_e^{\infty} \frac{(\ln y)^{\lambda_2 - 1}}{y(\ln xy)^{\lambda}} dy = \int_{\frac{1}{\ln x}}^{\infty} \frac{t^{\lambda_2 - 1}}{(1+t)^{\lambda}} dt \\ &= \quad B(\lambda_1, \lambda_2)(1 - \theta_\lambda(x)) > 0, \end{aligned}$$

$$\begin{aligned} 0 \quad &< \quad \theta_\lambda(x) = \frac{1}{B(\lambda_1, \lambda_2)} \int_0^{\frac{1}{\ln x}} \frac{t^{\lambda_2 - 1}}{(1+t)^{\lambda}} dt \\ &< \quad \frac{1}{B(\lambda_1, \lambda_2)} \int_0^{\frac{1}{\ln x}} t^{\lambda_2 - 1} dt = \frac{1}{\lambda_2 B(\lambda_1, \lambda_2)} \frac{1}{(\ln x)^{\lambda_2}}, \end{aligned}$$

and then (2.3) is valid. The lemma is proved. □

Lemma 2.2. *Let the assumptions of Lemma 2.1 be fulfilled and additionally,* $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, a_n \geq 0, n \in \mathbf{N} \backslash \{1\}, f(x)$ *is a non-negative measurable function in* $(1, \infty)$. *Then we have the following inequalities (Note: in this paper, if* $a_n = 0$, *then we think* $a_n^q = 0$ $(q < 0)$*):*

$$\begin{aligned} J \quad &: \quad = \left\{ \sum_{n=2}^{\infty} \frac{(\ln n)^{p\lambda_2 - 1}}{n} \left[\int_1^{\infty} \frac{f(x)}{(\ln xn)^{\lambda}} dx \right]^p \right\}^{\frac{1}{p}} \\ &\geq \quad [B(\lambda_1, \lambda_2)]^{\frac{1}{q}} \left\{ \int_1^{\infty} \varpi(x) x^{p-1} (\ln x)^{p(1-\lambda_1)-1} f^p(x) dx \right\}^{\frac{1}{p}}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} L_1 \quad &: \quad = \left\{ \int_1^{\infty} \frac{(\ln x)^{q\lambda_1 - 1}}{x[\varpi(x)]^{q-1}} \left[\sum_{n=2}^{\infty} \frac{a_n}{(\ln xn)^{\lambda}} \right]^q dx \right\}^{\frac{1}{q}} \\ &\geq \quad \left\{ B(\lambda_1, \lambda_2) \sum_{n=2}^{\infty} n^{q-1} (\ln n)^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}}. \end{aligned} \tag{2.5}$$

Proof. (i) By the reverse Hölder's inequality with weight (cf. [7]) and (2.3), it follows

$$\left[\int_1^\infty \frac{f(x)dx}{(\ln xn)^\lambda}\right]^p = \left\{\int_1^\infty \frac{1}{(\ln xn)^\lambda}\left[\frac{(\ln x)^{(1-\lambda_1)/q}}{(\ln n)^{(1-\lambda_2)/p}}\frac{x^{1/q}}{n^{1/p}}f(x)\right]\right.$$
$$\left.\times\left[\frac{(\ln n)^{(1-\lambda_2)/p}}{(\ln x)^{(1-\lambda_1)/q}}\frac{n^{1/p}}{x^{1/q}}\right]dx\right\}^p$$
$$\geq \int_1^\infty \frac{1}{(\ln xn)^\lambda}\frac{x^{p-1}(\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln n)^{1-\lambda_2}}f^p(x)dx$$
$$\times\left\{\int_1^\infty \frac{1}{(\ln xn)^\lambda}\frac{n^{q-1}(\ln n)^{(1-\lambda_2)(q-1)}}{x(\ln x)^{1-\lambda_1}}dx\right\}^{p-1}$$
$$= \left\{\omega(n)\frac{(\ln n)^{q(1-\lambda_2)-1}}{n^{1-q}}\right\}^{p-1}\int_1^\infty \frac{x^{p-1}(\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln xn)^\lambda(\ln n)^{1-\lambda_2}}f^p(x)dx$$
$$= \frac{[B(\lambda_1,\lambda_2)]^{p-1}n}{(\ln n)^{p\lambda_2-1}}\int_1^\infty \frac{x^{p-1}(\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln xn)^\lambda(\ln n)^{1-\lambda_2}}f^p(x)dx.$$

Then by Lebesgue term by term integration theorem (cf. [6]), we have

$$J \geq [B(\lambda_1,\lambda_2)]^{\frac{1}{q}}\left\{\sum_{n=2}^\infty\int_1^\infty \frac{x^{p-1}(\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln xn)^\lambda(\ln n)^{1-\lambda_2}}f^p(x)dx\right\}^{\frac{1}{p}}$$
$$= [B(\lambda_1,\lambda_2)]^{\frac{1}{q}}\left\{\int_1^\infty\sum_{n=2}^\infty \frac{x^{p-1}(\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln xn)^\lambda(\ln n)^{1-\lambda_2}}f^p(x)dx\right\}^{\frac{1}{p}}$$
$$= [B(\lambda_1,\lambda_2)]^{\frac{1}{q}}\left\{\int_1^\infty \varpi(x)x^{p-1}(\ln x)^{p(1-\lambda_1)-1}f^p(x)dx\right\}^{\frac{1}{p}},$$

and (2.4) follows. Still by the reverse Hölder's inequality with weight ($q<0$), we have

$$\left[\sum_{n=2}^\infty \frac{a_n}{(\ln xn)^\lambda}\right]^q = \left\{\sum_{n=2}^\infty \frac{1}{(\ln xn)^\lambda}\left[\frac{(\ln x)^{(1-\lambda_1)/q}}{(\ln n)^{(1-\lambda_2)/p}}\frac{x^{1/q}}{n^{1/p}}\right]\right.$$
$$\left.\times\left[\frac{(\ln n)^{(1-\lambda_2)/p}}{(\ln x)^{(1-\lambda_1)/q}}\frac{n^{1/p}}{x^{1/q}}a_n\right]\right\}^q$$
$$\leq \left\{\sum_{n=2}^\infty \frac{1}{(\ln xn)^\lambda}\frac{x^{p-1}(\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln n)^{1-\lambda_2}}\right\}^{q-1}$$
$$\times\sum_{n=2}^\infty \frac{1}{(\ln xn)^\lambda}\frac{n^{q-1}(\ln n)^{(1-\lambda_2)(q-1)}}{x(\ln x)^{1-\lambda_1}}a_n^q$$
$$= \frac{x[\varpi(x)]^{q-1}}{(\ln x)^{q\lambda_1-1}}\sum_{n=2}^\infty \frac{(\ln x)^{\lambda_1-1}}{x(\ln xn)^\lambda}n^{q-1}(\ln n)^{(q-1)(1-\lambda_2)}a_n^q.$$

Then by Lebesgue term by term integration theorem, we have

$$L_1 \geq \left\{\int_1^\infty\sum_{n=2}^\infty \frac{(\ln x)^{\lambda_1-1}}{x(\ln xn)^\lambda}n^{q-1}(\ln n)^{(q-1)(1-\lambda_2)}a_n^q dx\right\}^{\frac{1}{q}}$$
$$= \left\{\sum_{n=2}^\infty\left[(\ln n)^{\lambda_2}\int_1^\infty \frac{(\ln x)^{\lambda_1-1}}{x(\ln xn)^\lambda}dx\right]n^{q-1}(\ln n)^{q(1-\lambda_2)-1}a_n^q\right\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{n=2}^{\infty} \omega(n) n^{q-1} (\ln n)^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}},$$

and then in view of (2.3), inequality (2.5) follows. □

3. MAIN RESULTS

In the following, for $0 < p < 1, q < 0$, we still use the normal expressions of $||f||_{p,\Phi}$ and $||a||_{q,\Psi}$. Setting $\Phi(x) := (1 - \theta_\lambda(x)) x^{p-1} (\ln x)^{p(1-\lambda_1)-1} (x \in (1, \infty))$, $\Psi(n) := n^{q-1} (\ln n)^{q(1-\lambda_2)-1} (n \in \mathbf{N} \backslash \{1\})$, we have

$$[\Phi(x)]^{1-q} = \frac{(\ln x)^{q\lambda_1 - 1}}{x(1-\theta_\lambda(x))^{q-1}}, [\Psi(n)]^{1-p} = \frac{(\ln n)^{p\lambda_2 - 1}}{n}$$

and

Theorem 3.1. *If* $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda_1 > 0, 0 < \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda, f(x), a_n \geq 0$, $f \in L_{p,\Phi}(1, \infty), a = \{a_n\}_{n=2}^{\infty} \in l_{q,\Psi}$, $||f||_{p,\Phi} > 0$ *and* $||a||_{q,\Psi} > 0$, *then we have the following equivalent inequalities:*

$$\begin{aligned} I &:= \sum_{n=2}^{\infty} a_n \int_1^{\infty} \frac{f(x)}{(\ln xn)^{\lambda}} dx \\ &= \int_1^{\infty} f(x) \sum_{n=2}^{\infty} \frac{a_n}{(\ln xn)^{\lambda}} dx > B(\lambda_1, \lambda_2) ||f||_{p,\Phi} ||a||_{q,\Psi}, \end{aligned} \tag{3.1}$$

$$J = \left\{ \sum_{n=2}^{\infty} \frac{(\ln n)^{p\lambda_2 - 1}}{n} \left[\int_1^{\infty} \frac{f(x)}{(\ln xn)^{\lambda}} dx \right]^p \right\}^{\frac{1}{p}} > B(\lambda_1, \lambda_2) ||f||_{p,\Phi}, \tag{3.2}$$

$$L := \left\{ \int_1^{\infty} \frac{(\ln x)^{q\lambda_1 - 1}}{x(1-\theta_\lambda(x))^{q-1}} \left[\sum_{n=2}^{\infty} \frac{a_n}{(\ln xn)^{\lambda}} \right]^q dx \right\}^{\frac{1}{q}} > B(\lambda_1, \lambda_2) ||a||_{q,\Psi}, \tag{3.3}$$

where the constant factor $B(\lambda_1, \lambda_2)$ *in the above inequalities is the best possible.*

Proof. By Lebesgue term by term integration theorem, there are two expressions for I in (3.1). In view of (2.4), for $\varpi(x) > B(\lambda_1, \lambda_2)(1 - \theta_\lambda(x))$, we have (3.2). By the reverse Hölder's inequality, we have

$$I = \sum_{n=2}^{\infty} \left[\Psi^{\frac{-1}{q}}(n) \int_1^{\infty} \frac{1}{(\ln xn)^{\lambda}} f(x) dx \right] [\Psi^{\frac{1}{q}}(n) a_n] \geq J ||a||_{q,\Psi}. \tag{3.4}$$

Then by (3.2), we have (3.1). On the other-hand, assuming that (3.1) is valid, setting

$$a_n := [\Psi(n)]^{1-p} \left[\int_1^{\infty} \frac{1}{(\ln xn)^{\lambda}} f(x) dx \right]^{p-1}, n \in \mathbf{N} \backslash \{1\},$$

then $J^{p-1} = ||a||_{q,\Psi}$. By (2.4), we find $J > 0$. If $J = \infty$, then (3.2) is naturally valid; if $J < \infty$, then by (3.1), we have

$$\begin{aligned} ||a||_{q,\Psi}^q &= J^p = I > B(\lambda_1, \lambda_2) ||f||_{p,\Phi} ||a||_{q,\Psi}, \\ ||a||_{q,\Psi}^{q-1} &= J > B(\lambda_1, \lambda_2) ||f||_{p,\Phi}, \end{aligned}$$

and we have (3.2), which is equivalent to (3.1).

In view of (2.5), for $[\varpi(x)]^{1-q} > [B(\lambda_1,\lambda_2)(1-\theta_\lambda(x))]^{1-q}$, we have (3.3). By the reverse Hölder's inequality, we find

$$I = \int_1^\infty [\Phi^{\frac{1}{p}}(x)f(x)]\left[\Phi^{\frac{-1}{p}}(x)\sum_{n=2}^\infty \frac{1}{(\ln xn)^\lambda}a_n\right]dx \geq ||f||_{p,\Phi}L. \tag{3.5}$$

Then by (3.3), we have (3.1). On the other-hand, assuming that (3.1) is valid, setting

$$f(x) := [\Phi(x)]^{1-q}\left[\sum_{n=2}^\infty \frac{1}{(\ln xn)^\lambda}a_n\right]^{q-1}, x \in (1,\infty),$$

then $L^{q-1} = ||f||_{p,\Phi}$. By (2.5), we find $L > 0$. If $L = \infty$, then (3.3) is naturally valid; if $L < \infty$, then by (3.1), we have

$$\begin{aligned} ||f||_{p,\Phi}^p &= L^q = I > B(\lambda_1,\lambda_2)||f||_{p,\Phi}||a||_{q,\Psi}, \\ ||f||_{p,\Phi}^{p-1} &= L > B(\lambda_1,\lambda_2)||a||_{q,\Psi}, \end{aligned}$$

and we have (3.3) which is equivalent to (3.1). Hence inequalities (3.1), (3.2) and (3.3) are equivalent.

For $0 < \varepsilon < p\lambda_1$, setting $\widetilde{f}(x) = 0, x \in (1,e); \widetilde{f}(x) = \frac{1}{x}(\ln x)^{\lambda_1 - \frac{\varepsilon}{p} - 1}, x \in [e,\infty)$, and $\widetilde{a}_n = \frac{1}{n}(\ln n)^{\lambda_2 - \frac{\varepsilon}{q} - 1}, n \in \mathbf{N}\backslash\{1\}$, if there exists a positive number $k(\geq B(\lambda_1,\lambda_2))$, such that (3.1) is valid as we replace $B(\lambda_1,\lambda_2)$ by k, then in particular, it follows

$$\begin{aligned} \widetilde{I} \ :&= \sum_{n=2}^\infty \int_1^\infty \frac{1}{(\ln xn)^\lambda}\widetilde{a}_n\widetilde{f}(x)dx > k||\widetilde{f}||_{p,\Phi}||\widetilde{a}||_{q,\Psi} \\ &= k\left\{\int_e^\infty (1 - O(\frac{1}{(\ln x)^{\lambda_2}}))\frac{dx}{x(\ln x)^{\varepsilon+1}}\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\frac{1}{2(\ln 2)^{\varepsilon+1}} + \sum_{n=3}^\infty \frac{1}{n(\ln n)^{\varepsilon+1}}\right\}^{\frac{1}{q}} \\ &> k(\frac{1}{\varepsilon} - O(1))^{\frac{1}{p}}\left\{\frac{1}{2(\ln 2)^{\varepsilon+1}} + \int_2^\infty \frac{1}{y(\ln y)^{\varepsilon+1}}dy\right\}^{\frac{1}{q}} \\ &= \frac{k}{\varepsilon}(1 - \varepsilon O(1))^{\frac{1}{p}}\left\{\frac{\varepsilon}{2(\ln 2)^{\varepsilon+1}} + \frac{1}{(\ln 2)^\varepsilon}\right\}^{\frac{1}{q}}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \widetilde{I} &= \sum_{n=2}^\infty (\ln n)^{\lambda_2 - \frac{\varepsilon}{q} - 1}\frac{1}{n}\int_e^\infty \frac{1}{x(\ln xn)^\lambda}(\ln x)^{\lambda_1 - \frac{\varepsilon}{p} - 1}dx \\ &\overset{t=(\ln x)/(\ln n)}{=} \sum_{n=2}^\infty \frac{1}{n(\ln n)^{\varepsilon+1}}\int_{1/\ln n}^\infty \frac{1}{(t+1)^\lambda}t^{\lambda_1 - \frac{\varepsilon}{p} - 1}dt \\ &\leq \int_0^\infty \frac{1}{(t+1)^\lambda}t^{\lambda_1 - \frac{\varepsilon}{p} - 1}dt\left[\frac{1}{2(\ln 2)^{\varepsilon+1}} + \sum_{n=3}^\infty \frac{1}{n(\ln n)^{\varepsilon+1}}\right] \\ &\leq B(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p})\left[\frac{1}{2(\ln 2)^{\varepsilon+1}} + \int_2^\infty \frac{dy}{y(\ln y)^{\varepsilon+1}}\right] \\ &= \frac{1}{\varepsilon}B(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p})\left[\frac{\varepsilon}{2(\ln 2)^{\varepsilon+1}} + \frac{1}{(\ln 2)^\varepsilon}\right]. \end{aligned} \tag{3.7}$$

Hence by (3.6) and (3.7), it follows

$$B(\lambda_1-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p})\left[\frac{\varepsilon}{2(\ln 2)^{\varepsilon+1}}+\frac{1}{(\ln 2)^{\varepsilon}}\right]$$
$$> \quad k(1-\varepsilon O(1))^{\frac{1}{p}}\left\{\frac{\varepsilon}{2(\ln 2)^{\varepsilon+1}}+\frac{1}{(\ln 2)^{\varepsilon}}\right\}^{\frac{1}{q}}, \tag{3.8}$$

and $B(\lambda_1,\lambda_2)\geq k(\varepsilon\to 0^+)$. Hence $k=B(\lambda_1,\lambda_2)$ is the best value of (3.1).

We conform that the constant factor $B(\lambda_1,\lambda_2)$ in (3.2) ((3.3)) is the best possible. Otherwise we can come to a contradiction by (3.4) ((3.5)) that the constant factor in (3.1) is not the best possible. □

Remark 3.1. For $\lambda=1,\lambda_1=\lambda_2=\frac{1}{2}$ in (3.1), (3.2) and (3.3), we have (1.7) and the following equivalent inequalities:

$$\left\{\sum_{n=2}^{\infty}\frac{(\ln n)^{\frac{p}{2}-1}}{n}\left[\int_1^{\infty}\frac{f(x)dx}{\ln xn}\right]^p\right\}^{\frac{1}{p}}>\pi\left\{\int_1^{\infty}\frac{(1-\theta_1(x))x^{p-1}}{(\ln x)^{1-\frac{p}{2}}}f^p(x)dx\right\}^{\frac{1}{p}}, \tag{3.9}$$

$$\left\{\int_1^{\infty}\frac{(\ln x)^{\frac{q}{2}-1}}{x(1-\theta_1(x))^{q-1}}\left[\sum_{n=2}^{\infty}\frac{a_n}{\ln xn}\right]^q dx\right\}^{\frac{1}{q}}>\pi\left\{\sum_{n=2}^{\infty}\frac{n^{q-1}}{(\ln n)^{1-\frac{q}{2}}}a_n^q\right\}^{\frac{1}{q}}. \tag{3.10}$$

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