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FIXED POINT THEOREMS IN SYMMETRIC $2-$CONE BANACH SPACE $\left(l_p, \|.,.\|_p^c\right)$

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ABSTRACT. The present article is concerned with l_p sequence spaces in point of their symmetric $2-$cone norm structure. Further, fixed point theorem for these spaces are proved.

KEYWORDS : Cone metric space; Cone normed space; $2-$Cone normed space; Fixed point theorems.
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1. INTRODUCTION

In [9] cone metric spaces were introduced by means of a partial ordering " $<$ " on a Banach space $(E, ||.||)$ via a cone P, where some fixed point theorems were proved to generalize the corresponding ones in metric spaces. In [10] Rezapour et al. proved that there were no normal cones with normal constant $M < 1$ and for each $k > 1$ there are cones with normal constant $M > k$. Abdeljawad et al. generalized the Banach spaces $\mathbb{R}^m$, l^∞ and $C[a, b]$ by defining $m-$Euclidean cone normed spaces E^m, E^∞ and the space $C_E(S)$ of continuous functions in cones [1].

It is well known that any metric space is paracompact. As a generalization of metric spaces, cone metric spaces play very important role in fixed point theory, computer science and some other research areas as well as in general topology.

Recently some interesting developments have occured in $2-$normed spaces, sequence spaces, and related topics in these nonlinear spaces (see [11],[6]).

In the followings we recall some preliminary notions which will be needed subsequently.

Definition 1.1. Let E be a real Banach space and P a subset of E. Then P is called cone if

(i) P is closed, non-empty, and $P \neq \{0\}$;
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b;
(iii) $P \cap (-P) = \{0\}$.

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For given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$, $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x << y$ will stand for $y - x \in int\ P$, where $int\ P$ denotes the interior of P.

The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } ||x|| \leq M||y||.$$

The least positive number satisfying the above is called the normal constant of P [10].

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that

$$x_1 \leq x_2 \leq ... \leq x_n \leq ... \leq y$$

for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} ||x_n - x|| = 0$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is normal cone.

In the following it is supposed that E is a Banach space, P is a cone in E with int $P \neq \varnothing$ and $\leq$ is partial ordering with respect to P.

Definition 1.2. A cone normed space is an ordered pair $(X, ||.||_c)$ where X is a vector space over R and $||.||_c : X \to (E, P, ||.||)$ is a function satisfying:

$(c1)$ $0 < ||x||_c$, for all $x \in X$;
$(c2)$ $||x||_c = 0$ if and only if $x = 0$;
$(c3)$ $||\alpha x||_c = |\alpha|\, ||x||_c$, for each $x \in X$ and $\alpha \in \mathbb{R}$;
$(c4)$ $||x + y||_c \leq ||x||_c + ||y||_c$, $x, y \in X$.

It is easy to see that each cone normed space is cone metric space. Namely, the cone metric is defined by $d(x, y) = ||x - y||_c$.

According to above definition a sequence $\{x_n\}$ of a cone normed space $(X, ||.||_c)$ over $(E, P, ||.||)$ is said to be convergent, if there exists $x \in X$ such that for all $c >> 0$, $c \in E$, there exists n_0 such that $||x - x_n||_c << c$ for all $n \geq n_0$. Also, we say that $\{x_n\}$ is Cauchy if for each $c >> 0$, there exists n_0 such that $||x_m - x_n||_c << 0$ for all $m, n \geq n_0$ [1].

Gahler introduced the concepts of $2-$metric spaces and linear $2-$normed spaces and their topological structures [2]. Many works can be found on the scientific literature related to $2-$normed spaces. The definition of a finite dimensional real $2-$normed space is given as the following:

Definition 1.3. [3] Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A $2-$norm on X is a function $\|.,.\| : X \times X \to R$ which satisfies

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent;
(ii) $\|x, y\| = \|y, x\|$;
(iii) $\|\alpha x, y\| = |\alpha|\, \|x, y\|$, $\alpha \in \mathbb{R}$;
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

The pair $(X, \|.,.\|)$ is then called a $2-$normed space.

Definition 1.4. [8] Let X and Y be real linear spaces. Denote by D a non-empty subset $X \times Y$ such that for every $x \in X$, $y \in Y$ the sets

$$D_x = \{y \in Y; (x, y) \in D\} \text{ and } D^y = \{x \in X; (x, y) \in D\}$$

are linear subspaces of the space Y and X, respectively.

A function $\|.,.\| : D \to [0; \infty)$ will be called a generalized $2-$norm on D if it satisfies the following conditions:

$(N1)$ $\|x, \alpha y\| = |\alpha| \, \|x, y\| = \|\alpha x, y\|$ for any real number α and all $(x, y) \in D$;
$(N2)$ $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for $x \in X$, $y, z \in Y$ such that$(x, y), (x, z) \in D$;
$(N3)$ $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for $x, y \in X$, $z \in Y$ such that$(x, z), (y, z) \in D$.

The set D is called a $2-$normed set.

In particular, if $D = X \times Y$, the function $\|., .\|$ will be called a generalized $2-$norm on $X \times Y$ and the pair $(X \times Y, \|., .\|)$ a generalized $2-$normed space. Moreover, if $X = Y$, then the generalized $2-$normed space will be denoted by $(X, \|., .\|)$.

Assume that the generalized $2-$norm satifies, in addition, the symmetry condition. Then the symmetric $2-$norm can be defined as follows:

Definition 1.5. [8] Let X be a real linear space. Denote by χ a non-empty subset $X \times X$ with the property $\chi = \chi^{-1}$ and such that the set $\chi^y = \{x \in X; (x, y) \in \chi\}$ is a linear subspace of X, for all $y \in X$.

A function $\|., .\| : \chi \to [0; \infty)$ satisfying the following conditions:
$(S1)$ $\|x, y\| = \|y, x\|$ for all $(x, y) \in \chi$;
$(S2)$ $\|x, \alpha y\| = |\alpha| \, \|x, y\|$ for any real number α and all $(x, y) \in \chi$;
$(S3)$ $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for $x, y, z \in X$ such that$(x, y), (x, z) \in \chi$;

will be called a generalized symmetric $2-$norm on χ. The set χ is called a symmetric $2-$normed set. In particular, if $\chi = X \times X$, the function $\|., .\|$ will be called a generalized symmetric $2-$norm on X and the pair $(X, \|., .\|)$ a generalized symmetric $2-$normed space.

Gunawan and Mashadi introduced the concepts of $n-$normed spaces and their topological structures [4]. Then Lewandowska defined generalized $2-$normed spaces and generalized symmetric $2-$normed spaces. In [5] Gunawan studied the space l_p, $1 \leq p \leq \infty$, its natural $n-$norm and proved a fixed point theorem for l_p as an $n-$normed space.

In this article, we introduce generalized symmetric $2-$cone normed space and a generalized symmetric $2-$cone Banach space and prove the fixed point theorem for some generalized symmetric $2-$cone Banach spaces.

In the main part of the article the results expressing under what conditions a self-mapping T of generalized symmetric $2-$cone Banach space $\left(l_p, \|., .\|_p^c\right)$ has a unique fixed point are also given.

2. MAIN RESULTS

In the following we give the definition of $2-$cone normed space.

Definition 2.1. Let X be linear space over $\mathbb{R}$ with dimension greater than or equal to 2, E be Banach space with the norm $\|.\|$and $P \subset E$ be a cone. If the function

$$\|., .\|_c : X \times X \to (E, P, \|.\|)$$

satisfies the following axioms:

(i) $\|x, y\|_c = 0 \Leftrightarrow x \; ve \; y$ are linearly dependent;
(ii) $\|x, y\|_c = \|y, x\|_c$;
(iii) $||\alpha x, y|| = |\alpha| \, ||x, y||_c$;
(iv) $\|x, y + z\|_c \leq \|x, y\|_c + \|x, z\|_c$;
then $(X, \|., .\|_c)$ is called a $2-$cone normed space.

Example 2.2. Let $X = \mathbb{R}^n$, $E = \mathbb{R}^2$ and $P = \{(x_1, x_2) \in \mathbb{R}^2 : x_i \geqslant 0, \; i = 1, 2\}$. Then the function $\|., .\|_c : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow (E, P, \|.\|)$ defined by

$$\|x_1, x_2\|_c = (A, A)$$

where

$$A = abs\left(\begin{vmatrix} x_{11} & x_{12} & ... & x_{1n} \\ x_{21} & x_{22} & ... & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}\right),$$

abs stands for absolute value and $|.|$ stands for determinant of a matrix, is a generalized symmetric $2-$cone norm and $(X, \|.,.\|_c)$ is a generalized symmetric $2-$cone normed space.

If we fix $\{u_1, u_2, ..., u_d\}$ to be a basis for X, we can give the following lemma.

Lemma 2.3. *Let $(X, \|.,.\|_c)$ be a $2-$cone normed space. Then a sequence $\{x_n\}$ converges to x in X if and only if for each $c \in E$ with $c >> \theta$ (θ is zero element of E) there exists an $N = N(c) \in \mathbb{N}$ such that $n > N$ implies $\|x_n - x, u_i\|_c << c$ for every $i = 1, 2, ..., d$.*

Proof. We prove the necessity since the sufficiency is clear. But, in this case there exists $N = N(c) \in \mathbb{N}$ such that $n > N$ implies $\|x_n - x, u_i\|_c << \frac{c}{d \max\limits_i |\alpha_i|}$ for every $i = 1, 2, ..., d$. Since $\{u_1, u_2, ..., u_d\}$ is a basis for X, every y can be written of the form $y = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_d u_d$ for some $\alpha_1, \alpha_2, ..., \alpha_d$ in $\mathbb{R}$. Hence,

$$\begin{aligned} \|x_n - x, y\|_c &\leq |\alpha_1| \|x_n - x, u_1\|_c + ... + |\alpha_d| \|x_n - x, u_d\|_c \\ &\ll c. \end{aligned}$$

□

This gives us the following.

Lemma 2.4. *Let $(X, \|.,.\|_c)$ be a $2-$cone normed space. Then a sequence $\{x_n\}$ converges to x in X if and only if $\lim\limits_{n\to\infty} \max \|x_n - x, u_i\|_c = \theta$.*

Now we are ready to define a cone norm with respect to the basis $\{u_1, u_2, ..., u_d\}$ on X. Really, the function $\|.\|_{\overset{\infty}{c}} : X \to (E, P, \|.\|)$ defined by

$$\|.\|_{\overset{\infty}{c}} := \{\max \|x, u_i\|_c : i = 1, 2, ..., d\}$$

is a cone norm on X.

Note that if we choose another basis $\{v_1, v_2, ..., v_d\}$ then resulting $\|.\|_{\overset{\infty}{c}}$ will be equivalent to the one defined with respect to the basis $\{u_1, u_2, ..., u_d\}$.

Lemma 2.5. *Let $(X, \|.,.\|_c)$ be a $2-$cone normed space. Then a sequence $\{x_n\}$ converges to x in X if and only if for each $c \in E$ with $c >> \theta$ (θ is zero element of E) there exists an $N = N(c) \in \mathbb{N}$ such that $n > N$ implies $\|x_n - x\|_{\overset{\infty}{c}} << c$.*

Definition 2.6. Let $\|.\|_{\overset{\infty}{c}} : X \to (E, P, \|.\|)$ and $r \in E$ with $r >> \theta$. Then the set

$$B_{\{u_1,u_2,...,u_d\}}(x; r) = \left\{y : \|y - x\|_{\overset{\infty}{c}} << r\right\}$$

is called (open) ball centered at x, with radius r.

Then we have the following:

Lemma 2.7. *Let $(X, \|.,.\|_c)$ be a $2-$cone normed space. Then a sequence $\{x_n\}$ converges to x in X if and only if for each $r \in E$ with $r >> \theta$ (θ is zero element of E) there exists an $N = N(r) \in \mathbb{N}$ such that $n > N$ implies $\|x_n - x\|_{\overset{\infty}{c}} \in B_{\{u_1,u_2,...,u_d\}}(x; r)$.*

Theorem 2.1. *Any $2-$cone normed space X is a cone normed space and its topology agrees with the norm generated by $\|.\|_{\overset{\infty}{c}}$.*

Now we introduce the notions of $2-$cone norm of the sequence space l_p, $1 \leq p \leq \infty$, consisting of all sequences $x = (x_k)$ such that $\sum\limits_k |x_k|^p < \infty$ and prove some fixed point theorems.

Recall from [5] that the functions

$$\|.,.\|_p := \left[\frac{1}{2} \sum_k \sum_l \left| \det \begin{pmatrix} x_k & x_l \\ x_k & x_l \end{pmatrix} \right|^p \right]^{\frac{1}{p}}$$

and

$$\|.,.\|_\infty := \sup_k \sup_l \left| \det \begin{pmatrix} x_k & x_l \\ x_k & x_l \end{pmatrix} \right|$$

define a $2-$norm on l_p for $1 \leq p < \infty$ and for $p = \infty$ respectively. Then we have the following:

If $X = l_p$, $E = \mathbb{R}^n$ and $P = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_i \geqslant 0,\ i = 1, ..., n\}$. Then each of the functions $\|.,.\|_p^c : l_p \times l_p \longrightarrow (E, P, \|.\|)$ and $\|.,.\|_\infty^c : l_p \times l_p \longrightarrow (E, P, \|.\|)$ defined by

$$\|.,.\|_p^c := (\alpha_1 A, ..., \alpha_n A)$$

and

$$\|.,.\|_\infty^c := (\alpha_1 B, ..., \alpha_n B)$$

defines a $2-$norm on l_p for $1 \leq p < \infty$ and for $p = \infty$ respectively, where

$$A := \left[\frac{1}{2} \sum_k \sum_l \left| \det \begin{pmatrix} x_k & x_l \\ x_k & x_l \end{pmatrix} \right|^p \right]^{\frac{1}{p}},$$

$$B := \sup_k \sup_l \left| \det \begin{pmatrix} x_k & x_l \\ x_k & x_l \end{pmatrix} \right|$$

and $\alpha_i \geq 0$, $i = 1, 2, ..., n$.

Remember from [5] that for any $2-$ normed space X with dimension ≥ 2 an arbitrary linearly independent set $\{a_1, a_2\}$ can be choosen in X and with respect to $\{a_1, a_2\}$ a norm $\|.\|_p$ on X can be defined by

$$\|x\|_p^* := [\|x, a_1\|^p + \|x, a_2\|^p]^{\frac{1}{p}}$$

for $1 \leq p < \infty$ or

$$\|x\|_\infty^* := \sup [\|x, a_1\|, \|x, a_2\|]$$

for $p = \infty$. For instance for l_p we can choose $a_1 = (1, 0, 0, ...)$ and $a_2 = (0, 1, 0, ...)$.

The above facts allow us to define $2-$cone norms on l_p by

$$\|x\|_{p_c}^* := \left(\alpha_1 [\|x, a_1\|^p + \|x, a_2\|^p]^{\frac{1}{p}}, ..., \alpha_n [\|x, a_1\|^p + \|x, a_2\|^p]^{\frac{1}{p}} \right)$$

and by

$$\|x\|_{\infty_c}^* := (\alpha_1 \sup [\|x, a_1\|, \|x, a_2\|], ..., \alpha_n \sup [\|x, a_1\|, \|x, a_2\|])$$

where $\alpha_i \geq 0$, $i = 1, 2, ..., n$, for $1 \leq p < \infty$ and for $p = \infty$ respectively. Remember also that

$$\|x\|_p \leq \|x\|_p^* \leq 2^{\frac{1}{p}} \|x\|_p$$

for all $x \in l_p$, where $\|.\|_p$ is the usual norm on l_p. In particular one has $\|x\|_\infty^* = \|x\|_\infty$. Hence, if we take $\alpha_i = 1$ for all $i = 1, 2, ..., n$ in the above corollary we have $2-$cone norms $\|.,.\|_p^c := (A, ..., A)$ and $\|.,.\|_\infty^c := (B, ..., B)$ of l_p for $1 \leq p < \infty$ and for $p = \infty$ respectively. Thus we have

$$\left(\|x\|_p, ..., \|x\|_p \right) \leq \|x\|_{p_c}^* \leq \left(2^{\frac{1}{p}} \|x\|_p, ..., 2^{\frac{1}{p}} \|x\|_p \right)$$

where $\|x\|_P = \left(\|x\|_p, ..., \|x\|_p\right)$ is usual $p-$norm-like cone norm on $\left(l_p, \|.,.\|_p^c\right)$.

In order to show that $(l_p, \|.,.\|_c)$ is complete we need the following.

Lemma 2.8. *If a sequence in l_p is convergent in the usual norm $\|.\|_p$ then it is convergent in $2-$cone norm $\|.,.\|_p^c$. Similarly, if a sequence in l_p is Cauchy with respect to $\|.\|_p$ then it is Cauchy with respect to $\|.,.\|_p^c$.*

Theorem 2.2. $\left(l_p, \|.,.\|_p^c\right)$ *is a $2-$cone Banach space.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in $\left(l_p, \|.,.\|_p^c\right)$. Then for each $c \in E$ with $c >> \theta$ there exists $N = N(c) \in \mathbb{N}$ such that $n > N$ implies $\|x_n - x_m, y\|_p^c << c$ for all y in l_p, if and only if for each $c \in E$ with $c >> \theta$ there exists $N = N(c) \in \mathbb{N}$ such that $n > N$ implies $\|x_n - x_m\|_{\substack{p\\c}}^* << c$. This proves that $\{x_n\}$ is a Cauchy sequence in $2-$cone normed space $\left(l_p, \|.,.\|_p^c\right)$ if and only if $\{x_n\}$ is a Cauchy sequence in $\left(l_p, \|.\|_{\substack{p\\c}}^*\right)$. □

Theorem 2.3. *Let T be a self-mapping of l_p such that*

$$\|Tx - Ty, z\|_p^c \leq K \|x - y, z\|_p^c$$

for all x, y, z in l_p, where $K \in (0,1)$ is a constant. Then T has a unit fixed point in $\left(l_p, \|.,.\|_p^c\right)$.

Proof. Clearly T satisfies

$$\|Tx - Ty\|_{\substack{p\\c}}^* \leq K \|x - y\|_{\substack{p\\c}}^*$$

for all x, y, z in l_p. Since $\left(l_p, \|.\|_{\substack{p\\c}}^*\right)$ is a cone Banach space T must have a unique fixed point. □

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