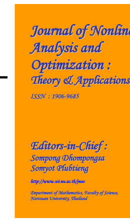

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THE EIGENVALUE PROBLEMS FOR DIFFERENTIAL PENCILS ON THE HALF LINE

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ABSTRACT. In this paper, we study the solution of the boundary value problem for second-order differential operator on the half-line having jump point in an interior point. Using of the fundamental system of solutions, we investigate the asymptotic distribution of eigenvalues.

KEYWORDS : Asymptotic form; Jump point; Eigenvalues.
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1. INTRODUCTION

We consider BVP(L) with the differential equation

$$y''(x) + (\rho^2 + i\rho q_1(x) + q_0(x))y(x) = 0, \quad x \geq 0, \tag{1.1}$$

on half-line and the boundary condition

$$U(y) := y'(0) + (\beta_1\rho + \beta_0)y(0) = 0, \tag{1.2}$$

and the jump condition

$$y(T-0) = a_1 y(T+0), \quad y'(T-0) = a_1^{-1} y'(T+0), \tag{1.3}$$

in an interior point $T > 0$. Here $a_1 > 0$, and the functions $q_j(x)$, $j = 0, 1$, are complex-valued, $q_1(x)$ is absolutely continuous and $(1+x)q_j^{(l)} \in L(0,\infty)$ for $0 \leq l \leq j \leq 1$.

Boundary value problems with discontinuities inside the interval often appear in mathematics, mechanics, physics, geophysics and other branches of natural sciences. For example, discontinuous inverse problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics (see [2]). The boundary value problems on the finite interval with turning points and without discontinuities have been studied in [3]. Also the boundary value problem on the half-line with turning points but without discontinuities has been studied in [4]. Indefinite differential equations with discontinuity

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produce essential qualitative modification in the investigation of the inverse problem. For classical Sturm-Liouville operators with discontinuities inverse problems on the half line have been considered in [5]. In this paper we study discontinuous BVP(L) on the half-line for indefinite pencil (1.1).

The presence of discontinues inverse problems helps to study the blowup behavior of solutions. In section 2, we determine the asymptotic form of the solutions of (1.1) and using these asymptotic estimates, derive characteristic function and give eigenvalues.

2. MAIN RESULTS

Let $y(x)$ and $z(x)$ be continuously differentiable functions on $[0,T]$ and $[T,\infty)$. If $y(x)$ and $z(x)$ satisfy the jump condition (1.3), then

$$< y, z >_{|x=T-0} = < y, z >_{|x=T+0},$$

where $< y, z >= yz' - y'z$, and is called the Wronskian of the functions $y(x)$ and $z(x)$. Denote $\Pi\pm := \{\rho : \pm Im\rho > 0\}$ and $\Pi_0 := \{\rho : Im\rho = 0\}$. By the well-known method (see [4, 5]), we get that for $x \geq T$, $\rho \in \Pi\pm$, there exists a solution $e(x,\rho)$ of Eq.(1.1) (which is called the Jost-type solution) with the following properties :

1. For each fixed $x \geq T$, the functions $e^{(\nu)}(x,\rho)$, $\nu = 0,1$, are holomorphic for $\rho \in \Pi_+$ and $\rho \in \Pi_-$(i.e., they are piecewise holomorphic).

2. The functions $e^{(\nu)}(x,\rho)$, $\nu = 0,1$, are continuous for $x \geq T$, $\rho \in \overline{\Pi}_+$ and $\rho \in \overline{\Pi}_-$ (we differ the sides of the cut Π_0). In the other words, for real ρ, there exist the finite limits

$$e_{\pm}^{(\nu)}(x,\rho) = \lim_{z \longrightarrow \rho,\ z \in \Pi_{\pm}} e^{(\nu)}(x,z).$$

Moreover, the functions $e^{(\nu)}(x,\rho)$, $\nu = 0,1$, are continuously differentiable with respect to $\rho \in \overline{\Pi}_+ \setminus \{0\}$ and $\rho \in \overline{\Pi}_- \setminus \{0\}$.

3. For $x \longrightarrow \infty$, $\rho \in \overline{\Pi}_{\pm} \setminus \{0\}$, $\nu = 0,1$,

$$e^{(v)}(x,\rho) = (\pm i\rho)^v exp(\pm(i\rho x - Q(x)))(1 + o(1)), \tag{2.1}$$

where

$$Q(x) = \frac{1}{2}\int_0^x q_1(t)dt. \tag{2.2}$$

4. For $|\rho| \longrightarrow \infty, \rho \in \overline{\Pi}_{\pm}$, $\nu = 0,1$, uniformly in $x \geq T$,

$$e^{(\nu)}(x,\rho) = (\pm i\rho)^{\nu} exp(\pm(i\rho x - Q(x)))[1], \tag{2.3}$$

where $[1] := 1 + O(\rho^{-1})$. Denote

$$\Delta(\rho) := U(e(x,\rho)). \tag{2.4}$$

The function $\Delta(\rho)$ is called the characteristic function for BVP(L). The function $\Delta(\rho)$ is holomorphic in Π_+ and Π_-, and for real ρ, there exist the finite limits

$$\Delta\pm(\rho)_= \lim_{z \longrightarrow \rho,\ z \in \Pi_{\pm}} \Delta(z).$$

Moreover, the function $\Delta(\rho)$ is continuously differentiable for $\rho \in \overline{\Pi}_{\pm} \setminus \{0\}$.

Theorem 2.1. For $|\rho| \longrightarrow \infty, \rho \in \overline{\Pi}_{\pm}$, the following asymptotical formula holds:

$$\Delta(\rho) = \frac{a_1\rho}{2} exp(\pm(i\rho T - Q(T)))((\beta_1 + i)exp(-i\rho T + Q(T))[1] + (\beta_1 - i)exp(i\rho T - Q(T))[1]).$$

Proof. Denote $\Pi^1_\pm := \{\rho : \pm Re\rho > 0\}$. Let $\{y_k(x,\rho)\}_{k=1,2}$ be the Birkhoff-type smooth fundamental system of solutions of Eq.(1.1) with the asymptotic forms

$$y_k^{(m)}(x,\rho) = ((-1)^{k-1}i\rho)^m exp((-1)^{k-1}(i\rho x - Q(x)))[1], \tag{2.5}$$

for $|\rho| \longrightarrow \infty, \rho \in \Pi^1_\pm$, m=0,1 (see[4, 5]). Then

$$e^{(m)}(x,\rho) = h_1(\rho)y_1^{(m)}(x,\rho) + h_2(\rho)y_2^{(m)}(x,\rho), \quad x \in [0,T]. \tag{2.6}$$

Using of solution (2.3) and the jump condition (1.3), we calculate coefficients $h_1(\rho)$ and $h_2(\rho)$ of the forms

$$h_1(\rho) = \frac{a_1\rho - a_1^{-1}i}{2\rho} exp(-i\rho T + Q(T))exp(\pm(i\rho T - Q(T)))[1],$$
$$h_2(\rho) = \frac{a_1\rho + a_1^{-1}i}{2\rho} exp(i\rho T - Q(T))exp(\pm(i\rho T - Q(T)))[1].$$

Substituting the results into (2.6), we obtain

$$e^{(m)}(x,\rho) = \frac{\rho a_1 i}{2} exp(\pm(i\rho T - Q(T)))(exp(-i\rho T + Q(T))exp(i\rho x - Q(x)) \\ +(-1)^m exp(i\rho T - Q(T))exp(-i\rho x + Q(x))), \qquad x \in [0,T].$$

Together with (1.2) and (2.4), this yields the characteristic function $\Delta(\rho)$. □

Theorem 2.2. i_1) For sufficiently large k, the function $\Delta(\rho)$ has zeros of the form:

$$\rho_k = \frac{1}{T}(k\pi + \frac{-Q(T)i}{2} + \kappa_1 i) + O(k^{-1}), \tag{2.7}$$

where

$$\kappa_1 = \frac{1}{2} ln \frac{i - \beta_1}{i + \beta_1}.$$

i_2) The zeros of BVP(L) are simple, that is, $\Delta_1(\rho_k) = \frac{d\Delta(\rho)}{2\rho d\rho}\ |_{\rho=\rho_k} \neq 0$.

Proof. Using Theorem 2.1 and Rouche's theorem [1], we obtain the zeros of the function $\Delta(\rho)$ of the form (2.7). The relations

$$\begin{cases} e''(x,\rho) + (i\rho q_1(x) + q_0(x))e(x,\rho) = -\rho^2 e(x,\rho), \\ \varphi''(x,\rho_k) + (i\rho_k q_1(x) + q_0(x))\varphi(x,\rho_k) = -\rho_k^2 \varphi(x,\rho_k), \end{cases}$$

result that

$$\frac{d}{dx} < e(x,\rho), \varphi(x,\rho_k) > + iq_1(x)e(x,\rho)\varphi(x,\rho_k)(\rho - \rho_k) = -(\rho^2 - \rho_k^2)e(x,\rho)\varphi(x,\rho_k).$$

Hence

$$-(\rho^2 - \rho_k^2)\int_0^\infty e(t,\rho)\varphi(t,\rho_k)dt = < e(x,\rho), \varphi(x,\rho_k) > (|_0^T + |_T^\infty) \\ +(\rho - \rho_k)\int_0^\infty iq_1(x)e(x,\rho)\varphi(x,\rho_k).$$

Since the Wronskian is continuous function, we have

$$\int_0^\infty e(t,\rho_k)\varphi(t,\rho_k)dt = -\Delta_1(\rho_k).$$

Using properties of the eigenfunctions i.e., $e(x,\rho_k) = \beta_k\varphi(x,\rho_k)$, $\beta_k \neq 0$ (see [4]), we arrive at $\beta_k \int_0^\infty \varphi^2(t,\rho_k)dt = \Delta_1(\rho_k) \neq 0$. □

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