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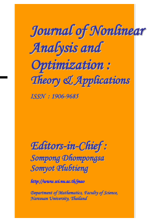

APPROXIMATION METHODS FOR FINITE FAMILY OF GENERALIZED NONEXPANSIVE MULTIVALUED MAPPINGS IN BANACH SPACES

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ABSTRACT. In the present paper, we define and study two new finite-step iterative process for a finite family of generalized nonexpansive multivalued mappings in Banach spaces. Several convergence theorems of the proposed iterations are established for this mappings.

KEYWORDS : Common fixed point; Generalized nonexpansive multivalued mapping; Iterative process, Convergence theorem.
AMS Subject Classification: 47H10,47H09.

1. INTRODUCTION

Let D be a nonempty subset of a Banach space X. Let $T : D \longrightarrow D$ be a singlevalued mapping. The Mann iteration process (see [1]), starting from $x_0 \in D$, is the sequence $\{x_n\}$ defined by

$$x_{n+1} = a_n x_n + (1 - a_n)Tx_n, \qquad a_n \in [0, 1], n \geq 0,$$

where a_n satisfies certain conditions. The Ishikawa iteration process (see [2]), starting from $x_0 \in D$, is the sequence $\{x_n\}$ defined by

$$\begin{cases} y_n = b_n x_n + (1 - b_n)Tx_n, & b_n \in [0, 1], n \geq 0, \\ x_{n+1} = a_n x_n + (1 - a_n)Ty_n, & a_n \in [0, 1], n \geq 0, \end{cases}$$

where a_n and b_n satisfy certain conditions.

Recently, Agarwal et al. [3] introduced the following iterative process which is both faster than and independent of the Ishikawa process.

$$\begin{cases} x_0 = x \in D, \\ y_n = b_n x_n + (1 - b_n)Tx_n, & b_n \in [0, 1], n \geq 0, \\ x_{n+1} = a_n Tx_n + (1 - a_n)Ty_n, & a_n \in [0, 1], n \geq 0. \end{cases}$$

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In the recent years, iterative process for approximating fixed point of nonexpansive multivalued mappings have been investigated by various authors (see, e.g., [4–11]) using the Mann iteration process or the Ishikawa iteration process.

In this paper, inspired by [3] we introduce two new iterative process for a finite family of generalized nonexpansive multivalued mappings in Banach spaces. Weak and strong convergence theorems of the proposed iteration processes to a common fixed point of a finite family of generalized nonexpansive multivalued mappings in uniformly convex Banach spaces are established. Our results are new even for single valued mappings.

2. PRELIMINARIES

A Banach space X is said to satisfy Opial's condition if $x_n \longrightarrow z$ weakly as $n \longrightarrow \infty$ and $z \neq y$ imply that

$$\limsup_{n\longrightarrow\infty} \|x_n - z\| < \limsup_{n\longrightarrow\infty} \|x_n - y\|.$$

All Hilbert spaces, all finite dimensional Banach spaces and $\ell^p (1 \leq p < \infty)$ have the Opial property.

A subset $D \subset X$ is called proximal if for each $x \in X$, there exists an element $y \in D$ such that

$$\| x - y \| = \text{dist}(x, D) = \inf\{\| x - z \| : z \in D\}.$$

It is known that every closed convex subset of a uniformly convex Banach space is proximal.

We denote by $CB(D), K(D)$ and $P(D)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of D respectively. The Hausdorff metric H on $CB(X)$ is defined by

$$H(A, B) := \max\{\sup_{x\in A} \text{dist}(x, B), \sup_{y\in B} \text{dist}(y, A)\},$$

for all $A, B \in CB(X)$.

Let $T : X \longrightarrow 2^X$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$. The set of fixed points of T will be denote by $F(T)$.

Definition 2.1. A multivalued mapping $T : X \longrightarrow CB(X)$ is called

(i) nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in X.$$

(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \| x - p \|$ for all $x \in X$ and all $p \in F(T)$.

J. Garcia et al. [12] introduced a new condition on singlevalued mappings, called condition (E), which is weaker than nonexpansiveness. Very recently, Abkar and Eslamian [13] used a modified condition for multivalued mappings as follows:

Definition 2.2. A multivalued mapping $T : X \longrightarrow CB(X)$ is said to satisfy condition (E_μ) provided that

$$\text{dist}(x, Ty) \leq \mu\, \text{dist}(x, Tx) + \|x - y\|, \quad x, y \in X.$$

We say that T satisfies condition (E) whenever T satisfies (E_μ) for some $\mu \geq 1$.

Lemma 2.3. *Let $T : X \longrightarrow CB(X)$ be a multivalued nonexpansive mapping. Then T satisfies the condition (E_1).*

The proof of the following Lemma is similar to that of Theorem 3.3 in [10], hence we omit the details.

Lemma 2.4. *([10]) Let D be a nonempty closed convex subset of uniformly convex Banach space X with the Opial property. Let $T : D \longrightarrow K(D)$ be a multivalued mapping satisfying the condition (E). If $x_n \longrightarrow x$ weakly and $\lim_{n\longrightarrow\infty} \operatorname{dist}(x_n, Tx_n) = 0$, then $x \in Tx$.*

The following Lemma can be found in [14].

Lemma 2.5. *Let X be a Banach space. Then X is uniformly convex if and only if for any given number $r > 0$ there exists a continuous, strictly increasing and convex function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ with $\varphi(0) = 0$ such that*

$$\| \alpha x + (1-\alpha)y \|^2 \leq \alpha \| x \|^2 + (1-\alpha) \| y \|^2 - \alpha(1-\alpha)\varphi(\| x - y \|),$$

for all $x, y \in B_r(0) = \{x \in X : \| x \| \leq r\}$, and $\alpha \in [0,1]$.

Definition 2.6. Let D be a nonempty closed subset of a Banach space X. A mapping $T : D \longrightarrow CB(D)$ is hemi-compact if for any bounded sequence $\{x_n\}$ in D such that $\operatorname{dist}(Tx_n, x_n) \longrightarrow 0 \quad as \quad n \longrightarrow \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \longrightarrow p \in D$. We note that if D is compact, then every multivalued mapping $T : D \longrightarrow CB(D)$ is hemi-compact.

3. CONVERGENCE THEOREMS

Let D be a nonempty convex subset of a Banach space X, let $T_1, T_2, ..., T_m$ be finite multivalued mappings from D into $CB(D)$. Let $\{x_n\}$ be the sequence defined by $x_1 \in D$ and

$$(\mathbf{A}) : \quad \begin{cases} y_{n,1} = (1-a_{n,1})x_n + a_{n,1}z_{n,1}, \\ y_{n,2} = (1-a_{n,2})z_{n,1} + a_{n,2}z_{n,2}, \\ ... \\ y_{n,m-1} = (1-a_{n,m-1})z_{n,m-2} + a_{n,m-1}z_{n,m-1}, \\ x_{n+1} = (1-a_{n,m})z_{n,m-1} + a_{n,m}z_{n,m}, \qquad n \geq 1, \end{cases}$$

where $z_{n,1} \in T_1(x_n)$, $z_{n,k} \in T_k(y_{n,k-1})$ for $k = 2, ..., m$, and $\{a_{n,i}\} \in [0,1]$.

Definition 3.1. A mapping $T : D \longrightarrow CB(D)$ is said to satisfy condition (I) if there is a nondecreasing function $g : [0,\infty) \longrightarrow [0,\infty)$ with $g(0) = 0$, $g(r) > 0$ for $r \in (0,\infty)$ such that

$$\operatorname{dist}(x, Tx) \geq g(\operatorname{dist}(x, F(T)).$$

Let $T_i : D \longrightarrow CB(D), (i = 1, 2, ..., m)$ be finite given mappings. The mappings $T_1, T_2, ..., T_m$ are said to satisfy condition (II) if there exists a nondecreasing function $g : [0,\infty) \longrightarrow [0,\infty)$ with $g(0) = 0$, $g(r) > 0$ for $r \in (0,\infty)$, such that

$$\max_{1\leq i\leq m} \operatorname{dist}(x, T_i x) \geq g(\operatorname{dist}(x, \mathcal{F})),$$

where $\mathcal{F} = \bigcap_{i=1}^m F(T_i)$.

Theorem 3.2. *Let D be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T_i : D \longrightarrow CB(D), (i = 1, 2, ..., m)$ be a finite family of quasi-nonexpansive multivalued mappings satisfy the condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, ..., m)$, for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_{n,i} \in [\alpha, \beta] \subset (0,1)$(i=1,2,...,m). Assume further that the mappings $T_1, T_2, ..., T_m$ satisfying the condition (II). Then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, .., T_m$.*

Proof. Let $p \in \mathcal{F}$. It follows from (A) and the quasi-nonexpansivness of $T_i, (i = 1, 2, ..., m)$ that

$$\begin{aligned} \| y_{n,1} - p \| &= \| (1-a_{n,1})x_n + a_{n,1}z_{n,1} - p \| \\ &\le (1-a_{n,1}) \| x_n - p \| + a_{n,1} \| z_{n,1} - p \| \\ &= (1-a_{n,1}) \| x_n - p \| + a_{n,1}\text{dist}(z_{n,1}, T_1(p)) \\ &\le (1-a_{n,1}) \| x_n - p \| + a_{n,1}H(T_1(x_n), T_1(p)) \\ &\le (1-a_{n,1}) \| x_n - p \| + a_{n,1} \| x_n - p \| \\ &\le \|x_n - p\| \end{aligned}$$

and

$$\begin{aligned} \| y_{n,2} - p \| &= \| (1-a_{n,2})z_{n,1} + a_{n,2}z_{n,2} - p \| \\ &\le (1-a_{n,2}) \| z_{n,1} - p \| + a_{n,2} \| z_{n,2} - p \| \\ &= (1-a_{n,2})\text{dist}(z_{n,1}, T_1(p)) + a_{n,2}\text{dist}(z_{n,2}, T_2(p)) \\ &\le (1-a_{n,2})H(T_1(x_n), T_1(p)) + a_{n,2}H(T_2(y_{n,1}), T_2(p)) \\ &\le (1-a_{n,2}) \| x_n - p \| + a_{n,2} \| y_{n,1} - p \| \\ &\le \|x_n - p\|. \end{aligned}$$

Continuing the above process, we get

$$\|y_{n,k} - p\| \le \|x_n - p\| \quad \text{for} \;\; k = 3, ..., m-1.$$

In particular

$$\begin{aligned} \| x_{n+1} - p \| &= \| (1-a_{n,m})z_{n,m-1} + a_{n,m}z_{n,m} - p \| \\ &\le (1-a_{n,m}) \| z_{n,m-1} - p \| + a_{n,m} \| z_{n,m} - p \| \\ &= (1-a_{n,m})\text{dist}(z_{n,m-1}, T_{m-1}(p)) + a_{n,m}\text{dist}(z_{n,m}, T_m(p)) \\ &\le (1-a_{n,m})H(T_{m-1}(y_{n,m-2}), T_{m-1}(p)) + a_{n,m}H(T_m(y_{n,m-1}), T_m(p)) \\ &\le (1-a_{n,m}) \| y_{n,m-2} - p \| + a_{n,m} \| y_{n,m-1} - p \| \\ &\le (1-a_{n,m}) \| x_n - p \| + a_{n,m} \| x_n - p \| \\ &\le \|x_n - p\|. \end{aligned}$$

Thus $\lim_{n \longrightarrow \infty} \| x_n - p \|$ exists for any $p \in \mathcal{F}$. Since the sequences $\{x_n\}$ and $\{y_{n,i}\}$, $(i = 1, ..., m-1)$ are bounded, we can find $r > 0$ depending on p such that $x_n - p$, $y_{n,i} - p \in B_r(0)$ for all $n \ge 0$. From Lemma 2.5, we get

$$\begin{aligned} \| y_{n,1} - p \|^2 &= \| (1-a_{n,1})x_n + a_{n,1}z_{n,1} - p \|^2 \\ &\le (1-a_{n,1}) \| x_n - p \|^2 + a_{n,1} \| z_{n,1} - p \|^2 \\ &\quad - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\ &= (1-a_{n,1}) \| x_n - p \|^2 + a_{n,1}\text{dist}(z_{n,1}, T_1(p))^2 \\ &\quad - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\ &\le (1-a_{n,1}) \| x_n - p \|^2 + a_{n,1}H(T_1(x_n), T_1(p))^2 \\ &\quad - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\ &\le (1-a_{n,1}) \| x_n - p \|^2 + a_{n,1} \| x_n - p \|^2 \\ &\quad - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\ &\le \|x_n - p\|^2 - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \end{aligned}$$

and

$$\| y_{n,2} - p \|^2 = \| (1-a_{n,2})z_{n,1} + a_{n,2}z_{n,2} - p \|^2$$

$$
\begin{aligned}
&\leq (1-a_{n,2}) \parallel z_{n,1}-p \parallel^2 + a_{n,2} \parallel z_{n,2}-p \parallel^2 \\
&\quad -a_{n,2}(1-a_{n,2})\varphi(\|z_{n,1}-z_{n,2}\|) \\
&= (1-a_{n,2})\mathrm{dist}(z_{n,1},T_1(p))^2 + a_{n,2}\mathrm{dist}(z_{n,2},T_2(p))^2 \\
&\quad -a_{n,2}(1-a_{n,2})\varphi(\|z_{n,1}-z_{n,2}\|) \\
&\leq (1-a_{n,2})H(T_1(x_n),T_1(p))^2 + a_{n,2}H(T_2(y_{n,1}),T_2(p))^2 \\
&\quad -a_{n,2}(1-a_{n,2})\varphi(\|z_{n,1}-z_{n,2}\|) \\
&\leq (1-a_{n,2}) \parallel x_n-p \parallel^2 + a_{n,2} \parallel y_{n,1}-p \parallel^2 \\
&\quad -a_{n,2}(1-a_{n,2})\varphi(\|z_{n,1}-z_{n,2}\|) \\
&\leq \|x_n-p\|^2 - a_{n,1}a_{n,2}(1-a_{n,1})\varphi(\|x_n-z_{n,1}\|) \\
&\quad -a_{n,2}(1-a_{n,2})\varphi(\|z_{n,1}-z_{n,2}\|).
\end{aligned}
$$

Applying Lemma 2.5 again, we have

$$
\begin{aligned}
\parallel y_{n,3}-p \parallel^2 &= \parallel (1-a_{n,3})z_{n,2}+a_{n,3}z_{n,3}-p \parallel^2 \\
&\leq (1-a_{n,3}) \parallel z_{n,2}-p \parallel^2 + a_{n,3} \parallel z_{n,3}-p \parallel^2 \\
&\quad -a_{n,3}(1-a_{n,3})\varphi(\|z_{n,2}-z_{n,3}\|) \\
&= (1-a_{n,3})\mathrm{dist}(z_{n,2},T_2(p))^2 + a_{n,3}\mathrm{dist}(z_{n,3},T_3(p))^2 \\
&\quad -a_{n,3}(1-a_{n,3})\varphi(\|z_{n,2}-z_{n,3}\|) \\
&\leq (1-a_{n,3})H(T_2(y_{n,1}),T_2(p))^2 + a_{n,3}H(T_3(y_{n,2}),T_3(p))^2 \\
&\quad -a_{n,3}(1-a_{n,3})\varphi(\|z_{n,2}-z_{n,3}\|) \\
&\leq (1-a_{n,3}) \parallel y_{n,1}-p \parallel^2 + a_{n,3} \parallel y_{n,2}-p \parallel^2 \\
&\quad -a_{n,3}(1-a_{n,3})\varphi(\|z_{n,2}-z_{n,3}\|) \\
&\leq (1-a_{n,3})\|x_n-p\|^2 + a_{n,3}\|x_n-p\|^2 \\
&\quad -a_{n,3}a_{n,2}a_{n,1}(1-a_{n,1})\varphi(\|x_n-z_{n,1}\|) \\
&\quad -a_{n,3}a_{n,2}(1-a_{n,2})\varphi(\|z_{n,1}-z_{n,2}\|) \\
&\quad -a_{n,3}(1-a_{n,3})\varphi(\|z_{n,2}-z_{n,3}\|) \\
&\leq \|x_n-p\|^2 - a_{n,1}a_{n,2}a_{n,3}(1-a_{n,1})\varphi(\|x_n-z_{n,1}\|) \\
&\quad -a_{n,3}a_{n,2}(1-a_{n,2})\varphi(\|z_{n,1}-z_{n,2}\|) \\
&\quad -a_{n,3}(1-a_{n,3})\varphi(\|z_{n,2}-z_{n,3}\|)
\end{aligned}
$$

By continuing this process we obtain

$$
\begin{aligned}
\parallel x_{n+1}-p \parallel^2 &= \parallel (1-a_{n,m})z_{n,m-1}+a_{n,m}z_{n,m}-p \parallel^2 \\
&\leq (1-a_{n,m}) \parallel z_{n,m-1}-p \parallel^2 + a_{n,m} \parallel z_{n,m}-p \parallel^2 \\
&\quad -a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1}-z_{n,m}\|) \\
&= (1-a_{n,m})\mathrm{dist}(z_{n,m-1},T_{m-1}(p))^2 + a_{n,m}\mathrm{dist}(z_{n,m},T_m(p))^2 \\
&\quad -a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1}-z_{n,m}\|) \\
&\leq (1-a_{n,m})H(T_{m-1}(y_{n,m-2}),T_{m-1}(p))^2 \\
&\quad +a_{n,m}H(T_m(y_{n,m-1}),T_m(p))^2 \\
&\quad -a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1}-z_{n,m}\|) \\
&\leq (1-a_{n,m}) \parallel y_{n,m-2}-p \parallel^2 + a_{n,m} \parallel y_{n,m-1}-p \parallel^2 \\
&\quad -a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1}-z_{n,m}\|) \\
&\leq \|x_n-p\|^2 - a_{n,m}a_{n,m-1}a_{n,m-2}...a_{n,1}(1-a_{n,1})\varphi(\|x_n-z_{n,1}\|)
\end{aligned}
$$

$$-\cdots - a_{n,m}a_{n,m-1}(1-a_{n,m-1})\varphi(\|z_{n,m-2}-z_{n,m-1}\|)$$
$$-a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1}-z_{n,m}\|).$$

Thus we have

$$a_{n,m}a_{n,m-1}(1-a_{n,m-1})\varphi(\|z_{n,m-2}-z_{n,m-1}\|) \le \|x_n-p\|^2 - \| x_{n+1}-p \|^2 .$$

It follows from our assumption that

$$\alpha^2(1-\beta)\varphi(\|z_{n,m-2}-z_{n,m-1}\|) \le \|x_n-p\|^2 - \| x_{n+1}-p \|^2 .$$

Since $\lim_{n\longrightarrow\infty}\|x_n-p\|$ exists for all $p\in\mathcal{F}$, taking the limit as $n\longrightarrow\infty$ yields that

$$\lim_{n\longrightarrow\infty}\varphi(\|z_{n,m-2}-z_{n,m-1}\|)=0.$$

Since φ is continuous at 0 and is strictly increasing, we have

$$\lim_{n\longrightarrow\infty}\|z_{n,m-2}-z_{n,m-1}\|=0.$$

Similarly, we can obtain that

$$\lim_{n\longrightarrow\infty}\|x_n-z_{n,1}\|=0,$$

and

$$\lim_{n\longrightarrow\infty}\|z_{n,k}-z_{n,k-1}\|=0 \qquad \text{for } \ k=2,...,m.$$

Using (A) we have

$$\lim_{n\longrightarrow\infty}\|x_n-y_{n,1}\| = a_{n,1}\lim_{n\longrightarrow\infty}\|x_n-z_{n,1}\|=0.$$

Also we have

$$\|x_n-z_{n,2}\| \le \|x_n-z_{n,1}\|+\|z_{n,1}-z_{n,2}\|\longrightarrow 0, \qquad n\longrightarrow\infty.$$

Hence

$$\lim_{n\longrightarrow\infty}\|x_n-y_{n,2}\|=(1-a_{n,2})\lim_{n\longrightarrow\infty}\|x_n-z_{n,1}\|+a_{n,2}\lim_{n\longrightarrow\infty}\|x_n-z_{n,2}\|=0.$$

Continuing the above process , for $k=2,...,m$ we have

$$\|x_n-z_{n,k}\| \le \|x_n-z_{n,k-1}\|+\|z_{n,k-1}-z_{n,k}\|\longrightarrow 0, \qquad n\longrightarrow\infty$$

and also for $k=2,...,m-1$

$$\lim_{n\longrightarrow\infty}\|x_n-y_{n,k}\|=(1-a_{n,k})\lim_{n\longrightarrow\infty}\|x_n-z_{n,k-1}\|+a_{n,k}\lim_{n\longrightarrow\infty}\|x_n-z_{n,k}\|=0.$$

Also we have

$$\text{dist}(x_n,T_1x_n)\le\|x_n-z_{n,1}\|\longrightarrow 0.$$

For $k=2,...,m$, since T_i satisfies the condition (E) we have

$$\begin{aligned}
\text{dist}(x_n,T_kx_n) &\le \|x_n-y_{n,k-1}\|+\text{dist}(y_{n,k-1},T_kx_n)\\
&\le \|x_n-y_{n,k-1}\|+\mu\,\text{dist}(y_{n,k-1},T_ky_{n,k-1})+\|x_n-y_{n,k-1}\|\\
&\le \|x_n-y_{n,k-1}\|+\mu\,\text{dist}(x_n,T_ky_{n,k-1})+\mu\,\|x_n-y_{n,k-1}\|\\
&\quad +\|x_n-y_{n,k-1}\|\\
&\le (\mu+2)\|x_n-y_{n,k-1}\|+\mu\|x_n-z_{n,k}\|\longrightarrow 0, \qquad n\longrightarrow\infty.
\end{aligned}$$

Note that by our assumption $\lim_{n\longrightarrow\infty}\text{dist}(x_n,\mathcal{F})=0$. Next we show that $\{x_n\}$ is a Cauchy sequence in D. Let $\varepsilon>0$ be arbitrarily chosen. Since $\lim_{n\longrightarrow\infty}\text{dist}(x_n,\mathcal{F})=0$, there exists N such that for all $n\ge N$, we have

$$\text{dist}(x_n,\mathcal{F})<\frac{\varepsilon}{4}.$$

In particular, $\inf\{\|x_N - p\| : p \in \mathcal{F}\} < \varepsilon/4$. Thus there exist a $q \in \mathcal{F}$ such that

$$\|x_N - q\| < \frac{\varepsilon}{2}.$$

Hence for $m, n \geq N$ we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - q\| + \|x_n - q\| \\ &\leq 2\|x_N - q\| < 2(\frac{\varepsilon}{2}) = \varepsilon.\end{aligned}$$

This implies that $\{x_n\}$ is Cauchy sequence in D and hence converges to some $w \in D$. For $i = 1, ..., m$, since T_i satisfies the condition (E), we have

$$\begin{aligned}\text{dist}(w, T_i w) &\leq \|w - x_n\| + \text{dist}(x_n, T_i w) \\ &\leq 2\|w - x_n\| + (\mu)\text{dist}(x_n, T_i x_n) \longrightarrow 0, \qquad n \longrightarrow \infty\end{aligned}$$

from which it follows that $\text{dist}(w, T_i w) = 0$ which in turn implies that $w \in T_i(w)$. Thus $w \in \mathcal{F}$. This completes the proof. □

Theorem 3.3. *Let D be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T_i : D \longrightarrow CB(D), (i = 1, 2, ..., m)$ be a finite family of quasi-nonexpansive multivalued mappings satisfying the condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, ..., m)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_{n,i} \in [\alpha, \beta] \subset (0, 1)$(i=1,2,...,m). If one of the mappings $T_i, (i = 1, 2, ..., m)$ is hemi-compact, then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, .., T_m$.*

Proof. As in the proof of Theorem 3.2, we have $\lim_{n\longrightarrow\infty} \text{dist}(T_i x_n, x_n) = 0, (i = 1, 2, ..., m)$. Without loss of generality, we assume that T_1 is hemi-compact. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim x_{n_k} = w$ for some $w \in D$. Since T_i satisfies the condition (E) for $i = 1, 2, ..., m$, we have

$$\begin{aligned}\text{dist}(w, T_i(w)) &\leq \|w - x_{n_k}\| + \text{dist}(x_{n_k}, T_i(w)) \\ &\leq (\mu)\,\text{dist}(x_{n_k}, T_i(x_{n_k})) + 2\|w - x_{n_k}\| \longrightarrow 0, \qquad \text{as} \quad k \longrightarrow \infty.\end{aligned}$$

This implies that $w \in \mathcal{F}$. Since $\{x_{n_k}\}$ converges strongly to w and $\lim_{n\longrightarrow\infty} \|x_n - w\|$ exists (as in the proof of Theorem 3.2), it follows that $\{x_n\}$ converges strongly to w. □

Theorem 3.4. *Let D be a nonempty closed convex subset of a uniformly convex Banach space X with the Opial property. Let $T_i : D \longrightarrow K(D), (i = 1, 2, ..., m)$ be finite family of quasi-nonexpansive multivalued mappings satisfying the condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, ..., m)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_{n,i} \in [\alpha, \beta] \subset (0, 1)$(i=1,2,...,m). Then $\{x_n\}$ converges weakly to a common fixed point of $T_1, T_2, .., T_m$.*

Proof. As in the proof of Theorem 3.2, $\{x_n\}$ is bounded and for $i = 1, 2, ..., m$,

$$\lim_{n\longrightarrow\infty} \text{dist}(x_n, T_i(x_n)) = 0.$$

Consider $w_1, w_2 \in D$, two weak cluster points of the sequence $\{x_n\}$. Then there exist two subsequences $\{y_n\}$ and $\{z_n\}$ of $\{x_n\}$ such that $y_n \longrightarrow w_1$ weakly and $z_n \longrightarrow w_2$ weakly. By Lemma 2.4 we have $w_1, w_2 \in \mathcal{F}$. As in the proof of Theorem 3.2, $\lim_{n\longrightarrow\infty} \|x_n - w_1\|$ and $\lim_{n\longrightarrow\infty} \|x_n - w_2\|$ exist. We claim that $w_1 = w_2$. Indeed, assume to the contrary that $w_1 \neq w_2$. By the Opial property we have

$$\begin{aligned}\lim_{n\longrightarrow\infty} \|x_n - w_1\| &= \lim_{n\longrightarrow\infty} \|y_n - w_1\| < \lim_{n\longrightarrow\infty} \|y_n - w_2\| \\ &= \lim_{n\longrightarrow\infty} \|x_n - w_2\| = \lim_{n\longrightarrow\infty} \|z_n - w_2\|\end{aligned}$$

$$< \lim_{n\longrightarrow\infty} \|z_n - w_1\| = \lim_{n\longrightarrow\infty} \|x_n - w_1\|,$$

which is a contradiction, hence $w_1 = w_2$. Thus the sequence $\{x_n\}$ has at most one weak cluster point. Since D is weakly sequentially compact, we deduce that the sequence $\{x_n\}$ has exactly one weak cluster point $w \in D$, that is $x_n \longrightarrow w$ weakly. By Lemma 2.4 we obtain that $w \in \mathcal{F}$. □

We now intend to remove the restriction that $T_i(p) = p$ for each $p \in \mathcal{F}$. We define the following iteration process.

Let D be a nonempty convex subset of a Banach space X, and let $T_1, T_2, ..., T_m$ be finite multivalued mappings from D into $P(D)$, and

$$P_{T_i}(x) = \{y \in T_i(x) : \| x - y \| = \text{dist}(x, T_i(x))\}.$$

Then, for $x_1 \in D$, we consider the following iterative process:

$$(\mathbf{B}) : \quad \begin{cases} y_{n,1} = (1 - a_{n,1})x_n + a_{n,1} z_{n,1}, \\ y_{n,2} = (1 - a_{n,2})z_{n,1} + a_{n,2} z_{n,2}, \\ ... \\ y_{n,m-1} = (1 - a_{n,m-1})z_{n,m-2} + a_{n,m-1} z_{n,m-1}, \\ x_{n+1} = (1 - a_{n,m})z_{n,m-1} + a_{n,m} z_{n,m}, \qquad n \geq 1, \end{cases}$$

where $z_{n,1} \in P_{T_1}(x_n)$, $z_{n,k} \in P_{T_k}(y_{n,k-1})$ for $k = 2, ..., m$, and $\{a_{n,i}\} \in [0, 1]$.

Theorem 3.5. *Let D be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T_i : D \longrightarrow P(D), (i = 1, 2, ..., m)$ be multivalued mappings such that P_{T_i} is quasi-nonexpansive and satisfies the condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the iterative process defined by (B), and $a_{n,i} \in [\alpha, \beta] \subset (0, 1)$ (i=1,2,...,m). Assume that the mappings $T_1, T_2, ..., T_m$ satisfy the condition (II). Then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, ..., T_m$.*

Proof. Let $p \in \mathcal{F}$, then, $P_{T_i}(p) = \{p\}$ for $i = 1, 2, ..., m$. It follows from (B) that

$$\begin{aligned} \| y_{n,1} - p \| &= \| (1 - a_{n,1})x_n + a_{n,1} z_{n,1} - p \| \\ &\leq (1 - a_{n,1}) \| x_n - p \| + a_{n,1} \| z_{n,1} - p \| \\ &= (1 - a_{n,1}) \| x_n - p \| + a_{n,1} \text{dist}(z_{n,1}, P_{T_1}(p)) \\ &\leq (1 - a_{n,1}) \| x_n - p \| + a_{n,1} H(P_{T_1}(x_n), P_{T_1}(p)) \\ &\leq (1 - a_{n,1}) \| x_n - p \| + a_{n,1} \| x_n - p \| \\ &\leq \|x_n - p\|. \end{aligned}$$

Continuing the above process, we get

$$\|y_{n,k} - p\| \leq \|x_n - p\| \quad \text{for } \; k = 2, ..., m - 1.$$

It follows that

$$\begin{aligned} \| x_{n+1} - p \| &= \| (1 - a_{n,m})z_{n,m-1} + a_{n,m} z_{n,m} - p \| \\ &\leq (1 - a_{n,m}) \| z_{n,m-1} - p \| + a_{n,m} \| z_{n,m} - p \| \\ &= (1 - a_{n,m})\text{dist}(z_{n,m-1}, P_{T_{m-1}}(p)) + a_{n,m} \text{dist}(z_{n,m}, P_{T_m}(p)) \\ &\leq (1 - a_{n,m}) H(P_{T_{m-1}}(y_{n,m-2}), P_{T_{m-1}}(p)) \\ &\quad + a_{n,m} H(P_{T_m}(y_{n,m-1}), P_{T_m}(p)) \\ &\leq (1 - a_{n,m}) \| y_{n,m-2} - p \| + a_{n,m} \| y_{n,m-1} - p \| \\ &\leq (1 - a_{n,m}) \| x_n - p \| + a_{n,m} \| x_n - p \| \\ &\leq \|x_n - p\|. \end{aligned}$$

Thus $\lim_{n\longrightarrow\infty} \parallel x_n - p \parallel$ exists for any $p \in \mathcal{F}$. Since the sequences $\{x_n\}$ and $\{y_{n,i}\}$, $i = 1, ..., m-1$ are bounded, we can find $r > 0$ depending on p such that $x_n - p$, $y_{n,i} - p \in B_r(0)$ for all $n \geq 0$. From Lemma 2.5, we get

$$\begin{aligned}
\parallel y_{n,1} - p \parallel^2 &= \parallel (1-a_{n,1})x_n + a_{n,1}z_{n,1} - p \parallel^2 \\
&\leq (1-a_{n,1}) \parallel x_n - p \parallel^2 + a_{n,1} \parallel z_{n,1} - p \parallel^2 \\
&\quad - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\
&= (1-a_{n,1}) \parallel x_n - p \parallel^2 + a_{n,1}\text{dist}(z_{n,1}, P_{T_1}(p))^2 \\
&\quad - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\
&\leq (1-a_{n,1}) \parallel x_n - p \parallel^2 + a_{n,1}H(P_{T_1}(x_n), P_{T_1}(p))^2 \\
&\quad - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\
&\leq (1-a_{n,1}) \parallel x_n - p \parallel^2 + a_{n,1} \parallel x_n - p \parallel^2 \\
&\quad - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\
&\leq \|x_n - p\|^2 - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|).
\end{aligned}$$

It follows from lemma 2.5 that

$$\begin{aligned}
\parallel x_{n+1} - p \parallel^2 &= \parallel (1-a_{n,m})z_{n,m-1} + a_{n,m}z_{n,m} - p \parallel^2 \\
&\leq (1-a_{n,m}) \parallel z_{n,m-1} - p \parallel^2 + a_{n,m} \parallel z_{n,m} - p \parallel^2 \\
&\quad - a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1} - z_{n,m}\|) \\
&= (1-a_{n,m})\text{dist}(z_{n,m-1}, P_{T_{m-1}}(p))^2 + a_{n,m}\text{dist}(z_{n,m}, P_{T_m}(p))^2 \\
&\quad - a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1} - z_{n,m}\|) \\
&\leq (1-a_{n,m})H(P_{T_{m-1}}(y_{n,m-2}), P_{T_{m-1}}(p))^2 \\
&\quad + a_{n,m}H(P_{T_m}(y_{n,m-1}), P_{T_m}(p))^2 \\
&\quad - a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1} - z_{n,m}\|) \\
&\leq (1-a_{n,m}) \parallel y_{n,m-2} - p \parallel^2 + a_{n,m} \parallel y_{n,m-1} - p \parallel^2 \\
&\quad - a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1} - z_{n,m}\|) \\
&\leq \|x_n - p\|^2 - a_{n,m}a_{n,m-1}a_{n,m-2}...a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\
&\quad - \cdots - a_{n,m}a_{n,m-1}(1-a_{n,m-1})\varphi(\|z_{n,m-2} - z_{n,m-1}\|) \\
&\quad - a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1} - z_{n,m}\|).
\end{aligned}$$

Now, by a similar argument as in the proof of Theorem 3.2 we have $\lim_{n\longrightarrow\infty} \|x_n - z_{n,k}\| = 0$ for $k = 2, ..., m-1$ and $\lim_{n\longrightarrow\infty} \|x_n - y_{n,k}\| = 0$ for $k = 2, ..., m$. Also we have

$$\text{dist}(x_n, T_1x_n) \leq \text{dist}(x_n, P_{T_1}x_n) \leq \|x_n - z_{n,1}\| \longrightarrow 0.$$

For $k = 2, ..., m$, since P_{T_i} satisfies the condition (E) we have

$$\begin{aligned}
\text{dist}(x_n, T_kx_n) &\leq \text{dist}(x_n, P_{T_k}x_n) \leq \|x_n - y_{n,k-1}\| + \text{dist}(y_{n,k-1}, P_{T_k}x_n) \\
&\leq \|x_n - y_{n,k-1}\| + \mu\,\text{dist}(y_{n,k-1}, P_{T_k}y_{n,k-1}) + \|x_n - y_{n,k-1}\| \\
&\leq \|x_n - y_{n,k-1}\| + \mu\,\text{dist}(x_n, P_{T_k}y_{n,k-1}) + \mu\,\|x_n - y_{n,k-1}\| \\
&\quad + \|x_n - y_{n,k-1}\| \\
&\leq (\mu+2)\|x_n - y_{n,k-1}\| + \mu\|x_n - z_{n,k}\| \longrightarrow 0, \qquad n \longrightarrow \infty.
\end{aligned}$$

Note that by our assumption $\lim_{n\longrightarrow\infty} \text{dist}(x_n, \mathcal{F}) = 0$. By a similar argument as in the proof of Theorem 3.2 we show that $\{x_n\}$ is Cauchy sequence in D and hence converges to $w \in D$. For $i = 1, ..., m$, since P_{T_i} satisfies the condition (E), we have

$$\text{dist}(w, T_iw) \leq \text{dist}(w, P_{T_i}w)$$

$$\leq \quad \|w - x_n\| + \text{dist}(x_n, P_{T_i}w)$$
$$\leq \quad 2\|w - x_n\| + (\mu)\text{dist}(x_n, P_{T_i}x_n) \longrightarrow 0, \qquad n \longrightarrow \infty$$

from which it follows that $\text{dist}(w, T_i w) = 0$, which in turn implies that $w \in T_i(w), (i = 1, 2, ..., m)$. Thus $w \in \mathcal{F}$. This completes the proof. □

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