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EXISTENCE RESULTS FOR A P-LAPLACIAN EQUATION WITH DIFFUSION, STRONG ALLEE EFFECT AND CONSTANT YIELD HARVESTING

G.A. AFROUZI*, S. SHAKERI

Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

ABSTRACT. We study a reaction-diffusion equation involving p-Laplacian with strong Allee effect type growth and constant yield harvesting (semipositone) in heterogeneous bounded habitats. An existence result under suitable assumptions is presented. We obtain our results via the method of sub-super solutions.

KEYWORDS : Semipositone; Sub-super solution; Allee effect; Harvesting.
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1. Introduction

The aim of this paper is to investigate the following the nonlinear boundary value problem

$$\begin{cases} -\Delta_p u = a(x)u + b(x)u^2 - h(x)u^3 - c\alpha(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 1)$ with boundary $\partial\Omega$ of class C^1, $\Delta_p z := \operatorname{div}(|\nabla z|^{p-2}\nabla z)$ is the p-Laplacian operator, $p > 1$, a, b, h and α are c^μ functions (Hölder continuous) such that h and b are strictly positive functions on Ω and a is negative at least for some $x \in \Omega$ (strong allee effect). Let a_0, a_1, b_0, b_1, h_0 and h_1 are defined as $a_0 := -\inf_{x\in\overline{\Omega}} a(x),\ a_1 := \sup_{x\in\overline{\Omega}} a(x),\ b_0 := \inf_{x\in\overline{\Omega}} b(x),\ h_1 := \sup_{x\in\overline{\Omega}} h(x)$ and $b_1 := \sup_{x\in\overline{\Omega}} b(x), h_0 := \inf_{x\in\overline{\Omega}} h(x)$. Here $c\alpha$ with $\alpha : \Omega \longrightarrow [0, 1]$ and $c \geq 0$ a parameter represents the constant yield harvesting. We prove the existence of positive solution under certain condition. Our approach is based on the method of sub and supersolutions. Recently, In the case when $p = 2$ the problem (1.1) have been studied by R.Shivaji et al. (see [13]). The main purpose of this paper is to extend it to the p-Laplacian case with constant yield harvesting.

* *Corresponding author.*
Email address : afrouzi@umz.ac.ir(G.A. Afrouzi) and s.shakeri@umz.ac.ir(S. Shakeri).
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A typical model of reaction diffusion equations that describes the spatiotemporal distribution and abundance of organisms is

$$u_t = d\Delta u + uf(x,u)$$

where $u(x,t)$ is the populations density, $d > 0$ is the diffusion coefficient, Δu is the Laplacian of u with respect to the x variable, and $f(x,u)$ is the per capita growth rate, which is affected by the heterogeneous environment. Such ecology models were first studied by Skellam (see [12]). A classic example is Fishers equation (see [4]) with

$$f(x,u) = (1-u).$$

Similar reaction diffusion biological models have been studied by Kolmogoroff, Petrovsky, and Piscounoff (see [9]) earlier. Since then reaction diffusion models have been used to describe various spatiotemporal phenomena in biology, physics, chemistry and ecology, Fife (see [3]), Okuba and Levin(see [5]), Murray (see [6]), and Cantrell and Cosner (see [2]). Since the pioneer work by Skellam (see [12]), the logistic growth rate

$$f(x,u) = m(x) - b(x)u$$

has been used in population dynamics to model the crowding effect (see Oruganti, Shi and Shivaji (see [10])). A more general logistic type model can be characterized by a declining growth rate per capita function, i.e., $f(x,u)$ is decreasing with respect to u. In this paper, we consider the dispersal and evolution of species on a bounded domain Ω (in R^N) when the per capita growth rate is

$$f(x,u) = a(x)u + b(x)u^2 - h(x)u^3.$$

Note that (1.1)is a semipositone problem due to the presence of the constant yield harvesting term. It is well known in the literature that the study of positive solutions to semipositone problems is mathematically challenging (see [1],[8], [10]). (see [10])where such a model was discussed for the logistic growth case with constant coefficients. Here we deal with the more difficult strong Allee effect growth.We prove the existence of positive solution via the method of sub-super solutions.

2. Preliminaries

Here and in what follows, $W_0^{1,p}(\Omega)$, $p > 1$, denotes the usual Sobolev space, We give the definition of sub-super solution of (1.1).

Definition 2.1. we say that (ψ) (*resp.* ϕ in $W_0^{1,p}(\Omega)$ are called a subsolution (*resp.* supersolution) of (1.1), if ψ satisfies

$$\begin{cases} \int_\Omega |\nabla\psi(x)|^{p-2}\nabla\psi(x)\nabla w(x)dx \le \int_\Omega \Big(a(x)\psi(x) + b(x)\psi^2 - h(x)\psi^3 - c\alpha(x)\Big)w(x)dx \\ \psi \le 0 \end{cases} \tag{2.1}$$

$$\begin{cases} \int_\Omega |\nabla\phi(x)|^{p-2}\nabla\phi(x)\nabla w(x)dx \ge \int_\Omega \Big(a(x)\phi(x) + b(x)\phi^2 - h(x)\phi^3 - c\alpha(x)\Big)w(x)dx \\ \phi \ge 0 \end{cases} \tag{2.2}$$

for all non-negative test functions $w \in W_0^{1,p}(\Omega)$.

Now, if there exists a subsolution and a supersolution ψ and ϕ, respectively, such that $\psi(x) \le 0 \le \phi(x)$ for all $x \in \Omega$, then (1.1) has a positive solution $u \in W_0^{1,p}(\Omega)$ such that $\psi(x) \le u(x) \le \phi(x)$ for all $x \in \Omega$. We shall obtain the existence of positive weak solution to the problem (1.1) by constructing a subsolution ψ and a positive supersolution ϕ.

Proposition 2.2.

: (a) *If* $\lambda_1 \geq \frac{b_1^2+4a_0h_0}{4h_0}$ *and* $c>0$, *then (1.1) has no positive solution.*
: (b) *for* c *large, the problem (1.1) has no positive solution.*

Proof. (a) Let λ_1 be the first eigenvalue of- Δ_p with Dirichlet boundary conditions and ϕ_1 the corresponding eigenfunction

$$\begin{cases} -\Delta_p\phi = \lambda_1|\phi|^{p-2}\phi & \text{in } \Omega,\\ \phi = 0 & \text{on } \partial\Omega \end{cases}$$

with $\|\phi_1\|_\infty = 1$. let u be a positive solution of (1.1). Then by using the green identity, we have

$$0 = \int (u\Delta_p\phi_1 - \phi_1\Delta_p u) = \int \phi_1(a(x)u + b(x)u^2 - h(x)u^3 - c\alpha(x)) - \lambda_1|\phi_1|^{p-2}u\phi_1 dx.$$

Let

$$\tilde{f}(s) = a_1 + b_1 s - h_0 s^2$$

and

$$f^*(s) = u\tilde{f}(u).$$

Then

$$\tilde{f}_0 = \sup_{u\in[0,\infty]} \tilde{f}(u) = \lambda_1 \geq \frac{b_1^2+4a_0h_0}{4h_0}$$

and

$$a(x)u + b(x)u^2 - h(x)u^3 - c\alpha(x) \leq f^*(u) - c\alpha(x) \quad \text{for } x \geq 0.$$

Rewriting we have

$$0 = \int (u\Delta_p\phi_1 - \phi_1\Delta_p u) \leq \int \phi_1 f^*(u) - \lambda_1|\phi_1|^{p-2}u\phi_1 - c\alpha(x)\phi_1 dx.$$

But for $\lambda_1 \geq \frac{b_1^2+4a_0h_0}{4h_0}$,

$$\int \phi_1 f^*(u) - \lambda_1|\phi_1|^{p-2}u\phi_1 - c\alpha(x)\phi_1 dx = \int (\tilde{f}(u) - \lambda_1|\phi_1|^{p-2})u\phi_1 dx - c\int \alpha(x)\phi_1 dx \leq 0$$

which is a contradiction. Hence (1.1) has no positive solution.
We now prove (b). We observe that

$$\begin{aligned} c\int \alpha(x)\phi_1 dx &= \int \phi_1\Delta_p u + \int \phi_1(a(x)u + b(x)u^2 - h(x)u^3)dx \\ &\leq -\lambda_1\int |\phi_1|^{p-2}u + \tilde{f}_0\int \phi_1 \\ &\leq \tilde{f}_0\int \phi_1. \end{aligned}$$

Clearly is not satiesfies for c large and Hence part (b) of Proposition holds. □

3. Existence results

In this section we prove the existence of solution by comparison method. we first prove an existence result for

$$\begin{cases} -\Delta_p u = -a_0 u + b_0 u^2 - h_1 u^3 - c\alpha(x) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Let λ_1 be the principal eigenvalue of the operator $-\Delta_p$ with Dirichlet boundary conditions and ϕ_1 the corresponding eigenfunction with ϕ_1 and $\|\phi_1\|_\infty = 1$. Hence there exist $\eta \geq 0$ and $\mu \in (0,1]$ and $k > 0$ such that

$$\begin{cases} |\nabla \phi_1|^p - \lambda_1 \phi_1^p \geq k & \text{in } \overline{\Omega}_\eta, \\ \phi_1 \geq \mu & \text{in } \Omega \setminus \overline{\Omega}_\eta, \end{cases}$$

where $\overline{\Omega}_\eta = \{x \in \Omega : d(x, \partial\Omega) \leq \eta\}$. To discuss our existence result, it is known that $\phi_1 > 0$ in Ω and $\frac{\partial \phi_1}{\partial n} < 0$ on $\partial\Omega$, where n denotes the outward unit normal to $\partial\Omega$.

Let

$$b_0 > 2\sqrt{a_0 h_1}$$

and

$$g(s) = -a_0 s + b_0 s^2 - h_1 s^3.$$

The zeros of g are 0,

$$r := \frac{b_0 - \sqrt{b_0^2 - 4a_0h_1}}{2h_1},$$

and

$$R := \frac{b_0 + \sqrt{b_0^2 - 4a_0h_1}}{2h_1},$$

and hence

$$g(s) := -h_1 s(s-R)(s-r).$$

Let r^* be the first positive zero of g'. In fact

$$r^* = \frac{b_0 - \sqrt{b_0^2 - 3a_0h_1}}{3h_1} < \frac{b_0}{3h_1}.$$

But g is convex on $(0, \frac{b_0}{3h_1})$. Hence

$$\sigma := -\inf_{s \in [0,R]} g(s) < a_0 [\frac{b_0 - \sqrt{b_0^2 - 3a_0h_1}}{3h_1}] = a_0 r^*$$

We first note that

$$\begin{aligned} \frac{\sigma}{R^{p-1}} &< a_0 \Big(\frac{b_0 - \sqrt{b_0^2 - 3a_0h_1}}{3h_1}\Big)\Big(\frac{b_0 + \sqrt{b_0^2 - 4a_0h_1}}{2h_1}\Big)^{1-p} \\ &= \frac{(2h_1)^{p-1} a_0}{[b_0 + \sqrt{b_0^2 - 4a_0h_1}]^{p-1}} \frac{b_0 - \sqrt{b_0^2 - 3a_0h_1}}{3h_1} \\ &= \frac{(2h_1)^{p-1} a_0^2}{b_0 + \sqrt{b_0^2 - 3a_0h_1}} \Big[b_0 + \sqrt{b_0^2 - 4a_0h_1}\Big]^{1-p} \end{aligned}$$

and thus RHS tends to zero as b_0 tends to infinity. Hence there exist $b_0^{(1)} := b_0^{(1)}(a_0, h_1, \Omega)$ such that for every $b_0 > b_0^{(1)}$, we have $m > R^{\frac{\sigma}{p-1}}$. Next we also note that

$$\frac{R}{r} = \frac{b_0 + \sqrt{b_0^2 - 4a_0h_1}}{b_0 - \sqrt{b_0^2 - 4a_0h_1}} = \frac{[b_0 + \sqrt{b_0^2 - 4a_0h_1}]^2}{4a_0h_1} \to \infty \quad \text{as} \quad b_0 \to \infty.$$

Hence there exists $b_0^{(2)} := b_0^{(2)}(a_0, h_1, \Omega)$ such that for every $b_0 > b_0^{(2)}$, we have

$$[R\frac{p-1}{p}\mu^{\frac{p}{p-1}}, R\frac{p-1}{p}] \subset (r, R)$$

and

$$k_\mu := \inf_{s \in [R\frac{p-1}{p}\mu^{\frac{p}{p-1}}, R\frac{p-1}{p}]} g(s) > 0.$$

Eventually

$$\begin{aligned} \frac{k_\mu}{R^{p-1}} &= \frac{\min\{g(R\frac{p-1}{p}\mu^{\frac{p}{p-1}}), g(R\frac{p-1}{p})\}}{R^{p-1}} \\ &= \min\{h_1 R\frac{p-1}{p}\mu^{\frac{p}{p-1}}(R\frac{p-1}{p}\mu^{\frac{p}{p-1}} - R)(R\frac{p-1}{p}\mu^{\frac{p}{p-1}} - r), \\ & h_1 R\frac{p-1}{p}(R\frac{p-1}{p} - R)(R\frac{p-1}{p} - r)\} \end{aligned} \quad (3.2)$$

tends to infinity as b_0 tends to infinity. Thus there exists $b_0^{(3)} := b_0^{(3)}(a_0, h_1, \Omega) > b_0^{(2)}$ such that such that for every $b_0 > b_0^{(3)}$ we have

$$\lambda_1 < \frac{k_\mu}{R^{p-1}}.$$

For a given $a_0 > 0, h_1 > 0$, define $\tilde{b_0} := \max\{b_0^{(3)}, b_0^{(1)}\} := \tilde{b_0}(a_0, h_1, \Omega)$. Then we have

Lemma 3.1. *Let $b_0 > \tilde{b_0}$ and $c^* := c^*(a_0, h_1, \Omega, b_0) := \min\{mR^{p-1} - \sigma, k_\mu - R^{p-1}\lambda_1\}$ for $c \le c^*$, (3.1) has a positive solution.*

Proof. We now construct subsolution

$$\psi := R(\frac{p}{p-1})\phi_1^{\frac{p}{p-1}}$$

is a sub-solution of (3.1). For $x \in \overline{\Omega}_\eta$ then

$$\nabla\psi = R\phi_1^{\frac{1}{p-1}}\nabla\phi_1$$

and ψ will be a subsolution if

$$\begin{cases} \int_\Omega |\nabla\psi(x)|^{p-2}\nabla\psi(x)\nabla w(x)dx \le \int_\Omega \Big(a(x)\psi(x) + b(x)\psi^2 - h(x)\psi^3 - c\alpha(x)\Big)w(x)dx \\ \psi \le 0 \end{cases} \quad (3.3)$$

But

$$\begin{aligned} \int_\Omega |\nabla\psi(x)|^{p-2}\nabla\psi(x)\nabla w(x)dx &= \int_\Omega \phi(x)|\nabla\phi(x)|^{p-2}\nabla\phi(x)\nabla w(x)dx \\ &= R^{p-1}[\int_\Omega |\nabla\phi(x)|^{p-2}\nabla\phi(x)\nabla(\phi(x)w(x))]dx \\ &- R^{p-1}[\int_\Omega |\nabla\phi(x)|^p w(x)]dx \\ &= R^{p-1}[\int_\Omega (\lambda_1\phi(x)^p - |\nabla\phi(x)|^p)w(x)]dx. \end{aligned}$$

Now for $x \in \overline{\Omega}_\eta$,

$$R^{p-1}(\lambda_1\phi(x)^p - |\nabla\phi(x)|^p) \le -R^{p-1}m \le -\sigma - c \le -a_0\psi + b_0\psi^2 - h_1\psi^3 - c\alpha(x).$$

Next for $x \in \Omega \setminus \overline{\Omega}_\eta$,

$$R^{p-1}(\lambda_1\phi(x)^p - |\nabla\phi(x)|^p) \le -R^{p-1}\lambda_1 \le k_\mu - c \le -a_0\psi + b_0\psi^2 - h_1\psi^3 - c\alpha(x).$$

Hence

$$\psi := R(\frac{p}{p-1})\phi_1^{\frac{p}{p-1}}$$

is a sub-solution of (3.1). We also note that $\phi = R$ is a super-solution. Hence (3.1) has a positive solution. ☐

Our main result is the following theorem.

Theorem 3.1. *There exists positive constants* $\tilde{b_0} := \tilde{b_0}^{(1)}(a_0, h_1, \Omega)$ *and*

$$c^* := c^*(a_0, h_1, \Omega, b_0)$$

such that for $b_0 \geq \tilde{b}_0$ *and* $c \leq c^*$ *the problem (1.1) has a positive solution. Further* c^* *is an increasing function of* b_0 *and*

$$\lim_{b_0 \to \infty} c^* = \infty.$$

Proof. We observe that $\psi := R(\frac{p}{p-1})\phi_1^{\frac{p}{p-1}}$ is a positive sub-solution for (1.1), since ψ satisfies

$$\begin{cases} -\Delta_p \psi \leq g(\psi) - c\alpha(x) \leq a(x)\psi + b(x)\psi^2 - h(x)\psi^3 - c\alpha(x) & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

It is easy to see that $\phi = M$ where $M > 0$ is sufficiently large so that $\phi \geq \psi$ is also satisfied. Hence (1.1) has a positive solution. We recall here that $\sigma = -g(r^*, b_0)$. Differentiating σ with respect to b_0, we have

$$\begin{aligned} \frac{d\sigma}{db_0} &= -\frac{\partial g(r^*, b_0)}{\partial r^*}\frac{dr^*}{db_0} - \frac{\partial g}{\partial b_0} \\ &= -\frac{\partial g}{\partial b_0} \\ &= -{r^*}^2 < 0. \end{aligned}$$

Also R is an increasing function of b_0. Thus $R^{p-1}m - \sigma$ is an increasing function of b_0. Next since $\frac{r}{R}$ decreases as b_0 increases, we deduce from (5) that $\frac{k_\mu}{R^{p-1}}$ increases as b_0 increases. Therefore $k_\mu - R^{p-1}\lambda_1 = R^{p-1}[\frac{k_\mu}{R^{p-1}} - \lambda_1]$ also increases as b_0. Hence by the definition of c^*, it is clear that c^* is an increasing function of b_0. Finally since $R \to \infty$, $\frac{\sigma}{R^{p-1}} \to 0$ and $\frac{k_\mu}{R^{p-1}} \to \infty$ as $b_0 \to \infty$, it is easy to see that $\lim_{b_0 \to \infty} c^* = \infty$. This completes the proof of Theorem. □

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