



ASYMPTOTIC BEHAVIOR OF INFINITE PRODUCTS OF PROJECTION AND NONEXPANSIVE OPERATORS WITH COMPUTATIONAL ERRORS

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ABSTRACT. We study the asymptotic behavior of infinite products of orthogonal projections and other (possibly nonlinear) nonexpansive operators in Hilbert space in the presence of computational errors.

KEYWORDS : Hilbert space; Infinite product; Nonexpansive operator; Orthogonal projection.
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1. INTRODUCTION

Consider m closed linear subspaces S_1, S_2, \dots, S_m of a given Hilbert space H and let S denote their intersection. Let the infinite product $\prod_{j=1}^{\infty} P_j := \cdots P_3 P_2 P_1$ only consist of orthogonal projections P_{S_k} , $1 \leq k \leq m$, onto these subspaces. We are concerned with the asymptotic behavior of the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_n = \prod_{j=1}^n P_j x_0 = P_n P_{n-1} \cdots P_1 x_0, \quad n = 1, 2, \dots$$

Denoting the norm of H by $\|\cdot\|$, we recall that the classical theorems of J. von Neumann [10] and I. Halperin [9] declare that, for any $x_0 \in H$,

$$\lim_{n \rightarrow \infty} \|(P_{S_m} P_{S_{m-1}} \cdots P_{S_1})^n x_0 - P_S x_0\| = 0. \quad (1.1)$$

We observe that the iterative process here is strongly cyclical and this condition is, in fact, essential for the proof of (1.1). The convergence in (1.1) may not be uniform (on bounded subsets of initial points) and these theorems do not provide any rate of convergence.

Except for some earlier partial results, general necessary and sufficient conditions for uniform convergence in (1.1) were found much later (see [6] for $m = 2$ and [2] for the general case). In addition, some estimates of the rate of this convergence were also obtained, mainly by using the notion of angles between subspaces [8].

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Traditionally, these estimates were asserted for cyclical products, although this restriction was not as essential as it was for proving (1.1) in [9]. As a matter of fact, these estimates were stated for one cycle, but then immediately generalized to the power n . Consequently, the possibility of concatenating different fragmentary products of projections into an infinite one was not considered.

We have recently [15–17] used a geometric approach (in particular, angles between subspaces) to establish useful estimates for proving convergence of infinite products involving not only orthogonal projections, but also other (possibly nonlinear) nonexpansive operators in Hilbert space. In [18] we provide sufficient conditions for the strong and uniform (on bounded subsets of initial points) convergence of such infinite products by applying new estimates of the inclination [1] of a finite tuple of closed linear subspaces.

In view of the diverse applications of infinite products (see, for instance, [7] and the references therein), it is natural to ask if these results continue to hold in the presence of computational errors. In the present paper we give affirmative answers to this question. Our main results, Theorems 2.1 and 2.2, are formulated in Section 2. Their proofs are given in Section 3.

Previous studies concerning inexact powers and infinite products of operators can be found, for example, in [3–5, 11–14, 19].

2. MAIN RESULTS

For each $x \in H$ and each $B \subset H$, set

$$\rho(x, B) = \inf\{||x - y|| : y \in B\}.$$

We consider an iterative process, presented as an infinite product $\prod_{i=1}^{\infty} A_i \mathcal{P}_i$, where all A_i are quasi-nonexpansive, possibly nonlinear, operators of arbitrary nature and each \mathcal{P}_i is a finite product of all the projections $P_{S_1}, P_{S_2}, \dots, P_{S_m}$ in any order and amount (that is, with possible repetitions). Here S_1, S_2, \dots, S_m are assumed to be closed linear subspaces of a given Hilbert space H . By S we denote the intersection of S_1, S_2, \dots, S_m ; the case $S = \{0\}$ is permitted as well. We assume that all the subspaces S_j are different. Recall that the principal question studied in this paper concerns the asymptotic behavior of the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_n = \prod_{i=1}^n A_i \mathcal{P}_i x_0$, $n = 1, 2, \dots$, in the presence of computational errors.

We are now ready to state our two main results.

Theorem 2.1. *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of self-mappings of a Hilbert space H . Assume that for all integers $n \geq 1$,*

$$A_n(S) \subset S \quad (2.1)$$

and

$$||A_n y - A_n x|| \leq ||y - x|| \text{ for all } y \in H \text{ and all } x \in S. \quad (2.2)$$

Assume further that for all integers $n \geq 1$ and all $x \in H$,

$$||\mathcal{P}_n x - P_S x|| \leq q_n ||x - P_S x|| \quad (2.3)$$

with the factors $q_n \in (0, 1]$ satisfying

$$\lim_{n \rightarrow \infty} \prod_{i=1}^n q_i = 0 \quad (2.4)$$

and

$$M_0 := \sup\{1 + \sum_{p=2}^k \prod_{i=p}^k q_i : k = 2, 3, \dots\} < \infty. \quad (2.5)$$

Let the positive numbers M, ϵ, δ and a natural number n_0 satisfy

$$\prod_{i=1}^{n_0} q_i M \leq \epsilon/2 \quad (2.6)$$

and

$$\delta M_0 \leq \epsilon/2. \quad (2.7)$$

Finally, assume that $\{x_i\}_{i=0}^{\infty} \subset H$,

$$\|x_0\| \leq M, \quad (2.8)$$

and that for any integer $n \geq 1$,

$$\|x_n - A_n \mathcal{P}_n x_{n-1}\| \leq \delta. \quad (2.9)$$

Then

$$\rho(x_i, S) \leq \epsilon \text{ for all integers } i \geq n_0.$$

Theorem 2.2. Assume that (2.1) and (2.2) hold for all integers $n \geq 1$, and that (2.3) holds for all integers $n \geq 1$ and all points $x \in H$ with the factors $q_n \in (0, 1]$ satisfying both (2.4) and (2.5). Assume further that $\{\delta_i\}_{i=1}^{\infty} \subset (0, \infty)$ and

$$\lim_{i \rightarrow \infty} \delta_i = 0. \quad (2.10)$$

Let $\epsilon, M > 0$ be given. Then there exists an integer $n_1 \geq 1$ such that for each sequence $\{x_n\}_{n=0}^{\infty} \subset H$ satisfying

$$\|x_0\| \leq M \quad (2.11)$$

and

$$\|x_n - A_n \mathcal{P}_n x_{n-1}\| \leq \delta_n \text{ for all integers } n \geq 1, \quad (2.12)$$

the following inequality holds:

$$\rho(x_n, S) \leq \epsilon \text{ for all integers } n \geq n_1. \quad (2.13)$$

Before proceeding to the proofs of these theorems in Section 3, we observe that both (2.4) and (2.5) clearly hold if for all $i = 1, 2, \dots$,

$$q_i \leq q < 1$$

for some constant q .

More generally, both (2.4) and (2.5) hold if there are a real number $q \in (0, 1)$ and a natural number p such that

$$q_{ip} \leq q \text{ for all natural numbers } i.$$

In this connection, recall that the number

$$l(S_1, S_2, \dots, S_m) = \inf_{x \notin S} \max_{1 \leq j \leq m} \rho(x, S_j) \rho(x, S)^{-1}$$

is called the inclination of the m -tuple (S_1, S_2, \dots, S_m) . Clearly, $0 \leq l \leq 1$.

By [1], for any set of integers $\{i_1, \dots, i_N\} = \{1, 2, \dots, m\}$ and any $x \in H$,

$$\|P_{S_{i_N}} P_{S_{i_{N-1}}} \cdots P_{S_{i_1}} x - P_S x\| \leq (1 - l^2 N^{-2})^{1/2} \|x - P_S x\|.$$

Thus (2.4) and (2.5) hold if there is a natural number p such that

$$\sup\{N_{ip} : i = 1, 2, \dots\} < \infty,$$

where N_k is the number of operators in the product \mathcal{P}_k .

3. PROOFS OF THEOREMS 2.1 AND 2.2

We begin with the following lemma.

Lemma 3.1. *Assume that both (2.1) and (2.2) hold for each integer $n \geq 1$, (2.3) holds for all integers $n \geq 1$ and all points $x \in H$, δ is a positive number, and the sequence $\{x_n\}_{n=0}^{\infty} \subset X$ satisfies*

$$\|x_n - A_n \mathcal{P}_n x_{n-1}\| \leq \delta \quad (3.1)$$

for all integers $n \geq 1$. Then for any integer $k \geq 2$,

$$\rho(x_k, S) \leq \left(\prod_{i=1}^k q_i \right) \rho(x_0, S) + \delta \left(1 + \sum_{p=2}^k \left(\prod_{i=p}^k q_i \right) \right). \quad (3.2)$$

Proof. In view of (3.1), for any integer $n \geq 1$,

$$\begin{aligned} \rho(x_n, S) &\leq \|x_n - A_n \mathcal{P}_n x_{n-1}\| + \rho(A_n \mathcal{P}_n x_{n-1}, S) \\ &\leq \delta + \|A_n \mathcal{P}_n x_{n-1} - A_n P_S x_{n-1}\|, \end{aligned}$$

and in view of (2.1) and (2.2),

$$\begin{aligned} \rho(x_n, S) &\leq \delta + \|\mathcal{P}_n x_{n-1} - P_S x_{n-1}\| \leq \delta + q_n \|x_{n-1} - P_S x_{n-1}\| \\ &\leq \delta + q_n \rho(x_{n-1}, S). \end{aligned} \quad (3.3)$$

By (3.3),

$$\rho(x_1, S) \leq \delta + q_1 \rho(x_0, S), \quad \rho(x_2, S) \leq q_1 q_2 \rho(x_0, S) + q_2 \delta + \delta. \quad (3.4)$$

We now show by induction that for any integer $k \geq 2$, inequality (3.2) holds. To this end, we first note that in view of (3.4), inequality (3.2) certainly holds for $k = 2$.

Assume that $n \geq 2$ is an integer and that (3.2) holds for $k = n$. Thus

$$\rho(x_n, S) \leq \left(\prod_{i=1}^n q_i \right) \rho(x_0, S) + \delta \left(1 + \sum_{p=2}^n \left(\prod_{i=p}^n q_i \right) \right). \quad (3.5)$$

By (3.3) and (3.5),

$$\begin{aligned} \rho(x_{n+1}, S) &\leq \delta + q_{n+1} \rho(x_n, S) \\ &\leq \delta + \left(\prod_{i=1}^{n+1} q_i \right) \rho(x_0, S) + \delta q_{n+1} \left(1 + \sum_{p=2}^n \left(\prod_{i=p}^n q_i \right) \right) \\ &= \left(\prod_{i=1}^{n+1} q_i \right) \rho(x_0, S) + \delta \left(1 + q_{n+1} + \sum_{p=2}^n \left(\prod_{i=p}^{n+1} q_i \right) \right) \\ &= \left(\prod_{i=1}^{n+1} q_i \right) \rho(x_0, S) + \delta \left(1 + \sum_{p=2}^{n+1} \left(\prod_{i=p}^{n+1} q_i \right) \right), \end{aligned}$$

so that (3.2) holds for $k = n + 1$. Thus we have shown by induction that inequality (3.2) holds for all integers $k \geq 2$, as claimed. Lemma 3.1 is proved. \square

Completion of the proof of Theorem 2.1. By Lemma 3.1, (2.8), (2.5), (2.6) and (2.7), for any integer $k \geq n_0$, we have

$$\begin{aligned} \rho(x_k, S) &\leq \left(\prod_{i=1}^k q_i \right) \rho(x_0, S) + \delta \left(1 + \sum_{p=2}^k \left(\prod_{i=p}^k q_i \right) \right) \\ &\leq \left(\prod_{i=1}^{n_0} q_i \right) M + \delta M_0 \leq \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Theorem 2.1 is proved.

Proof of Theorem 2.2. Let M_0 be defined by (2.5). Choose a positive number $\bar{\delta}$ such that

$$\bar{\delta}M_0 < \epsilon/2. \quad (3.6)$$

By (2.10), there is a natural number n_0 such that

$$\delta_i < \bar{\delta} \text{ for all integers } i \geq n_0. \quad (3.7)$$

Choose

$$\hat{\delta} = \sup\{\delta_i : i = 1, 2, \dots\} \quad (3.8)$$

and set

$$M_1 = \hat{\delta}M_0 + M. \quad (3.9)$$

By (2.4), there is a natural number $n_1 > n_0$ such that

$$M_1 \prod_{i=n_0+1}^{n_1} q_i < \epsilon/2. \quad (3.10)$$

Assume that a sequence $\{x_i\}_{i=0}^{\infty} \subset H$ satisfies (2.11) and (2.12). By (3.8), (2.11), (2.12), Lemma 3.1 and (3.9),

$$\rho(x_{n_0}, S) \leq \prod_{i=1}^{n_0} q_i \rho(x_0, S) + \hat{\delta}M_0 < M_1. \quad (3.11)$$

For each integer $i \geq 1$, define

$$y_{i-1} = x_{i-1+n_0}, \quad \tilde{A}_i = A_{i+n_0}, \quad \tilde{\mathcal{P}}_i = \mathcal{P}_{i+n_0}. \quad (3.12)$$

By (3.11), (3.12), (2.12), (2.5), (3.6), (3.7), (3.10) and by Lemma 3.1 applied to $\{y_i\}_{i=0}^{\infty}$, $\{\tilde{A}_i\}_{i=1}^{\infty}$ and $\{\tilde{\mathcal{P}}_i\}_{i=1}^{\infty}$, we have for all integers $k \geq n_1$,

$$\begin{aligned} \rho(x_k, S) &= \rho(y_{k-n_0}, S) \\ &\leq \left(\prod_{p=n_0+1}^k q_p \right) \rho(y_0, S) + \hat{\delta} \left(1 + \sum_{p=n_0+1}^k \left(\prod_{i=p}^k q_i \right) \right) \\ &\leq \prod_{p=n_0+1}^{n_1} q_p M_1 + \bar{\delta} M_0 < \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

as asserted. Theorem 2.2 is proved.

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