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STRONG CONVERGENCE THEOREMS FOR STRONGLY RELATIVELY NONEXPANSIVE SEQUENCES AND APPLICATIONS

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ABSTRACT. The aim of this paper is to establish strong convergence theorems for a strongly relatively nonexpansive sequence in a smooth and uniformly convex Banach space. Then we employ our results to approximate solutions of the zero point problem for a maximal monotone operator and the fixed point problem for a relatively nonexpansive mapping.

KEYWORDS : Strongly relatively nonexpansive sequence; Common fixed point; Strong convergence theorem.
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1. INTRODUCTION

Let E be a smooth and uniformly convex Banach space, E^* the dual of E, $A \subset E \times E^*$ a maximal monotone operator with a zero point, and $\{r_n\}$ a sequence of positive real numbers. Assume that $\{x_n\}$ is a sequence defined as follows: $x_1 \in E$ and

$$x_{n+1} = J^{-1}\left(\frac{1}{n}Jx + \left(1 - \frac{1}{n}\right)J(J + r_n A)^{-1}Jx_n\right)$$

for $n \in \mathbb{N}$, where J and J^{-1} are the duality mappings of E and E^*, respectively. It is known [9] that if $r_n \longrightarrow \infty$, then $\{x_n\}$ converges strongly to some zero point of A. However, we have not known whether $\{x_n\}$ converges strongly or not without the assumption that $r_n \longrightarrow \infty$. In §5 we present an affirmative answer to this problem; see Theorem 5.2 and Remark 5.3.

Furthermore, a more general result is proved; see Theorem 4.1, which is a strong convergence theorem for a strongly relatively nonexpansive sequence introduced

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in [3]. In the proofs of Theorem 4.1, we use modifications of ideas developed in [10, 16]. In particular, Lemma 3.2 due to Maingé [10] is a fundamental tool; see also Example 3.3 and Lemma 3.4.

In §5, using Theorem 4.1, we also show Theorem 5.5 which is a strong convergence theorem for a relatively nonexpansive mapping in the sense of Matsushita and Takahashi [11].

2. PRELIMINARIES

Throughout the present paper, E denotes a real Banach space with norm $\|\cdot\|$, E^* the dual of E, $\langle x, x^*\rangle$ the value of $x^* \in E^*$ at $x \in E$, and $\mathbb{N}$ the set of positive integers. The norm of E^* is also denoted by $\|\cdot\|$. Strong convergence of a sequence $\{x_n\}$ in E to $x \in E$ is denoted by $x_n \longrightarrow x$ and weak convergence by $x_n \rightharpoonup x$. The (normalized) duality mapping of E is denoted by J, that is, it is a set-valued mapping of E into E^* defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in E$.

Let S_E denote the unit sphere of E, that is, $S_E = \{x \in E : \|x\| = 1\}$. The norm $\|\cdot\|$ of E is said to be Gâteaux differentiable if the limit

$$\lim_{t \longrightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S_E$. In this case E is said to be smooth and it is known that the duality mapping J of E is single-valued. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in S_E$ the limit (2.1) is attained uniformly for $x \in S_E$. A Banach space E is said to be uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in S_E$. In this case it is known that J is uniformly norm-to-norm continuous on each bounded subset of E; see [17] for more details.

A Banach space E is said to be strictly convex if $x, y \in S_E$ and $x \neq y$ imply $\|x + y\| < 2$. A Banach space E is said to be uniformly convex if for any $\epsilon > 0$ there exists $\delta > 0$ such that $x, y \in S_E$ and $\|x - y\| \geq \epsilon$ imply $\|x + y\|/2 \leq 1 - \delta$. It is known that E is reflexive and strictly convex if E is uniformly convex; E is uniformly smooth if and only if E^* is uniformly convex; see [17] for more details.

In the rest of this section, unless otherwise stated, we assume that E is a smooth, strictly convex, and reflexive Banach space. In this case it is known that the duality mapping J of E is single-valued and bijective, and J^{-1} is the duality mapping of E^*.

We deal with a real-valued function ϕ on $E \times E$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$; see [1, 8]. From the definition of ϕ, it is clear that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \tag{2.2}$$

for all $x, y \in E$. Since $\|\cdot\|^2$ is convex,

$$\phi\Big(w, J^{-1}\big(\lambda Jx + (1 - \lambda)Jy\big)\Big) \leq \lambda\phi(w, x) + (1 - \lambda)\phi(w, y) \tag{2.3}$$

holds for all $x, y, w \in E$ and $\lambda \in [0, 1]$. It is known that

$$\phi(x, J^{-1}x^*) \leq \phi(x, J^{-1}(x^* - y^*)) + 2\langle J^{-1}x^* - x, y^*\rangle \tag{2.4}$$

holds for all $x \in E$ and $x^*, y^* \in E^*$; see [9, Lemma 3.2].

Lemma 2.1. *([8, Proposition 2]) Let E be a smooth and uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. If $\phi(x_n, y_n) \longrightarrow 0$, then $x_n - y_n \longrightarrow 0$.*

Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. Then it is obvious from the definition of ϕ that $\phi(x_n, y_n) \longrightarrow 0$ if $x_n - y_n \longrightarrow 0$. From this fact and Lemma 2.1, we deduce the following: If E is a uniformly convex and uniformly smooth Banach space, then

$$x_n - y_n \longrightarrow 0 \Leftrightarrow Jx_n - Jy_n \longrightarrow 0 \Leftrightarrow \phi(x_n, y_n) \longrightarrow 0. \tag{2.5}$$

In the rest of this section, we assume that C is a nonempty closed convex subset of E.

Let $T\colon C \longrightarrow E$ be a mapping. The set of fixed points of T is denoted by $F(T)$. A point $p \in C$ is said to be an asymptotic fixed point of T [6, 14] if there exists a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $x_n - Tx_n \longrightarrow 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$. A mapping T is said to be of type (r) if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$; T is said to be relatively nonexpansive [11, 12] if T is of type (r) and $F(T) = \hat{F}(T)$. We know that if $T\colon C \longrightarrow E$ is of type (r), then $F(T)$ is closed and convex; see [12, Proposition 2.4].

It is known that, for each $x \in E$, there exists a unique point $x_0 \in C$ such that

$$\phi(x_0, x) = \min\{\phi(y, x) : y \in C\}.$$

Such a point x_0 is denoted by $Q_C(x)$ and Q_C is called the generalized projection of E onto C; see [1, 8]. It is known that

$$\langle z - Q_C(x), Jx - JQ_C(x)\rangle \leq 0 \tag{2.6}$$

or equivalently

$$\phi(z, Q_C(x)) + \phi(Q_C(x), x) \leq \phi(z, x) \tag{2.7}$$

holds for all $x \in E$ and $z \in C$. It is obvious from (2.7) that the generalized projection Q_C is of type (r).

Let A be a set-valued mapping of E into E^*, which is denoted by $A \subset E \times E^*$. The effective domain of A is denoted by $\operatorname{dom}(A)$ and the range of A by $R(A)$, that is, $\operatorname{dom}(A) = \{x \in E : Ax \neq \emptyset\}$ and $R(A) = \bigcup_{x \in \operatorname{dom}(A)} Ax$. A set-valued mapping $A \subset E \times E^*$ is said to be a monotone operator if $\langle x - y, x^* - y^*\rangle \geq 0$ for all $(x, x^*), (y, y^*) \in A$. A monotone operator $A \subset E \times E^*$ is said to be maximal if $A = A'$ whenever $A' \subset E \times E^*$ is a monotone operator such that $A \subset A'$. It is known that if A is a maximal monotone operator, then $A^{-1}0$ is closed and convex, where $A^{-1}0 = \{x \in E : Ax \ni 0\}$.

Let $A \subset E \times E^*$ be a maximal monotone operator and $r > 0$. Then it is known that $R(J + rA) = E^*$; see [15]. Thus a single-valued mapping $L_r = (J + rA)^{-1}J$ of E onto $\operatorname{dom}(A)$ is well defined and is called the resolvent of A. It is also known that $F(L_r) = A^{-1}0$ and

$$\phi(u, L_r x) + \phi(L_r x, x) \leq \phi(u, x) \tag{2.8}$$

for all $x \in E$ and $u \in F(L_r)$; see [7, 9]. It is obvious from (2.8) that the resolvent L_r of A is of type (r) for all $r > 0$ whenever $A^{-1}0$ is nonempty.

The following lemma is well known; see [2, 18].

Lemma 2.2. *Let $\{\xi_n\}$ be a sequence of nonnegative real numbers, $\{\gamma_n\}$ a sequence of real numbers, and $\{\alpha_n\}$ a sequence in $[0, 1]$. Suppose that $\xi_{n+1} \leq (1-\alpha_n)\xi_n + \alpha_n\gamma_n$ for every $n \in \mathbb{N}$, $\limsup_{n\longrightarrow\infty} \gamma_n \leq 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\xi_n \longrightarrow 0$.*

3. EVENTUALLY INCREASING FUNCTIONS AND STRONGLY RELATIVELY NONEXPANSIVE SEQUENCES

In this section, we provide some needed lemmas about an eventually increasing function and a strongly relatively nonexpansive sequence.

A function $\tau\colon \mathbb{N} \longrightarrow \mathbb{N}$ is said to be *eventually increasing* if $\lim_{n\longrightarrow\infty} \tau(n) = \infty$ and $\tau(n) \leq \tau(n+1)$ for all $n \in \mathbb{N}$. By definition, we easily obtain the following:

Lemma 3.1. *Let $\tau\colon \mathbb{N} \longrightarrow \mathbb{N}$ be an eventually increasing function and $\{\alpha_n\}$ a sequence of real numbers such that $\alpha_n \longrightarrow 0$. Then $\alpha_{\tau(n)} \longrightarrow 0$.*

We need the following lemma:

Lemma 3.2. *(Maingé [10, Lemma 3.1]) Let $\{\xi_n\}$ be a sequence of real numbers. Suppose that there exists a subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ such that $\xi_{n_i} < \xi_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exist $N \in \mathbb{N}$ and a function $\tau\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that $\tau(n) \leq \tau(n+1)$, $\xi_{\tau(n)} \leq \xi_{\tau(n)+1}$, and $\xi_n \leq \xi_{\tau(n)+1}$ for all $n \geq N$ and $\lim_{n\longrightarrow\infty} \tau(n) = \infty$.*

Under the assumptions of Lemma 3.2, we can not choose a strictly increasing function τ; see the following example:

Example 3.3. Let $\{\xi_n\}$ be a sequence of real numbers define by

$$\xi_n = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ 1/n & \text{if } n \text{ is even.} \end{cases}$$

Then the following hold:

(1) There exists a subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ such that $\xi_{n_i} < \xi_{n_i+1}$ for all $i \in \mathbb{N}$;
(2) there does not exist a subsequence $\{\xi_{m_k}\}$ of $\{\xi_n\}$ such that $\xi_{m_k} \leq \xi_{m_k+1}$ and $\xi_k \leq \xi_{m_k+1}$ for all $k \in \mathbb{N}$.

Proof. Define $n_i = 2i - 1$ for each $i \in \mathbb{N}$. Then it is clear that

$$\xi_{n_i} = \xi_{2i-1} = 0 < \frac{1}{2i} = \xi_{2i} = \xi_{n_i+1}$$

for every $i \in \mathbb{N}$. Thus (1) holds.

Let $\{\xi_{m_k}\}$ be a subsequence of $\{\xi_n\}$. Suppose that $\xi_{m_k} \leq \xi_{m_k+1}$ for all $k \in \mathbb{N}$. Then it is easy to check that m_k is odd and $m_k + 1$ is even for every $k \in \mathbb{N}$. We now assume that $\xi_k \leq \xi_{m_k+1}$ for all $k \in \mathbb{N}$. Then it follows that

$$\frac{1}{k} = \xi_k \leq \xi_{m_k+1} = \frac{1}{m_k+1}$$

if k is even. This implies that $k \geq m_k + 1 \geq k + 1$, which is a contradiction. □

Using Lemma 3.2, we obtain the following:

Lemma 3.4. *Let $\{\xi_n\}$ be a sequence of nonnegative real numbers which is not convergent. Then there exist $N \in \mathbb{N}$ and an eventually increasing function $\tau\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that $\xi_{\tau(n)} \leq \xi_{\tau(n)+1}$ for all $n \in \mathbb{N}$ and $\xi_n \leq \xi_{\tau(n)+1}$ for all $n \geq N$.*

Proof. Since $\{\xi_n\}$ is not convergent, for any $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $m \geq n$ and $\xi_m < \xi_{m+1}$, and hence there exists a subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ such that $\xi_{n_i} < \xi_{n_i+1}$ for every $i \in \mathbb{N}$. Lemma 3.2 implies that there exist $N \in \mathbb{N}$ and a function $\sigma\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that $\sigma(n) \leq \sigma(n+1)$, $\xi_{\sigma(n)} \leq \xi_{\sigma(n)+1}$, and $\xi_n \leq \xi_{\sigma(n)+1}$ for every $n \geq N$ and $\lim_{n\longrightarrow\infty} \sigma(n) = \infty$. Let us define $\tau\colon \mathbb{N} \longrightarrow \mathbb{N}$ by $\tau(n) = \sigma(N)$ for $n \in \{1, 2, \ldots, N\}$ and $\tau(n) = \sigma(n)$ for $n > N$, which completes the proof. □

In the rest of this section, unless otherwise stated, we assume that E is a smooth, strictly convex, and reflexive Banach space and C is a nonempty closed convex subset of E.

Let $\{T_n\}$ be a sequence of mappings of C into E such that $F = \bigcap_{n=1}^\infty F(T_n)$ is nonempty. Then

- $\{T_n\}$ is said to be a *strongly relatively nonexpansive sequence* [3] if each T_n is of type (r) and $\phi(T_n x_n, x_n) \longrightarrow 0$ whenever $\{x_n\}$ is a bounded sequence in E and $\phi(p, x_n) - \phi(p, T_n x_n) \longrightarrow 0$ for some point $p \in F$;
- $\{T_n\}$ satisfies the *condition (Z)* if every weak cluster point of $\{x_n\}$ belongs to F whenever $\{x_n\}$ is a bounded sequence in C such that $T_n x_n - x_n \longrightarrow 0$.

Let $A \subset E \times E^*$ be a maximal monotone operator with a zero point and $\{r_n\}$ a sequence of positive real numbers. Then (2.8) shows that the sequence $\{L_{r_n}\}$ of resolvents of A is a strongly relatively nonexpansive sequence; see [3] for more details.

In order to prove our main result in §4, we need the following lemmas:

Lemma 3.5. *Let $\{T_n\}$ be a sequence of mappings of C into E such that $F = \bigcap_{n=1}^\infty F(T_n)$ is nonempty, $\tau\colon \mathbb{N} \longrightarrow \mathbb{N}$ an eventually increasing function, and $\{z_n\}$ a bounded sequence in C such that $\phi(p, z_n) - \phi(p, T_{\tau(n)} z_n) \longrightarrow 0$ for some $p \in F$. If $\{T_n\}$ is a strongly relatively nonexpansive sequence, then $\phi(T_{\tau(n)} z_n, z_n) \longrightarrow 0$.*

Proof. Suppose that $\phi(T_{\tau(n)} z_n, z_n) \not\longrightarrow 0$. Then there exist $\epsilon > 0$ and a strictly increasing function $\sigma\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that $\tau \circ \sigma$ is also strictly increasing and

$$\phi(T_{\tau\circ\sigma(n)} z_{\sigma(n)}, z_{\sigma(n)}) \geq \epsilon \tag{3.1}$$

for all $n \in \mathbb{N}$. Set $\mu = \tau \circ \sigma$ and $R(\mu) = \{\mu(n) : n \in \mathbb{N}\}$. Define a sequence $\{y_n\}$ in C as follows: For each $n \in \mathbb{N}$,

$$y_n = \begin{cases} z_{\sigma\circ\mu^{-1}(n)} & \text{if } n \in R(\mu); \\ p & \text{if } n \notin R(\mu). \end{cases}$$

It is clear that $\{y_n\}$ is bounded,

$$\phi(p, y_n) - \phi(p, T_n y_n) = \phi(p, z_{\sigma\circ\mu^{-1}(n)}) - \phi(p, T_{\tau(\sigma\circ\mu^{-1}(n))} z_{\sigma\circ\mu^{-1}(n)})$$

for $n \in R(\mu)$, and $\phi(p, y_n) - \phi(p, T_n y_n) = 0$ for $n \notin R(\mu)$. Since $\sigma \circ \mu^{-1}$ is strictly increasing, it follows that $\phi(p, y_n) - \phi(p, T_n y_n) \longrightarrow 0$, so $\phi(T_n y_n, y_n) \longrightarrow 0$ because $\{T_n\}$ is a strongly relatively nonexpansive sequence. Therefore, noting that $y_{\mu(n)} = z_{\sigma(\mu^{-1}(\mu(n)))} = z_{\sigma(n)}$ and μ is strictly increasing, we have

$$\phi(T_{\tau\circ\sigma(n)} z_{\sigma(n)}, z_{\sigma(n)}) = \phi(T_{\mu(n)} y_{\mu(n)}, y_{\mu(n)}) \longrightarrow 0,$$

which contradicts to (3.1). □

Lemma 3.6. *Let $\{T_n\}$ be a sequence of mappings of C into E such that $F = \bigcap_{n=1}^\infty F(T_n)$ is nonempty, $\tau\colon \mathbb{N} \longrightarrow \mathbb{N}$ an eventually increasing function, and $\{z_n\}$ a bounded sequence in C such that $T_{\tau(n)} z_n - z_n \longrightarrow 0$. Suppose that $\{T_n\}$ satisfies the condition (Z). Then every weak cluster point of $\{z_n\}$ belongs to F.*

Proof. Let z be a weak cluster point of $\{z_n\}$. Then there exists a strictly increasing function $\sigma\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that $z_{\sigma(n)} \rightharpoonup z$ as $n \longrightarrow \infty$ and $\tau \circ \sigma$ is strictly increasing. Set $\mu = \tau \circ \sigma$ and $R(\mu) = \{\mu(n) : n \in \mathbb{N}\}$. Define a sequence $\{y_n\}$ in C as follows: For each $n \in \mathbb{N}$,

$$y_n = \begin{cases} z_{\sigma\circ\mu^{-1}(n)} & \text{if } n \in R(\mu); \\ p & \text{if } n \notin R(\mu), \end{cases}$$

where p is a point in F. Then it is clear that $\{y_n\}$ is bounded,

$$y_n - T_n y_n = z_{\sigma\circ\mu^{-1}(n)} - T_{\tau(\sigma\circ\mu^{-1}(n))} z_{\sigma\circ\mu^{-1}(n)}$$

for $n \in R(\mu)$, and $y_n - T_n y_n = 0$ for $n \mathbb{N}otin R(\mu)$. Since $z_n - T_{\tau(n)} z_n \longrightarrow 0$ and $\sigma \circ \mu^{-1}$ is strictly increasing, it follows that $y_n - T_n y_n \longrightarrow 0$. Noting that μ is strictly increasing and $y_{\mu(n)} = z_{\sigma\circ\mu^{-1}(\mu(n))} = z_{\sigma(n)}$ for every $n \in \mathbb{N}$, we know that $\{z_{\sigma(n)}\}$ is a subsequence of $\{y_n\}$, and hence z is a weak cluster point of $\{y_n\}$. Since $\{T_n\}$ satisfies the condition (Z), we conclude that $z \in F$. □

Lemma 3.7. *Let $\{T_n\}$ be a sequence of mappings of C into E, F a nonempty closed convex subset of E, $\{z_n\}$ a bounded sequence in C such that $z_n - T_n z_n \longrightarrow 0$, and $u \in E$. Suppose that every weak cluster point of $\{z_n\}$ belongs to F. Then*

$$\limsup_{n\longrightarrow\infty} \langle T_n z_n - w, Ju - Jw\rangle \le 0,$$

where $w = Q_F(u)$.

Proof. Since $z_n - T_n z_n \longrightarrow 0$ and $\{z_n\}$ is bounded, there exists a weakly convergent subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\begin{aligned}\limsup_{n\longrightarrow\infty} \langle T_n z_n - w, Ju - Jw\rangle &= \limsup_{n\longrightarrow\infty} \langle z_n - w, Ju - Jw\rangle \\ &= \lim_{i\longrightarrow\infty} \langle z_{n_i} - w, Ju - Jw\rangle.\end{aligned}$$

Let z be the weak limit of $\{z_{n_i}\}$. By assumption, we see that $z \in F$. Thus (2.6) shows that

$$\lim_{i\longrightarrow\infty} \langle z_{n_i} - w, Ju - Jw\rangle = \langle z - w, Ju - Jw\rangle \le 0,$$

which is the desired result. □

4. STRONG CONVERGENCE THEOREMS FOR STRONGLY RELATIVELY NONEXPANSIVE SEQUENCES

In this section, we prove the following strong convergence theorem:

Theorem 4.1. *Let E be a smooth and uniformly convex Banach space, C a nonempty closed convex subset of E, $\{S_n\}$ a sequence of mappings of C into E such that $F = \bigcap_{n=1}^{\infty} F(S_n)$ is nonempty, and $\{\alpha_n\}$ a sequence in $[0,1]$ such that $\alpha_n \longrightarrow 0$. Let u be a point in E and $\{x_n\}$ a sequence defined by $x_1 \in C$ and*

$$x_{n+1} = Q_C J^{-1}\big(\alpha_n Ju + (1-\alpha_n) JS_n x_n\big) \tag{4.1}$$

for $n \in \mathbb{N}$. Suppose that

- *$\{S_n\}$ is a strongly relatively nonexpansive sequence;*
- *$\{S_n\}$ satisfies the condition (Z);*
- *$\alpha_n > 0$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.*

Then $\{x_n\}$ converges strongly to $w = Q_F(u)$.

First, we show some lemmas; then we prove Theorem 4.1. In the rest of this section, we set

$$y_n = J^{-1}\big(\alpha_n Ju + (1-\alpha_n) JS_n x_n\big)$$

for $n \in \mathbb{N}$, so (4.1) is reduced to $x_{n+1} = Q_C(y_n)$.

Lemma 4.2. *Both $\{x_n\}$ and $\{S_n x_n\}$ are bounded, and moreover, the following hold:*

(1) $y_n - S_n x_n \longrightarrow 0$;
(2) $\phi(w, x_{n+1}) \le \alpha_n \phi(w, u) + \phi(w, S_n x_n)$ *for every* $n \in \mathbb{N}$;
(3) $\phi(w, x_{n+1}) \le (1-\alpha_n)\phi(w, x_n) + 2\alpha_n \langle y_n - w, Ju - Jw\rangle$ *for every* $n \in \mathbb{N}$.

Proof. Since Q_C and S_n are of type (r) and $w \in F(S_n) \subset C$, it follows from (2.3) that

$$\begin{aligned}\phi(w, x_{n+1}) &\le \phi(w, y_n) \\ &\le \alpha_n \phi(w, u) + (1-\alpha_n)\phi(w, S_n x_n) \\ &\le \alpha_n \phi(w, u) + (1-\alpha_n)\phi(w, x_n)\end{aligned} \tag{4.2}$$

for every $n \in \mathbb{N}$. Thus, by induction on n, we have

$$\phi(w, S_n x_n) \le \phi(w, x_n) \le \max\{\phi(w, x_1), \phi(w, u)\}.$$

Therefore, by virtue of (2.2), it turns out that $\{x_n\}$ and $\{S_n x_n\}$ are bounded.

By $\alpha_n \longrightarrow 0$, it is clear that $Jy_n - JS_n x_n = \alpha_n(Ju - JS_n x_n) \longrightarrow 0$. This shows that

$$y_n - S_n x_n = J^{-1}Jy_n - J^{-1}JS_n x_n \longrightarrow 0$$

because E^* is uniformly smooth and J^{-1} is uniformly continuous on every bounded set. Thus (1) holds.

(2) follows from (4.2).

Since S_n is of type (r), it follows from (4.2), (2.4), and (2.3) that

$$\begin{aligned}\phi(w, x_{n+1}) &\le \phi(w, y_n) \\ &\le \phi\Big(w, J^{-1}\big(\alpha_n Ju + (1-\alpha_n)JS_n x_n - \alpha_n(Ju - Jw)\big)\Big) \\ &\qquad + 2\langle y_n - w, \alpha_n(Ju - Jw)\rangle \\ &\le (1-\alpha_n)\phi(w, S_n x_n) + \alpha_n \phi(w, w) + 2\alpha_n \langle y_n - w, Ju - Jw\rangle \\ &\le (1-\alpha_n)\phi(w, x_n) + 2\alpha_n \langle y_n - w, Ju - Jw\rangle\end{aligned} \tag{4.3}$$

for every $n \in \mathbb{N}$. Therefore, (3) holds. □

Lemma 4.3. *Suppose that*

$$\limsup_{n\longrightarrow\infty}\big(\phi(w, x_n) - \phi(w, x_{n+1})\big) \le 0. \tag{4.4}$$

Then $\{x_n\}$ converges strongly to w.

Proof. We first show that $S_n x_n - x_n \longrightarrow 0$. Since S_n is of type (r), it follows from (2) in Lemma 4.2 that

$$0 \le \phi(w, x_n) - \phi(w, S_n x_n) \le \phi(w, x_n) - \phi(w, x_{n+1}) + \alpha_n \phi(w, u)$$

for every $n \in \mathbb{N}$, so $\phi(w, x_n) - \phi(w, S_n x_n) \longrightarrow 0$ by (4.4) and $\alpha_n \longrightarrow 0$. Since $\{S_n\}$ is a strongly relatively nonexpansive sequence and $\{x_n\}$ is bounded by Lemma 4.2, $\phi(S_n x_n, x_n) \longrightarrow 0$. Using Lemma 2.1, we conclude that $S_n x_n - x_n \longrightarrow 0$.

We know that $y_n - S_n x_n \longrightarrow 0$ by (1) in Lemma 4.2 and $\{S_n\}$ satisfies the condition (Z) by assumption, so Lemma 3.7 implies that

$$\limsup_{n\longrightarrow\infty}\langle y_n - w, Ju - Jw\rangle = \limsup_{n\longrightarrow\infty}\langle S_n x_n - w, Ju - Jw\rangle \le 0.$$

It follows from (3) in Lemma 4.2 that

$$\phi(w, x_{n+1}) \le (1-\alpha_n)\phi(w, x_n) + 2\alpha_n \langle y_n - w, Ju - Jw\rangle$$

for every $n \in \mathbb{N}$. Therefore, noting that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and using Lemma 2.2, we conclude that $\phi(w, x_n) \longrightarrow 0$, and hence $x_n \longrightarrow w$ by Lemma 2.1. □

Lemma 4.4. *The real number sequence $\{\phi(w, x_n)\}$ is convergent.*

Proof. We assume, to obtain a contraction, that $\{\phi(w, x_n)\}$ is not convergent. Then Lemma 3.4 implies that there exist $N \in \mathbb{N}$ and an eventually increasing function $\tau\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$\phi(w, x_{\tau(n)}) \le \phi(w, x_{\tau(n)+1}) \tag{4.5}$$

for every $n \in \mathbb{N}$ and

$$\phi(w, x_n) \le \phi(w, x_{\tau(n)+1}) \tag{4.6}$$

for every $n \ge N$.

We show that $S_{\tau(n)} x_{\tau(n)} - x_{\tau(n)} \longrightarrow 0$. Since $S_{\tau(n)}$ is of type (r), it follows from (4.5), (2) in Lemma 4.2, and Lemma 3.1 that

$$\begin{aligned} 0 &\le \phi(w, x_{\tau(n)}) - \phi(w, S_{\tau(n)} x_{\tau(n)}) \\ &\le \phi(w, x_{\tau(n)+1}) - \phi(w, S_{\tau(n)} x_{\tau(n)}) \\ &\le \alpha_{\tau(n)} \phi(w, u) \longrightarrow 0 \end{aligned}$$

as $n \longrightarrow \infty$. Since $\{x_{\tau(n)}\}$ is bounded and $\{S_n\}$ is a strongly relatively nonexpansive sequence, it follows from Lemma 3.5 that $\phi(S_{\tau(n)} x_{\tau(n)}, x_{\tau(n)}) \longrightarrow 0$, so we conclude that $S_{\tau(n)} x_{\tau(n)} - x_{\tau(n)} \longrightarrow 0$ by Lemma 2.1.

Finally, we obtain a contradiction that $\phi(w, x_n) \longrightarrow 0$. From (3) in Lemma 4.2 and (4.5), we know that

$$\begin{aligned} \phi(w, x_{\tau(n)+1}) &\le (1 - \alpha_{\tau(n)}) \phi(w, x_{\tau(n)}) + 2\alpha_{\tau(n)} \langle y_{\tau(n)} - w, Ju - Jw \rangle \\ &\le (1 - \alpha_{\tau(n)}) \phi(w, x_{\tau(n)+1}) + 2\alpha_{\tau(n)} \langle y_{\tau(n)} - w, Ju - Jw \rangle \end{aligned} \tag{4.7}$$

for every $n \in \mathbb{N}$, where $y_{\tau(n)} = J^{-1}\big(\alpha_{\tau(n)} Ju + (1 - \alpha_{\tau(n)}) J S_{\tau(n)} x_{\tau(n)}\big)$ for $n \in \mathbb{N}$. Thus, by $\alpha_{\tau(n)} > 0$, (4.7) shows that

$$\phi(w, x_{\tau(n)+1}) \le 2\langle y_{\tau(n)} - w, Ju - Jw \rangle \tag{4.8}$$

for every $n \in \mathbb{N}$. Since $\{S_n\}$ satisfies the condition (Z), it follows from Lemma 3.6 that every weak cluster point of $\{x_{\tau(n)}\}$ belongs to F. Using (4.8), (1) in Lemma 4.2, and Lemma 3.7, we have

$$\begin{aligned} \limsup_{n \longrightarrow \infty} \phi(w, x_{\tau(n)+1}) &\le 2 \limsup_{n \longrightarrow \infty} \langle y_{\tau(n)} - w, Ju - Jw \rangle \\ &= 2 \limsup_{n \longrightarrow \infty} \langle S_{\tau(n)} x_{\tau(n)} - w, Ju - Jw \rangle \le 0. \end{aligned}$$

Therefore, by virtue of (4.6), we conclude that

$$\limsup_{n \longrightarrow \infty} \phi(w, x_n) \le \limsup_{n \longrightarrow \infty} \phi(w, x_{\tau(n)+1}) \le 0,$$

and hence $\phi(w, x_n) \longrightarrow 0$, which is a contradiction. □

Proof of Theorem 4.1. Using Lemmas 4.3 and 4.4, we get the conclusion. □

5. APPLICATIONS

In this section, we study the zero point problem for a maximal monotone operator and the fixed point problem for a relatively nonexpansive mapping. We employ Theorem 4.1 to approximate solutions of these problems.

To prove the first theorem, we need the following lemma:

Lemma 5.1. *([4, Lemma 3.5]) Let E be a strictly convex and reflexive Banach space whose norm is uniformly Gâteaux differentiable, $\{r_n\}$ a sequence of positive real numbers, and L_{r_n} the resolvent of a maximal monotone operator $A \subset E \times E^*$. Suppose that $\inf_n r_n > 0$ and $A^{-1}0$ is nonempty. Then $\{L_{r_n}\}$ satisfies the condition (Z).*

We adopt a modified proximal point algorithm introduced by Kohsaka and Takahashi [9] in the following theorem:

Theorem 5.2. *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, $A \subset E \times E^*$ a maximal monotone operator, $\{\alpha_n\}$ a sequence in $(0,1]$, and $\{r_n\}$ a sequence of positive real numbers. Suppose that $A^{-1}0$ is nonempty, $\alpha_n \longrightarrow 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\inf_n r_n > 0$. Let u be a point in E and $\{x_n\}$ a sequence defined by $x_1 \in E$ and*

$$x_{n+1} = J^{-1}\big(\alpha_n Ju + (1-\alpha_n)JL_{r_n}x_n\big) \tag{5.1}$$

for $n \in \mathbb{N}$, where $L_{r_n} = (J + r_n A)^{-1}J$. Then $\{x_n\}$ converges strongly to $Q_{A^{-1}0}(u)$.

Proof. Set $S_n = L_{r_n}$ for $n \in \mathbb{N}$. It is known that $F(S_n) = A^{-1}0$ and L_{r_n} is a type (r) self-mapping of E for each $n \in \mathbb{N}$. Hence $\bigcap_{n=1}^{\infty} F(S_n) = A^{-1}0$ is nonempty. It is also known that $\{S_n\}$ is a strongly relatively nonexpansive sequence by [3, Example 3.2] and $\{S_n\}$ satisfies the condition (Z) by Lemma 5.1. It is clear that Q_E is the identity mapping on E. Therefore, Theorem 4.1 implies the conclusion. □

Remark 5.3. Theorem 5.2 is similar to [9, Theorem 3.3]. In [9, Theorem 3.3], E is assumed to be smooth and uniformly convex and $\{\alpha_n\}$ in $[0,1]$ while $\{r_n\}$ is assumed to diverge to infinity.

To prove the next theorem, we need the following lemma:

Lemma 5.4. *([3, Lemma 2.1]) Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a uniformly convex Banach space E and $\{\lambda_n\}$ a sequence in $[0,1]$ such that $\liminf_{n\longrightarrow\infty} \lambda_n > 0$. Suppose that*

$$\lambda_n\|x_n\|^2 + (1-\lambda_n)\|y_n\|^2 - \|\lambda_n x_n + (1-\lambda_n)y_n\|^2 \longrightarrow 0.$$

Then $(1-\lambda_n)(x_n - y_n) \longrightarrow 0$.

The following is a strong convergence theorem for a relatively nonexpansive mapping; see [11, 12] for other convergence theorems and see also [3].

Theorem 5.5. *([13, Theorem 3.4]) Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty closed convex subset of E, $T\colon C \longrightarrow E$ a relatively nonexpansive mapping, $\{\alpha_n\}$ a sequence in $(0,1]$, and $\{\beta_n\}$ a sequence in $[0,1]$. Suppose that $\alpha_n \longrightarrow 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $0 < \liminf_{n\longrightarrow\infty} \beta_n \le \limsup_{n\longrightarrow\infty} \beta_n < 1$. Let u be a point in E and $\{x_n\}$ a sequence defined by $x_1 \in C$ and*

$$x_{n+1} = Q_C J^{-1}\Big(\alpha_n Ju + (1-\alpha_n)\big(\beta_n Jx_n + (1-\beta_n)JTx_n\big)\Big) \tag{5.2}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $Q_{F(T)}(u)$.

Proof. Set $S_n = J^{-1}\big(\beta_n J + (1-\beta_n)JT\big)$ for $n \in \mathbb{N}$. Then it is easy to check that each S_n is a mapping of type (r) and $\bigcap_{n=1}^{\infty} F(S_n) = F(T)$; see [5, Corollary 3.8]. Moreover, it is clear that (5.2) coincides with (4.1). To finish the proof, it is enough to show that $\{S_n\}$ is a strongly relatively nonexpansive sequence and $\{S_n\}$ satisfies the condition (Z).

Let $\{y_n\}$ be a bounded sequence in C such that $\phi(p, y_n) - \phi(p, S_n y_n) \longrightarrow 0$ for some $p \in \bigcap_{n=1}^{\infty} F(S_n)$. Since T is of type (r), we have

$$\begin{aligned}
&\beta_n\|Jy_n\|^2 + (1-\beta_n)\|JTy_n\|^2 - \|JS_n y_n\|^2 \\
&\quad = \beta_n\phi(p, y_n) + (1-\beta_n)\phi(p, Ty_n) - \phi(p, S_n y_n) \\
&\quad \le \beta_n\phi(p, y_n) + (1-\beta_n)\phi(p, y_n) - \phi(p, S_n y_n) \\
&\quad = \phi(p, y_n) - \phi(p, S_n y_n) \longrightarrow 0.
\end{aligned}$$

Using Lemma 5.4 and (2.5), it turns out that

$$Jy_n - JS_ny_n = (1-\beta_n)(Jy_n - JTy_n) \longrightarrow 0,$$

and hence $\phi(S_ny_n, y_n) \longrightarrow 0$. Thus $\{S_n\}$ is a strongly relatively nonexpansive sequence.

Let $\{z_n\}$ be a bounded sequence in C such that $z_n - S_nz_n \longrightarrow 0$. Then it follows from (2.5) that

$$(1-\beta_n)(Jz_n - JTz_n) = Jz_n - JS_nz_n \longrightarrow 0,$$

so we conclude that $z_n - Tz_n \longrightarrow 0$ by $\limsup_{n\longrightarrow\infty} \beta_n < 1$ and (2.5). Since T is relatively nonexpansive, every weak cluster point of $\{z_n\}$ belongs to $F(T)$. This means that $\{S_n\}$ satisfies the condition (Z). Consequently, Theorem 4.1 implies the conclusion. □

Remark 5.6. In [13, Theorem 3.4], $\{\alpha_n\}$ and $\{\beta_n\}$ are assumed to be sequences in $(0,1)$.

References

1. Y. I. Alber, Metric and generalized projection operators in Banach spaces: properties and applications, Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math. 178, Dekker, New York, 1996, 15–50.
2. K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, Nonlinear Anal. 67 (2007), 2350–2360.
3. K. Aoyama, F. Kohsaka and W. Takahashi, Strongly relatively nonexpansive sequences in Banach spaces and applications, J. Fixed Point Theory Appl. 5 (2009), 201–224.
4. K. Aoyama, F. Kohsaka and W. Takahashi, Proximal point methods for monotone operators in Banach spaces, Taiwanese J. Math. 15 (2011), 259–281.
5. K. Aoyama, F. Kohsaka and W. Takahashi, Strong convergence theorems by shrinking and hybrid projection methods for relatively nonexpansive mappings in Banach spaces, Nonlinear analysis and convex analysis, Yokohama Publ., Yokohama, 2009, 7–26.
6. Y. Censor and S. Reich, Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization, Optimization 37 (1996), 323–339.
7. S. Kamimura, F. Kohsaka and W. Takahashi, Weak and strong convergence theorems for maximal monotone operators in a Banach space, Set-Valued Anal. 12 (2004), 417–429.
8. S. Kamimura and W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, SIAM J. Optim. 13 (2002), 938–945.
9. F. Kohsaka and W. Takahashi, Strong convergence of an iterative sequence for maximal monotone operators in a Banach space, Abstr. Appl. Anal. 2004, 239–249.
10. P.-E. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Anal. 16 (2008), 899–912.
11. S. Matsushita and W. Takahashi, Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces, Fixed Point Theory Appl. 2004, 37–47.

12. S. Matsushita and W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space, J. Approx. Theory 134 (2005), 257–266.
13. W. Nilsrakoo and S. Saejung, Strong convergence theorems by Halpern-Mann iterations for relatively nonexpansive mappings in Banach spaces, Appl. Math. Comput. 217 (2011), 6577–6586.
14. S. Reich, A weak convergence theorem for the alternating method with Bregman distances, Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math. 178, Dekker, New York, 1996, 313–318.
15. R. T. Rockafellar, On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 149 (1970), 75–88.
16. S. Saejung, Halpern's iteration in Banach spaces, Nonlinear Anal. 73 (2010), 3431–3439.
17. W. Takahashi, Nonlinear functional analysis, Yokohama Publishers, Yokohama, 2000.
18. H.-K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. (2) 66 (2002), 240–256.