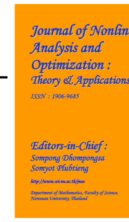

Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 3, No. 1, (2012), 55-66
ISSN : 1906-9685
http://www.math.sci.nu.ac.th

COMMON FIXED POINT THEOREMS OF INTEGRAL TYPE IN MENGER PM SPACES

I. ALTUN[1], M. TANVEER[2,*] AND M. IMDAD[3]

Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey

[2] School of Computer Systems Sciences, Jawaharlal Nehru University New Delhi 110 067, India

[3] Department of Mathematics, Aligarh Muslim University Aligarh 202 002, India

ABSTRACT. In this paper, we propose integral type common fixed point theorems in Menger spaces satisfying common property $(E.A)$. Our results generalize several previously known results in Menger as well as metric spaces. Some related results are also derived besides furnishing an illustrative example.

KEYWORDS : Menger space; Common property $(E.A)$; Weakly compatible pair of mappings and t-norm.

1. INTRODUCTION AND PRELIMINARIES

Menger [24] initiated the study of probabilistic metric space (often abbreviated as PM space) in 1942 and by now the theory of probabilistic metric spaces has already made a considerable progress in several directions (see[29]). The idea of Menger was to use distribution functions (instead of non-negative real numbers) as values of a probabilistic metric. This new notion (i.e.PM space) can cover even those situations wherein one can not exactly ascertain a distance between two points, but can only know the possibility of a possible value for the distance (between a pair of points). This probabilistic generalization of metric space is well utilized in the investigations of physiological thresholds besides physical quantities particularly in connections with both string and E-infinity theory (cf.[10]).

In 1986, Jungck [18] introduced the notion of compatible mappings and utilized the same to improve commutativity conditions in common fixed point theorems. This concept has been frequently employed to prove existence theorems on common fixed points. However, the study of common fixed points of non-compatible mappings is also equally interesting which was initiated by Pant [29]. Recently, Aamri and Moutawakil [1] and Liu et al. [23] respectively defined the property

* *Corresponding author.*
Email address : ialtun@kku.tr(I. Altun), tanveer_gouri@yahoo.co.in(M. Tanveer) and mhimdad@yahoo.co.in(M. Imdad).
Article history : Received 15 November 2010. Accepted 27 December 2011.

$(E.A)$ and the common property $(E.A)$ and proved interesting common fixed point theorems in metric spaces. Most recently, Kubiaczyk and Sharma [21] adopted the property $(E.A)$ in PM spaces and used it to prove results on common fixed points wherein authors claim their results for strict contractions which are merely for contractions. Recently, Imdad et al. [16] adopted the common property $(E.A)$ in PM spaces and proved some coincidence and common fixed point results in Menger spaces.

The theory of fixed points in PM spaces is a part of probabilistic analysis and continues to be an active area of mathematical research. Thus far, several authors studied fixed point and common fixed point theorems in PM spaces which include [2, 3, 7, 12, 13, 15, 16, 20, 27, 28, 30, 31, 33, 34, 35] besides many more. In 2002, Branciari [5] obtained a fixed point result for a mapping satisfying an integral analogue of Banach contraction principle. The authors of the papers [4, 9, 16, 32, 37, 38] proved a host of fixed point theorems involving relatively more general integral type contractive conditions. In an interesting note, Suzuki [36] showed that Meir-Keeler contractions of integral type are still Meir-Keeler contractions.

The aim of this paper is to prove integral type fixed point theorems in Menger PM spaces satisfying common property $(E.A)$. Our results substantially improve the corresponding theorems contained in [5, 8, 16, 32, 38] along with some other relevant results in Menger as well as metric spaces. Some related results are also derived besides furnishing an illustrative example.

Definition 1.1. [33] A mapping $F : \Re \to \Re^+$ is called distribution function if it is non-decreasing, left continuous with $\inf\{F(t) : t \in \Re\} = 0$ and $\sup\{F(t) : t \in \Re\} = 1$.

Let L be the set of all distribution functions whereas H be the set of specific distribution functions (also known as Heaviside function) defined by

$$H(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0. \end{cases}$$

Definition 1.2. [24] Let X be a non-empty set. An ordered pair $(X, \mathcal{F})$ is called a PM space if $\mathcal{F}$ is a mapping from $X \times X$ into L satisfying the following conditions:

(i) $F_{p,q}(x) = H(x)$ if and only if $p = q$,
(ii) $F_{p,q}(x) = F_{q,p}(x)$,
(iii) $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$, then $F_{p,r}(x + y) = 1$, for all $p, q, r \in X$ and $x, y \geq 0$.

Every metric space (X, d) can always be realized as a PM space by considering $\mathcal{F} : X \times X \to L$ defined by $F_{p,q}(x) = H(x - d(p, q))$ for all $p, q \in X$. So PM spaces offer a wider framework (than that of the metric spaces) and are general enough to cover even wider statistical situations.

Definition 1.3. [33] A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if

(i) $\Delta(a, 1) = a, \Delta(0, 0) = 0$,
(ii) $\Delta(a, b) = \Delta(b, a)$,
(iii) $\Delta(c, d) \geq \Delta(a, b)$ for $c \geq a, d \geq b$,
(iv) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$ for all $a, b, c \in [0, 1]$.

Example 1.4. The following are the four basic t-norms:

(i) The minimum t-norm: $T_M(a, b) = \min\{a, b\}$.

(ii) The product t-norm: $T_P(a,b) = a.b$.

(iii) The Lukasiewicz t-norm: $T_L(a,b) = \max\{a+b-1, 0\}$.

(iv) The weakest t-norm, the drastic product:

$$T_D(a,b) = \begin{cases} \min\{a,b\}, \text{ if } \max\{a,b\} = 1 \\ \\ 0, \text{ otherwise.} \end{cases}$$

In respect of above mentioned t-norms, we have the following ordering:

$T_D < T_L < T_P < T_M$.

Definition 1.5. [24] A Menger PM space $(X, \mathcal{F}, \Delta)$ is a triplet where $(X, \mathcal{F})$ is a PM space and Δ is a t-norm satisfying the following condition:

$$F_{p,r}(x+y) \geq \Delta(F_{p,q}(x), F_{q,r}(y)).$$

Definition 1.6. [12] A sequence $\{p_n\}$ in a Menger PM space $(X, \mathcal{F}, \Delta)$ is said to be convergent to a point p in X if for every $\epsilon > 0$ and $\lambda > 0$, there is an integer $M(\epsilon, \lambda)$ such that $F_{p_n,p}(\epsilon) > 1 - \lambda$, for all $n \geq M(\epsilon, \lambda)$.

Lemma 1.7. [33, 26] *Let $(X, \mathcal{F}, \Delta)$ be a Menger space with a continuous t-norm Δ with $\{x_n\}, \{y_n\} \subset X$ such that $\{x_n\}$ converges to x and $\{y_n\}$ converges to y. If $F_{x,y}(.)$ is continuous at the point t_0, then $\lim_{n \longrightarrow \infty} F_{x_n,y_n}(t_0) = F_{x,y}(t_0)$.*

Definition 1.8. Let (A, S) be a pair of maps from a Menger PM space $(X, \mathcal{F}, \Delta)$ into itself. Then the pair of maps (A, S) is said to be weakly commuting if

$$F_{ASx,SAx}(t) \geq F_{Ax,Sx}(t),$$

for each $x \in X$ and $t > 0$.

Definition 1.9. [28] A pair (A, S) of self mappings of a Menger PM space $(X, \mathcal{F}, \Delta)$ is said to be compatible if $F_{ASp_n,SAp_n}(x) \longrightarrow 1$ for all $x > 0$, whenever $\{p_n\}$ is a sequence in X such that $Ap_n, Sp_n \longrightarrow t$, for some t in X as $n \longrightarrow \infty$.

Clearly, a weakly commuting pair is compatible but every compatible pair need not be weakly commuting.

Definition 1.10. [11] A pair (A, S) of self mappings of a Menger PM space $(X, \mathcal{F}, \Delta)$ is said to be non-compatible if and only if there exists at least one sequence $\{x_n\}$ in X such that

$$\lim_{n \longrightarrow \infty} Ax_n = \lim_{n \longrightarrow \infty} Sx_n = t \in X \text{ for some } t \in X,$$

implies that $\lim_{n \longrightarrow \infty} F_{ASx_n,SAx_n}(t_0)$ (for some $t_0 > 0$) is either less than 1 or non-existent.

Definition 1.11. [21] A pair (A, S) of self mappings of a Menger PM space $(X, \mathcal{F}, \Delta)$ is said to satisfy the property $(E.A)$ if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \longrightarrow \infty} Ax_n = \lim_{n \longrightarrow \infty} Sx_n = t \in X.$$

Clearly, a pair of compatible mappings as well as non-compatible mappings satisfies the property $(E.A)$.

Inspired by Liu et al. [23], Imdad et al. [16] defined the following:

Definition 1.12. Two pairs (A, S) and (B, T) of self mappings of a Menger PM space $(X, \mathcal{F}, \Delta)$ are said to satisfy the common property $(E.A)$ if there exist two sequences $\{x_n\}, \{y_n\}$ in X and some t in X such that

$$\lim_{n\longrightarrow\infty} Ax_n = \lim_{n\longrightarrow\infty} Sx_n = \lim_{n\longrightarrow\infty} Ty_n = \lim_{n\longrightarrow\infty} By_n = t.$$

Definition 1.13. [19] A pair (A, S) of self mappings of a nonempty set X is said to be weakly compatible if the pair commutes on the set of their coincidence points i.e. $Ap = Sp$ (for some $p \in X$) implies $ASp = SAp$.

Definition 1.14. [15] Two finite families of self mappings $\{A_i\}$ and $\{B_j\}$ are said to be pairwise commuting if:

(i) $A_iA_j = A_jA_i,\ i, j \in \{1, 2...m\}$,
(ii) $B_iB_j = B_jB_i,\ i, j \in \{1, 2...n\}$,
(iii) $A_iB_j = B_jA_i,\ i \in \{1, 2...m\}, j \in \{1, 2...n\}$.

2. MAIN RESULTS

The following lemma is crucial in the proof of succeeding theorems.

Lemma 2.1. *Let $(X, \mathcal{F}, \Delta)$ be a Menger space. If there exists some $k \in (0, 1)$ such that for all $p, q \in X$ and all $x > 0$,*

$$\int_0^{F_{p,q}(kx)} \phi(t)dt \geq \int_0^{F_{p,q}(x)} \phi(t)dt, \tag{1.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a summable nonnegative Lebesgue integrable function such that $\int_\epsilon^1 \phi(t)dt > 0$ for each $\epsilon \in [0, 1)$, then $p = q$.

Proof. As

$$\int_0^{F_{p,q}(kx)} \phi(t)dt \geq \int_0^{F_{p,q}(x)} \phi(t)dt$$

implies

$$\int_0^{F_{p,q}(x)} \phi(t)dt \geq \int_0^{F_{p,q}(k^{-1}x)} \phi(t)dt,$$

one can inductively write (for $m \in N$)

$$\begin{aligned}\int_0^{F_{p,q}(x)} \phi(t)dt \geq \int_0^{F_{p,q}(k^{-1}x)} \phi(t)dt \geq ... &\geq \int_0^{F_{p,q}(k^{-m}x)} \phi(t)dt \\ &\geq ... \to \int_0^1 \phi(t)dt \text{ as } m \to \infty.\end{aligned}$$

Therefore

$$\int_0^{F_{p,q}(x)} \phi(t)dt - \int_0^1 \phi(t)dt \geq 0$$

and henceforth

$$\int_0^{F_{p,q}(x)} \phi(t)dt - \left(\int_0^{F_{p,q}(x)} \phi(t)dt + \int_{F_{p,q}(x)}^1 \phi(t)dt\right) \geq 0$$

or

$$\int_{F_{p,q}(x)}^1 \phi(t)dt \leq 0$$

which amounts to say that $F_{p,q}(x) \geq 1$ for all $x \geq 0$. Thus, we get $p = q$. □

Remark 2.2. By setting $\phi(t) = 1$ (for each $t \geq 0$) in (1.1) of Lemma 2.1, we have

$$\int_0^{F_{p,q}(kx)} \phi(t)dt = F_{p,q}(kx) \geq F_{p,q}(x) = \int_0^{F_{p,q}(x)} \phi(t)dt,$$

which shows that Lemma 2.1 is a generalization of the Lemma 2 (contained in [35])

In what follows, Δ is a continuous t-norm (in the product topology).

Lemma 2.3. *Let A, B, S and T be four self mappings of a Menger space $(X, \mathcal{F}, \Delta)$ which satisfy the following conditions: (i) the pair (A, S) (or (B, T)) satisfies the property $(E.A)$,*

(ii) $B(y_n)$ converges for every sequence $\{y_n\}$ in X whenever $T(y_n)$ converges,

(iii) for any $p, q \in X$ and for all $x > 0$,

$$\int_0^{F_{Ap,Bq}(kx)} \phi(t)dt \geq \int_0^{m(x,y)} \phi(t)dt \tag{2.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a non-negative summable Lebesgue integrable function such that $\int_\epsilon^1 \phi(t)dt > 0$ for each $\epsilon \in [0, 1)$, where $0 < k < 1$ and

$$m(x, y) = \min\{F_{Sp,Tq}(x), F_{Sp,Ap}(x), F_{Tq,Bq}(x), F_{Sp,Bq}(x), F_{Tq,Ap}(x)\},$$

(iv) $A(X) \subset T(X)$ (or $B(X) \subset S(X)$).

Then the pairs (A, S) and (B, T) share the common property $(E.A)$.

Proof. Suppose that the pair (A, S) enjoys the property $(E.A)$, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \longrightarrow \infty} Ax_n = \lim_{n \longrightarrow \infty} Sx_n = t, \text{ for some } t \in X.$$

Since $A(X) \subset T(X)$, for each x_n there exists $y_n \in X$ such that $Ax_n = Ty_n$, and henceforth

$$\lim_{n \to \infty} Ty_n = \lim_{n \to \infty} Ax_n = t.$$

Thus in all, we have $Ax_n \to t, Sx_n \to t$ and $Ty_n \to t$. Now we assert that $By_n \to t$. To accomplish this, using (2.1), with $p = x_n, q = y_n$, one gets

$$\int_0^{F_{Ax_n,By_n}(kx)} \phi(t)dt \geq \int_0^{m(x,y)} \phi(t)dt$$
$$\geq \int_0^{\min(F_{Sx_n,Ty_n}(x), F_{Sx_n,Ax_n}(x), F_{Ty_n,By_n}(x), F_{Sx_n,By_n}(x), F_{Ty_n,Ax_n}(x))} \phi(t)dt.$$

Let $l = \lim_{n \longrightarrow \infty} B(y_n)$. Also, let $x > 0$ be such that $F_{t,l}(\cdot)$ is continuous in x and kx. Then, on making $n \to \infty$ in the above inequality, we obtain

$$\int_0^{F_{t,l}(kx)} \phi(t)dt \geq \int_0^{\min(F_{t,t}(x), F_{t,t}(x), F_{t,l}(x), F_{t,l}(x), F_{t,t}(x))} \phi(t)dt$$

or

$$\int_0^{F_{t,l}(kx)} \phi(t)dt \geq \int_0^{F_{t,l}(x)} \phi(t)dt.$$

This implies that $l = t$ (in view of Lemma 2.1) which shows that the pairs (A, S) and (B, T) share the common property $(E.A)$. □

Remark 2.4. The converse of Lemma 2.3 is not true in general. For a counter example, one can see Example 3.4 furnished in the end of this paper.

Theorem 2.5. *Let A, B, S and T be self mappings of a Menger space $(X, \mathcal{F}, \Delta)$ which satisfy the inequality (2.1) together with*

(i) the pairs (A, S) and (B, T) share the common property $(E.A)$,

(ii) $S(X)$ and $T(X)$ are closed subsets of X.

Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. Since the pairs (A, S) and (B, T) share the common property $(E.A)$, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t, \text{ for some } t \in X.$$

Since $S(X)$ is a closed subset of X, hence $\lim_{n\to\infty} Sx_n = t \in S(X)$. Therefore, there exists a point $u \in X$ such that $Su = t$. Now, we assert that $Au = Su$. To prove this, on using (2.1) with $p = u, q = y_n$, one gets

$$\int_0^{F_{Au,By_n}(kx)} \phi(t)dt \geq \int_0^{\min(F_{Su,Ty_n}(x),F_{Su,Au}(x),F_{Ty_n,By_n}(x),F_{Su,By_n}(x),F_{Ty_n,Au}(x))} \phi(t)dt$$

which on making $n \to \infty$, reduces to

$$\int_0^{F_{Au,t}(kx)} \phi(t)dt \geq \int_0^{\min(F_{t,t}(x),F_{t,Au}(x),F_{t,t}(x),F_{t,t}(x),F_{t,Au}(x))} \phi(t)dt$$

or

$$\int_0^{F_{Au,t}(kx)} \phi(t)dt \geq \int_0^{F_{Au,t}(x)} \phi(t)dt.$$

Now appealing to Lemma 2.1, we have $Au = t$ and henceforth $Au = Su$. Therefore, u is a coincidence point of the pair (A, S).

Since $T(X)$ is a closed subset of X, therefore $\lim_{n\to\infty} Ty_n = t \in T(X)$ and hence one can find a point $w \in X$ such that $Tw = t$. Now we show that $Bw = Tw$. To accomplish this, on using (2.1) with $p = x_n, q = w$, we have

$$\int_0^{F_{Ax_n,Bw}(kx)} \phi(t)dt \geq \int_0^{\min(F_{Sx_n,Tw}(x),F_{Sx_n,Ax_n}(x),F_{Tw,Bw}(x),F_{Sx_n,Bw}(x),F_{Tw,Ax_n}(x))} \phi(t)dt$$

which on making $n \to \infty$, reduces to

$$\int_0^{F_{t,Bw}(kx)} \phi(t)dt \geq \int_0^{\min(F_{t,t}(x),F_{t,t}(x),F_{t,Bw}(x),F_{t,Bw}(x),F_{t,t}(x))} \phi(t)dt$$

or

$$\int_0^{F_{t,Bw}(kx)} \phi(t)dt \geq \int_0^{F_{t,Bw}(x)} \phi(t)dt.$$

On employing Lemma 2.1, we have $Bw = t$ and henceforth $Tw = Bw$. Therefore, w is a coincidence point of the pair (B, T).

Since the pair (A, S) is weakly compatible and $Au = Su$, therefore

$$At = ASu = SAu = St.$$

Again, on using (2.1) with $p = t, q = w$, we have

$$\int_0^{F_{At,Bw}(kx)} \phi(t)dt \geq \int_0^{\min(F_{St,Tw}(x),F_{St,At}(x),F_{Tw,Bw}(x),F_{St,Bw}(x),F_{Tw,At}(x))} \phi(t)dt$$

or

$$\int_0^{F_{At,t}(kx)} \phi(t)dt \geq \int_0^{\min(F_{At,t}(x),F_{t,t}(x),F_{t,t}(x),F_{At,t}(x),F_{t,At}(x))} \phi(t)dt$$

or

$$\int_0^{F_{At,t}(kx)} \phi(t)dt \geq \int_0^{F_{At,t}(x)} \phi(t)dt.$$

Appealing to Lemma 2.1, we have $At = St = t$ which shows that t is a common fixed point of the pair (A, S).

Also the pair (B, T) is weakly compatible and $Bw = Tw$, hence

$$Bt = BTw = TBw = Tt.$$

Next, we show that t is a common fixed point of the pair (B, T). In order to accomplish this, using (2.1) with $p = u, q = t$, we have

$$\int_0^{F_{Au,Bt}(kx)} \phi(t)dt \geq \int_0^{\min(F_{Su,Tt}(x),F_{Su,Au}(x),F_{Tt,Bt}(x),F_{Su,Bt}(x),F_{Tt,Au}(x))} \phi(t)dt$$

or

$$\int_0^{F_{t,Bt}(kx)} \phi(t)dt \geq \int_0^{\min(F_{t,Bt}(x),F_{t,t}(x),F_{Bt,Bt}(x),F_{t,Bt}(x),F_{Bt,t}(x))} \phi(t)dt$$

or

$$\int_0^{F_{t,Bt}(kx)} \phi(t)dt \geq \int_0^{F_{t,Bt}(x)} \phi(t)dt.$$

Using Lemma 2.1, we have $Bt = t$ which shows that t is a common fixed point of the pair (B, T). Hence t is a common fixed point of both the pairs (A, S) and (B, T). Uniqueness of common fixed point is an easy consequence of the inequality (2.1). This completes the proof. □

Remark 2.6. Theorem 2.5 extends the main result of Ciric [8] to Menger spaces. Theorem 2.5 also generalizes the main result of Kubiaczyk and Sharma [21] for two pairs of mappings without conditions on containments amongst range sets of the involved mappings.

Theorem 2.7. *The conclusions of Theorem 2.5 remain true if the condition (ii) of Theorem 2.5 is replaced by the following: (iii)′ $\overline{A(X)} \subset T(X)$ and $\overline{B(X)} \subset S(X)$.*

Corollary 2.8. *The conclusions of Theorems 2.5 and 2.7 remain true if the condition (ii) (of Theorem 2.5) and (iii)′ (of Theorem 2.7)are replaced by the following: (iv) $A(X)$ and $B(X)$ are closed subsets of X provided $A(X) \subset T(X)$ and $B(X) \subset S(X)$.*

Theorem 2.9. *Let A, B, S and T be self mappings of a Menger space $(X, \mathcal{F}, \Delta)$ satisfying the inequality (2.1). Suppose that*

(i) the pair (A, S) (or (B, T)) has property $(E.A)$,

(ii) $B(y_n)$ converges for every sequence $\{y_n\}$ in X whenever $T(y_n)$ converges,

(iii) $A(X) \subset T(X)$ (or $B(X) \subset S(X)$),

(iii) $S(X)$ (or $T(X)$) is a closed subset of X.

Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. In view of Lemma 2.3, the pairs (A, S) and (B, T) share the common property $(E.A)$, i.e. there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t, \text{ for some } t \in X.$$

If $S(X)$ is a closed subset of X, then proceeding on the lines of Theorem 2.5, one can show that the pair (A, S) has a coincidence point, say u, i.e. $Au = Su = t$. Since $A(X) \subset T(X)$ and $Au \in A(X)$, there exists $w \in X$ such that $Au = Tw$. Now, we assert that $Bw = Tw$.
On using (2.1) with $p = x_n, q = w$, one gets

$$\int_0^{F_{Ax_n,Bw}(kx)} \phi(t)dt \geq \int_0^{\min(F_{Sx_n,Tw}(x), F_{Sx_n,Ax_n}(x), F_{Tw,Bw}(x), F_{Sx_n,Bw}(x), F_{Tw,Ax_n}(x))} \phi(t)dt$$

which on making $n \to \infty$, reduces to

$$\int_0^{F_{t,Bw}(kx)} \phi(t)dt \geq \int_0^{\min(F_{t,t}(x), F_{t,t}(x), F_{t,Bw}(x), F_{t,Bw}(x), F_{t,t}(x))} \phi(t)dt$$

or

$$\int_0^{F_{t,Bw}(kx)} \phi(t)dt \geq \int_0^{F_{t,Bw}(x)} \phi(t)dt.$$

Owing to Lemma 2.1, we have $t = Bw$ and hence $Tw = Bw$ which shows that w is a coincidence point of the pair (B, T). Rest of the proof can be completed on the lines of the proof of Theorem 2.5. This completes the proof.

By choosing A, B, S and T suitably, one can deduce corollaries involving two or three mappings. As a sample, we deduce the following natural result for a pair of self mappings. □

Corollary 2.10. *Let A and S be self mappings on a Menger space $(X, \mathcal{F}, \Delta)$. Suppose that*

(i) the pair (A, S) enjoys the property $(E.A)$,

(ii) for all $p, q \in X$ and for all $x > 0$,

$$\int_0^{F_{Ap,Aq}(kx)} \phi(t)dt \geq \int_0^{m(x,y)} \phi(t)dt \tag{2.2}$$

where $m(x, y) = \min\{F_{Sp,Sq}(x), F_{Sp,Ap}(x), F_{Sq,Aq}(x), F_{Sp,Aq}(x), F_{Sq,Ap}(x)\}, 0 < k < 1$

(iii) $S(X)$ is a closed subset of X.

Then A and S have a coincidence point. Moreover, if the pair (A, S) is weakly compatible, then A and S have a unique common fixed point.

As an application of Theorem 2.5, we have the following result for four finite families of self mappings. While proving our result, we utilize Definition 1.14 which is a natural extension of commutativity condition to two finite families of mappings.

Theorem 2.11. *Let $\{A_1, A_2, ...A_m\}, \{B_1, B_2, ...B_p\}, \{S_1, S_2, ...S_n\}$ and $\{T_1, T_2, ...T_q\}$ be four finite families of self mappings of a Menger space $(X, \mathcal{F}, \Delta)$ with $A = A_1A_2...A_m, B = B_1B_2...B_p, S = S_1S_2...S_n$ and $T = T_1T_2...T_q$ satisfying the condition (2.1). If $S(X)$ and $T(X)$ are closed subsets of X, and the pairs (A, S) and (B, T) share the common property $(E.A)$, then*

(i) *the pair (A, S) as well as (B, T) has a coincidence point,*
(ii) *A_i, B_k, S_r and T_t have a unique common fixed point provided the pair of families $(\{A_i\}, \{S_r\})$ and $(\{B_k\}, \{T_t\})$ commute pairwise.*

Proof. The proof follows on the lines of Theorem 4.1 due to Imdad and Ali [14] and Theorem 3.1 due to Imdad et al. [15]. □

Remark 2.12. By restricting four families as $\{A_1, A_2\}, \{B_1, B_2\}, \{S_1\}$ and $\{T_1\}$ in Theorem 2.11 we get improved version of results due to Chugh and Rathi [7], Kutukcu and Sharma [22], Rashwan and Hedar [30], Singh and Jain [35] and others. Theorem 2.11 also generalizes the main result of Razani and Shirdaryazdi [31] for any finite number of mappings.

By setting $A_1 = A_2 = ... = A_m = G, B_1 = B_2 = ... = B_p = H, S_1 = S_2 = ...S_n = I$ and $T_1 = T_2 = ... = T_q = J$ in Theorem 2.11, we deduce the following:

Corollary 2.13. *Let G, H, I and J be self mappings of a Menger space $(X, \mathcal{F}, \Delta)$ such that the pairs (G^m, I^n) and (H^p, J^q) share the common property $(E.A)$ and also satisfies the condition*

$$\int_0^{F_{G^m x, H^p y}(kz)} \phi(t)dt \geq \int_0^{m(x,y)} \phi(t)dt$$

(where $m(x, y) = \min\{F_{I^n x, J^q y}(z), F_{I^n x, G^m x}(z), F_{I^n x, H^p y}(z), F_{J^q y, H^p y}(z), F_{J^q y, G^m x}(z)\}$) for all $x, y \in X$, $\forall\ z > 0$ where $k \in (0, 1)$ and m, n, p and q are fixed positive integers. If $I^n(X)$ and $J^q(X)$ are closed subsets of X, then G, H, I and J have a unique common fixed point provided $GI = IG$ and $HJ = JH$.

Remark 2.14. Corollary 2.13 is a slight but partial generalization of Theorem 2.5 as the commutativity requirements (i.e. $GI = IG$ and $HJ = JH$) in this corollary are relatively stronger as compared to weak compatibility (in Theorem 2.5). Corollary 2.13 also presents a generalized and improved form of a result due to Bryant [6] in Menger PM spaces.

3. RELATED RESULTS AND AN EXAMPLE

In this section, we utilize Theorem 2.5 and Theorem 2.9 [16, 14] as means to derive corresponding common fixed point theorems in metric spaces.

Theorem 3.1. *Let A, B, S and T be self mappings of a metric space (X, d). Suppose that*

(i) the pairs (A, S) and (B, T) share the common property $(E.A)$,

(ii) $S(X)$ and $T(X)$ are closed subsets of X,

(iii) for all $x, y \in X$

$$\int_0^{d(Ax,By)} \phi(t)dt \leq k \int_0^{m(x,y)} \phi(t)dt \tag{3.1}$$

where $m(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ax, Ty)\}$, and $0 < k < 1$.

Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. Define $F_{x,y}(t) = H(t - d(x, y))$ and $\Delta(a, b) = \min\{a, b\}$. Then $(X, \mathcal{F}, \Delta)$ is a Menger space induced by the metric space (X, d). It is straight forward to notice that the conditions (i) and (ii) of Theorem 3.1 respectively imply conditions (i) and (ii) of Theorem 2.5. Also inequality (3.1) of Theorem 3.1 implies inequality (2.1) of Theorem 2.5. To accomplish this notice that (for any $x, y \in X$ and $t > 0$), $F_{Ax,By}(kt) = 1$ provided $kt > d(Ax, By)$ which amounts to say that (2.1) holds. Otherwise, if $kt \leq d(Ax, By)$, then

$$t \leq \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ax, Ty)\},$$

and hence in all the cases, condition (2.1) holds. Thus, all the conditions of Theorem 2.5 are satisfied and conclusions follow immediately in view of Theorem 2.5. □

Remark 3.2. Theorem 3.1 improves the main result of Ciric [8] and several other similar common fixed point theorems especially those contained in [14, 15, 22, 28]) as we never require any condition on the containment of ranges amongst involved mappings besides weakening the completeness of the space to closedness of suitable subsets along with improvement in commutativity considerations. Here, one may also notice that all the involved mappings can be discontinuous (at the same time).

Remark 3.3. Similarly, we can also apply our other results (i.e. Theorems 2.7-2.11 and Corollaries 2.8-2.13) to derive the corresponding common fixed point theorems in metric spaces but here details are avoided.

We conclude this paper by furnishing an illustrative example to demonstrate the validity of the hypotheses of Theorem 2.5.

Example 3.4. Consider $X = [-1, 1]$ and define $F_{x,y}(t) = H(t - |x - y|)$ for all $x, y \in X$. Then $(X, \mathcal{F}, \Delta)$ is a Menger PM space with $\Delta(a, b) = \min\{a, b\}$. Define self mappings A, B, S and T on X as

$$A(x) = \begin{cases} \frac{3}{5}, & \text{if } x \in \{-1, 1\} \\ \frac{x}{4}, & \text{if } x \in (-1, 1), \end{cases} \qquad B(x) = \begin{cases} \frac{3}{5}, & \text{if } x \in \{-1, 1\} \\ \frac{-x}{4}, & \text{if } x \in (-1, 1), \end{cases}$$

$$S(x) = \begin{cases} \frac{1}{2}, & \text{if } x = -1 \\ \frac{x}{2}, & \text{if } x \in (-1, 1) \\ \frac{-1}{2}, & \text{if } x = 1 \end{cases} \quad \text{and } T(x) = \begin{cases} \frac{-1}{2}, & \text{if } x = -1 \\ \frac{-x}{2}, & \text{if } x \in (-1, 1) \\ \frac{1}{2}, & \text{if } x = 1. \end{cases}$$

Then with sequences as $\{x_n = \frac{1}{n}\}$ and $\{y_n = \frac{-1}{n}\}$ in X, we have

$$\lim_{n \longrightarrow \infty} Sx_n = \lim_{n \longrightarrow \infty} Ax_n = \lim_{n \longrightarrow \infty} By_n = \lim_{n \longrightarrow \infty} Ty_n = 0$$

which shows that pairs (A, S) and (B, T) share the common property (E.A). By a routine calculation, one can verify the contraction condition (2.1) with $k = \frac{1}{2}$. Also,

$$A(X) = B(X) = \left\{\frac{3}{5}\right\} \cup \left(-\frac{1}{4}, \frac{1}{4}\right) \not\subset \left[-\frac{1}{2}, \frac{1}{2}\right] = S(X) = T(X).$$

Thus, all the conditions of Theorem 2.5 are satisfied and 0 is a unique common fixed point of the pairs (A, S) and (B, T) which is their coincidence point as well.

Here it is worth noting that majority of earlier established theorems (with rare possible exceptions) cannot be used in the context of this example as Theorem 2.5 never requires any condition on the containment of ranges of the involved mappings. Also the completeness condition is replaced by the closedness of the subspaces. Moreover, the continuity requirements of all the involved mappings are completely relaxed whereas most of the earlier theorems require the continuity of at least one involved mapping.

References

1. A. Aamri and D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270(2002), 181-188.
2. J. Ali, M. Imdad and D. Bahuguna, Common fixed point theorems in Menger spaces with common property (E.A), Comp. Math. Appl., 60(2010), 3152-3159.
3. J. Ali, M. Imdad, D. Miheţ, and M. Tanveer, Common fixed points of strict contractions in Menger spaces, Acta Math. Hung., 132(4)(2011), 367-386.
4. A. Aliouche, A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type, J. Math. Anal. Appl., 322(2)(2006), 796-802.
5. A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci., 29(9)(2002), 531-536.
6. V.W. Bryant, A remark on fixed point theorem for iterated mappings, Amer. Math. Monthly, 75 (1968), 399-400.
7. R. Chugh and S. Rathi, Weakly compatible maps in probabilistic metric spaces, J. Indian Math. Soc., 72 (2005), 131-140.
8. Lj.B. Cirić, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc., 45 (1974), 267-273.
9. A. Djoudi and A. Aliouche, Common fixed point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type, J. Math. Anal. Appl., 329 (1) (2007), 31-45.
10. MS El Naschie, A review of applications and results of E-infinity theory, Int. J. Nonlinear Sci. Numer. Simul. 8(2007), 11-20.
11. J.X. Fang and Y. Gao, Common fixed point theorems under strict contractive conditions in Menger spaces, Nonlinear Analysis, 70 (2009). 184-193.
12. O. Hadzić and E. Pap, Fixed point theory in probabilistic metric spaces, Kluwer Academic Publishers, Dordrecht, 2001.
13. T.L. Hicks, Fixed point theory in probabilistic metric spaces, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 13 (1983), 63-72.
14. M. Imdad and J. Ali, Jungck's common fixed point theorem and E.A property, Acta Math. Sinica, 24 (2008), 87-94.
15. M. Imdad, Javid Ali and M. Tanveer, Coincidence and common fixed point theorems for nonlinear contractions in Menger PM spaces, Chaos, Solitons & Fractals, 42 (2009), 3121-3129.
16. M. Imdad, M. Tanveer and M. Hasan, Some commn fixed point theorems in Menger PM spaces, Fixed Point Theory Appl., Vol. 2010, 14 pages.
17. M. Imdad, M. Tanveer and M. Hasan, Erratum to "Some common fixed point theorems in Menger PM Spaces", Fixed Point Theory Appl., 2011, 2011:28.
18. G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci., 9 (4) (1986), 771-779.
19. G. Jungck, Common fixed points for noncontinuous nonself maps on nonmetric spaces. Far East J. Math. Sci., 4 (2) (1996), 199-215.
20. J.K. Kohli and S. Vashistha, Common fixed point theorems in probabilistic metric spaces, Acta Math Hung., 115 (2007), 37-47.
21. I. Kubiaczyk and S. Sharma, Some common fixed point theorems in Menger space under strict contractive conditions, Southeast Asian Bull. Math., 32 (2008), 117-124.
22. S. Kutukcu and S. Sharma, Compatible maps and common fixed points in Menger probabilistic metric spaces, Commun. Korean Math. Soc., 24 (2009), 17-27.
23. Y. Liu, Jun Wu and Z. Li, Common fixed points of single-valued and multi-valued maps, Int. J. Math. Math. Sci., 19 (2005), 3045-3055.
24. K. Menger, Statistical metrics, Proc. Nat. Acad. Sci. USA, 28 (1942), 535-537.
25. K. Menger, Probabilistic geometry, Proc. Nat. Acad. Sci. USA, 37 (1951), 226-229.

26. D. Miheţ, A generalization of a contraction principle in probabilistic metric spaces (II), Int. J. Math. Math. Sci 5 (2005), 729-736.
27. D. Miheţ, Fixed point theorems in fuzzy metric spaces using property E.A., Nonlinear Anal., 73 (2010), 2184-2188.
28. S.N. Mishra, Common fixed points of compatible mappings in PM-spaces, Math. Japon., 36 (1991), 283-289.
29. R.P. Pant, Common fixed points of noncommuting mappings, J. Math. Anal. Appl., 188 (1994), 436-440.
30. R.A. Rashwan and A. Hedar, On common fixed point theorems of compatible maps in Menger spaces, Demonst. Math., 31 (1998), 537-546.
31. A. Razani and M. Shirdaryazdi, A common fixed point theorem of compatible maps in Menger space, Chaos, Solitions and Fractals, 32 (2007), 26-34.
32. B. E. Rhoades, Two fixed-point theorems for mappings satisfying a general contractive condition of integral type, Internat. J. Math. Math. Sci., 63(2003), 4007-4013.
33. B. Schweizer and A. Sklar, Probabilistic metric spaces, Elsevier, North Holand, New York, 1983.
34. V.M. Sehgal and A.T. Bharucha-Reid, Fixed point of contraction mappings on probabilistic metric spaces, Math. Systems Theory, 6 (1972), 97-102.
35. B. Singh and S. Jain, A fixed point theorem in Menger spaces through weak compatibility, J. Math. Anal. Appl., 301 (2005), 439-448.
36. T. Suzuki, Meir-Keeler contractions of integral type are still Meir-Keeler contractions, Internat. J. Math. Math. Sci., (2007), Article ID 39281, 6 pages.
37. D. Turkoglu and I. Altun, A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying an implicit relation, Matematica Mexicana. Boletin.Tercera Serie., 13 (1) (2007), 195-205.
38. P. Vijayaraju, B.E. Rhoades and R. Mohanraj, A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type, Internat. J. Math. Math. Sci., 15(2005), 2359-2364.