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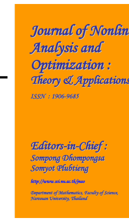


# BEST APPROXIMATION IN INTUITIONISTIC FUZZY $n$-NORMED LINEAR SPACES

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**ABSTRACT.** The central issues that are addressed by the theory of best approximation are related to the questions of existence, uniqueness, characterizations and qualitative properties of the minimizing functions. The theory can also estimate the rapidity of convergence of a sequence of functions converging to a minimizing function. The aim of this article is to introduce and study the notions of best approximation, proximal set, Chebyshev set and approximatively compact set in the new setup of intuitionistic fuzzy $n$-normed linear spaces.

**KEYWORDS** : Intuitionistic fuzzy $n$-normed linear space; Best approximation; $n$-normed space.
**AMS Subject Classification**: 40A05; 60B99.

## 1. INTRODUCTION

Since the introduction of fuzzy set theory by Zadeh [40] in 1965, fuzzy logic became an important area of research in various branches of mathematics such as metric and topological spaces [8, 12, 16], theory of functions [15, 38], approximation theory [1], etc. Fuzzy set theory also found applications for modeling uncertainty and vagueness in various fields of science and engineering. The notion of intuitionistic fuzzy set (IFS) introduced by Atanassov [4] has triggered a lot of debate (for details, see [6, 7, 13]) regarding the use of the terminology "intuitionistic" and the term is considered to be a misnomer on the following account:

- The algebraic structure of IFSs in not intuitionistic, since negation is involutive in IFS theory.
- Intuitionistic logic obeys the law of contradiction, IFSs do not.

Also IFSs are considered to be equivalent to interval-valued fuzzy sets and they are particular cases of $L$-fuzzy sets. In response to this debate, Attanassov justified the terminology in [2]. Apart from the terminological issues, research in intuitionistic fuzzy setting remains well motivated as IFSs give us a very natural tool for modeling imprecision in real life situations and found its application in various area of science

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and engineering, e.g., this theory has been extensively used in decision making problems [3] and E-infinity theory of high energy physics [22].

The theory of 2-norm and $n$-norm on a linear space was introduced by S. Gähler [10, 11], which was developed by S.S. Kim and Y.J. Cho [18], R. Malceski [20], A. Misiak [21], H. Gunawan and M. Mashadi [14]. Fuzzy norm on a linear space was first introduced by Katsaras [17] and studied by various authors from different points of view [5, 9, 16, 19, 39]. Vijayabalaji and Narayanan [36] extended $n$-normed linear space to fuzzy $n$-normed linear space. Saadati and Park [24] introduced the notion of intuitionistic fuzzy normed space while the notion of intuitionistic fuzzy $n$-normed linear space was introduced by Vijayabalaji et al. [37].

The best approximation problems were introduced by P. L. Chebyshev in 1853. Such problems deal with the search for a function in a prescribed class which has the least deviation from a given function, as measured in a prescribed metric. The theory also takes into account the continuity properties of the metric projection and can estimate the rapidity of convergence of a sequence of functions converging to a minimizing function. Some works in approximation theory can be found in [27, 28, 34, 35]. Some interesting works on the theory of best approximation has been done by Sintunavarat, W. and his coauthors in [29–33]. In the present paper, we propose to define and study the notions of $t$-best approximation, $t$-proximal set, $t$-Chebyshev set, $t$-approximatively compact set and prove some useful results related to those concepts in an intuitionistic fuzzy $n$-normed linear space. Most of the results in this article are closely linked with the notion of convergence of a sequence in intuitionistic fuzzy $n$-normed linear space and a new and unambiguous definition of the same has been given in [25, 26]. Here we develop the results based on that new definition.

## 2. PRELIMINARIES

Throughout this paper $\mathbb{R}$ and $\mathbb{N}$ will denote the set of real numbers and the set of natural numbers respectively. First we recall some definitions.

**Definition 2.1.** [14] Let $n \in \mathbb{N}$ and $X$ be a real linear space of dimension $d \geq n$ ($d$ may be infinite). A real valued function $\|.\|$ on $\underbrace{X \times X \times \cdots \times X}_{n} = X^n$ is called an $n$-norm on $X$ if it satisfies the following properties:

(i) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,
(ii) $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation,
(iii) $\|x_1, x_2, \ldots, \alpha x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$ for any $\alpha \in \mathbb{R}$,
(iv) $\|x_1, x_2, \ldots, x_{n-1}, y + z\| \leq \|x_1, x_2, \ldots, x_{n-1}, y\| + \|x_1, x_2, \ldots, x_{n-1}, z\|$, and the pair $(X, \|.\|)$ is called an $n$-normed linear space.

**Definition 2.2.** [25] An intuitionistic fuzzy $n$-normed linear space (in short IFnNLS) is the five-tuple $(X, \mu, \nu, *, \circ)$, where $X$ is a linear space of dimension $d \geq n$ over a field $F$, $*$ is a continuous $t$-norm, $\circ$ is a continuous $t$-conorm, $\mu$, $\nu$ are fuzzy sets on $X^n \times (0, \infty)$, $\mu$ denotes the degree of membership and $\nu$ denotes the degree of non-membership of $(x_1, x_2, \ldots, x_n, t) \in X^n \times (0, \infty)$ satisfying the following conditions for every $x_1, x_2, \ldots, x_n \in X$ and $s, t > 0$:

(i) $\mu(x_1, x_2, \ldots, x_n, t) + \nu(x_1, x_2, \ldots, x_n, t) \leq 1$,
(ii) $\mu(x_1, x_2, \ldots, x_n, t) > 0$,
(iii) $\mu(x_1, x_2, \ldots, x_n, t) = 1$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,
(iv) $\mu(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$,

(v) $\mu(x_1, x_2, \ldots, cx_n, t) = \mu(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$ if $c \neq 0, c \in F$,
(vi) $\mu(x_1, x_2, \ldots, x_n, s) * \mu(x_1, x_2, \ldots, x_n^{'}, t) \leq \mu(x_1, x_2, \ldots, x_n + x_n^{'}, s + t)$,
(vii) $\mu(x_1, x_2, \ldots, x_n, t) : (0, \infty) \to [0, 1]$ is continuous in $t$,
(viii) $\lim_{t\to\infty} \mu(x_1, x_2, \ldots, x_n, t) = 1$ and $\lim_{t\to 0} \mu(x_1, x_2, \ldots, x_n, t) = 0$,
(ix) $\nu(x_1, x_2, \ldots, x_n, t) < 1$,
(x) $\nu(x_1, x_2, \ldots, x_n, t) = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,
(xi) $\nu(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$,
(xii) $\nu(x_1, x_2, \ldots, cx_n, t) = \nu(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$ if $c \neq 0, c \in F$,
(xiii) $\nu(x_1, x_2, \ldots, x_n, s) \circ \nu(x_1, x_2, \ldots, x_n^{'}, t) \geq \nu(x_1, x_2, \ldots, x_n + x_n^{'}, s + t)$,
(xiv) $\nu(x_1, x_2, \ldots, x_n, t) : (0, \infty) \to [0, 1]$ is continuous in $t$,
(xv) $\lim_{t\to\infty} \nu(x_1, x_2, \ldots, x_n, t) = 0$ and $\lim_{t\to 0} \nu(x_1, x_2, \ldots, x_n, t) = 1$.

**Definition 2.3.** [25] Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. We say that a sequence $x = \{x_k\}$ in $X$ is convergent to $L \in X$ with respect to the intuitionistic fuzzy $n$-norm $(\mu, \nu)^n$ if, for every $\epsilon \in (0, 1)$, $t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that $\mu(y_1, y_2, \ldots, y_{n-1}, x_k - L, t) > 1 - \epsilon$ and $\nu(y_1, y_2, \ldots, y_{n-1}, x_k - L, t) < \epsilon$ for all $k \geq k_0$. It is denoted by $x_k \overset{(\mu,\nu)^n}{\to} L$ as $k \to \infty$.

**Definition 2.4.** The closure of a subset $B$ in an IFnNLS $(X, \mu, \nu, *, \circ)$ is denoted by $\overline{B}$ and defined by the set of all $x \in X$ such that there exists a sequence $\{x_k\}$ in $B$ such that $x_k \overset{(\mu,\nu)^n}{\to} x$. We say that $B$ is closed whenever $\overline{B} = B$.

## 3. MAIN RESULTS

Now we obtain our main results.

**Definition 3.1.** Let $A$ be a non-empty subset of an IFnNLS $(X, \mu, \nu, *, \circ)$. For $x \in X, t > 0$ and a linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of $X$, denote $\mu^Y(x, t) = \mu(y_1, y_2, \ldots, y_{n-1}, x, t)$ and $\nu^Y(x, t) = \nu(y_1, y_2, \ldots, y_{n-1}, x, t)$. Let

$$\mu^Y(x - A, t) = \sup\{\mu^Y(x - y, t) : y \in A\},$$
$$\nu^Y(x - A, t) = \inf\{\nu^Y(x - y, t) : y \in A\}.$$

An element $u \in A$ is said to be a $t$-best approximation to $x$ from $A$ if

$$\mu^Y(x - u, t) = \mu^Y(x - A, t) \text{ and } \nu^Y(x - u, t) = \nu^Y(x - A, t).$$

By $P_A^Y(x, t)$, we denote the set of elements of $t$-best approximation of $x$ by elements of the set $A$, i.e.,

$$P_A^Y(x, t) = \{y \in A : \mu^Y(x - A, t) = \mu^Y(x - y, t) \text{ and } \nu^Y(x - A, t) = \nu^Y(x - y, t)\}.$$

**Definition 3.2.** Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. For $\alpha \in (0, 1)$ and a linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of $X$ we define the open ball $B_x^Y(\alpha, t)$ and the closed ball $B_x^Y[\alpha, t]$ with center $x \in X$ and radius $t > 0$ as follows:

$$B_x^Y(\alpha, t) = \{y \in X : \mu^Y(x - y, t) > 1 - \alpha \text{ and } \nu^Y(x - y, t) < \alpha\},$$
$$B_x^Y[\alpha, t] = \{y \in X : \mu^Y(x - y, t) \geq 1 - \alpha \text{ and } \nu^Y(x - y, t) \leq \alpha\}.$$

**Definition 3.3.** Let $A$ be a non-empty subset of an IFnNLS $(X, \mu, \nu, *, \circ)$. Then $A$ is said to be a $t$-proximal set if $P_A^Y(x, t)$ is non-empty for every $x \in X \setminus A$ and a linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of $X$. $A$ is called a $t$-Chebyshev set if $P_A^Y(x, t)$ contains exactly one element for every $x \in X$ and some linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of $X$. Also $A$ is called a $t$-quasi-Chebyshev set if $P_A^Y(x, t)$ is a compact set.

**Example 3.4.** Let $X = \mathbb{R}^n$ $(n \geq 2)$ with

$$\|x_1, x_2, \ldots, x_n\| = abs\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right),$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$. Define $\mu, \nu : X^n \times (0, \infty) \to [0, 1]$ by

$$\mu(x_1, x_2, \ldots, x_n, t) = \frac{1}{e^{\|x_1, x_2, \ldots, x_n\|/t}} \text{ and } \nu(x_1, x_2, \ldots, x_n, t) = 1 - \frac{1}{e^{\|x_1, x_2, \ldots, x_n\|/t}}.$$

Also let $a * b = ab$ and $a \circ b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$. Then $(X, \mu, \nu, *, \circ)$ is an IFnNLS. Let

$$A = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -1 \leq x_1 \leq 1, 0 \leq x_2 \leq x_1^2, x_3 = \ldots = x_n = 0\}.$$

Consider $x_1 = (0, 3, 0, \ldots, 0), x_2 = (0, 2, 0, \ldots, 0), x_3 = (0, 0, 1, \ldots, 0), \ldots, x_n = (0, 0, \ldots, 1) \in \mathbb{R}^n$. Then for every $t > 0$,

$$\begin{aligned} &\mu((-1, 1, 0, \ldots, 0) - (0, 3, 0, \ldots, 0), (0, 2, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, \\ &\quad (0, 0, \ldots, 1), t) \\ &= \mu((1, 1, 0, \ldots, 0) - (0, 3, 0, \ldots, 0), (0, 2, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, \\ &\quad (0, 0, \ldots, 1), t) \\ &= \frac{1}{e^{2/t}}. \end{aligned}$$

Again

$$\begin{aligned} &\mu((0, 3, 0, \ldots, 0) - A, (0, 2, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, \\ &\quad (0, 0, \ldots, 1), t) \\ &= \sup\{\mu((0, 3, 0, \ldots, 0) - (u_1, u_2, \ldots, u_n), (0, 2, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \\ &\quad \ldots, (0, 0, \ldots, 1), t) : -1 \leq u_1 \leq 1, 0 \leq u_2 \leq u_1^2, u_3 = \ldots = u_n = 0\} \\ &= \frac{1}{e^{2/t}}. \end{aligned}$$

Similarly,

$$\begin{aligned} &\nu((-1, 1, 0, \ldots, 0) - (0, 3, 0, \ldots, 0), (0, 2, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, \\ &\quad (0, 0, \ldots, 1), t) \\ &= \nu((1, 1, 0, \ldots, 0) - (0, 3, 0, \ldots, 0), (0, 2, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, \\ &\quad (0, 0, \ldots, 1), t) \\ &= \nu((0, 3, 0, \ldots, 0) - A, (0, 2, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1), t) \\ &= 1 - \frac{1}{e^{2/t}}. \end{aligned}$$

Thus for every $t > 0$, $y_0 = (-1, 1, 0, \ldots, 0), y_1 = (1, 1, 0, \ldots, 0)$ are $t$-best approximations to $(0, 3, 0, \ldots, 0)$ from $A$. Therefore, $A$ is a $t$-proximal set but not a $t$-Chebyshev set.

Saadati and Park [23] investigated several properties of intuitionistic fuzzy topological spaces. Every IFnNLS $X$ induces a topology $\tau$ such that for some $A \subseteq X$, $A \in \tau$ if and only if for every $x \in A$ and a linearly independent set $Y = \{y_1, y_2, \ldots, y_{n-1}\} \subseteq X$, there exist $t > 0$ and $\alpha \in (0, 1)$ such that $B_x^Y(\alpha, t) \subseteq A$. It is not difficult to see that the family $\{B_x^Y(\frac{1}{n}, \frac{1}{n}) : n = 1, 2, \ldots\}$ is a countable local basis at $x$ and consequently $\tau$ is a first countable topology.

**Lemma 3.5.** *Let $A$ be a non-empty subset of an IFnNLS $(X, \mu, \nu, *, \circ)$ and $x \in X$. Then $x \in \overline{A}$ if and only if for all $t > 0$ and a linearly independent subset $Y = \{y_1, y_2, \dots, y_{n-1}\}$ of $X$,*

$$\mu^Y(x - A, t) = 1 \text{ and } \nu^Y(x - A, t) = 0.$$

**Proof** Let $x \in \overline{A}$. As $X$ is first countable, there exists a sequence $\{x_k\}$ in $A$ such that $x_k \overset{(\mu,\nu)^n}{\to} x$ as $k \to \infty$. Then for every $t > o, \lambda \in (0, 1)$ and a linearly independent subset $Y = \{y_1, y_2, \dots, y_{n-1}\}$ of $X$ there exists $n_0 \in \mathbb{N}$ such that $\mu^Y(x - x_k, t) > 1 - \lambda$ and $\nu^Y(x - x_k, t) < \lambda$ for all $k \geq n_0$. Thus

$$1 - \lambda < \mu^Y(x - x_k, t) \leq \mu^Y(x - A, t) \leq 1$$

and

$$\lambda > \nu^Y(x - x_k, t) \geq \mu^Y(x - A, t) \geq 0,$$

for all $k \geq n_0$. Hence $\mu^Y(x - A, t) = 1$ and $\nu^Y(x - A, t) = 0$.

Conversely, suppose $\mu^Y(x - A, t) = 1$ and $\nu^Y(x - A, t) = 0$ for all $t > 0$ and linearly independent subset $Y = \{y_1, y_2, \dots, y_{n-1}\}$ of $X$. We know that $\{B^Y(x, \lambda, t) : t > o, \lambda \in (0, 1), Y = \{y_1, y_2, \dots, y_{n-1}\}\}$ is a local base at $x$. By definition, there exists a sequence $\{x_k\} \subseteq A$ such that $\mu^Y(x - x_k, \frac{1}{k}) \geq 1 - \frac{1}{k}$ and $\nu^Y(x - x_k, \frac{1}{k}) \leq \frac{1}{k}$, which implies that $\{x_k\} \subseteq B^Y(x, \frac{1}{k}, \frac{1}{k})$. Given $t > 0$ and $\lambda \in (0, 1)$, choose $k \in \mathbb{N}$ such that $t, \lambda > \frac{1}{k}$, then $B^Y(x, \frac{1}{k}, \frac{1}{k}) \subseteq B^Y(x, \lambda, t)$. So we have $B^Y(x, \lambda, t) \cap A \neq \phi$. Hence $x \in \overline{A}$.

**Theorem 3.6.** *Let $A$ be a non-empty subset of an IFnNLS $(X, \mu, \nu, *, \circ)$. Then for all $x, y \in X, t > 0$ and a linearly independent subset $Y = \{y_1, y_2, \dots, y_{n-1}\}$ of $X$, we have*

(i) $P^Y_{A+y}(x + y, t) = P^Y_A(x, t) + y$,
(ii) $P^Y_{\alpha A}(\alpha x, |\alpha|t) = \alpha P^Y_A(x, t)$, $\alpha \in \mathbb{R} \setminus \{0\}$,
(iii) *$A$ is $t$-proximal ($t$-Chebyshev) if and only if $A + y$ is $t$-proximal ($t$-Chebyshev).*

**Proof** (i) Let $u \in P^Y_{A+y}(x + y, t)$. Then

$$\mu^Y(x - A, t) = \mu^Y(x + y - (A + y), t) = \mu^Y(x - (u - y), t)$$

and

$$\nu^Y(x - A, t) = \nu^Y(x + y - (A + y), t) = \nu^Y(x - (u - y), t),$$

which implies that $u - y \in P^Y_A(x, t)$. Hence $u \in P^Y_A(x, t) + y$.

The converse is obvious.

(ii) Let $u \in P^Y_{\alpha A}(\alpha x, |\alpha|t)$ for some $x \in X, t > 0, \alpha \in \mathbb{R} \setminus \{0\}$ and a linearly independent subset $Y = \{y_1, y_2, \dots, y_{n-1}\}$ of $X$. Then

$$\mu^Y(x - A, t) = \mu^Y(\alpha x - \alpha A, |\alpha|t) = \mu^Y(\alpha x - u, |\alpha|t) = \mu^Y(x - \frac{u}{\alpha}, t)$$

and similarly, $\nu^Y(x - A, t) = \nu^Y(x - \frac{u}{\alpha}, t)$, which implies $\frac{u}{\alpha} \in P^Y_A(x, t)$ and hence $u \in \alpha P^Y_A(x, t)$.

The converse is obvious.

(iii) follows from (i).

**Lemma 3.7.** *Let $(X, \|.\|)$ be an $n$-normed space and $\mu, \nu$ be fuzzy sets on $X^n \times (0, \infty)$ such that $\mu(x_1, x_2, \dots, x_n, t) = \frac{t}{t + \|x_1, x_2, \dots, x_n\|}$ and $\nu(x_1, x_2, \dots, x_n, t) = \frac{\|x_1, x_2, \dots, x_n\|}{t + \|x_1, x_2, \dots, x_n\|}$, also let $a * b = ab$ and $a \circ b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$. Let $A$ be a non-empty subset of the IFnNLS $(X, \mu, \nu, *, \circ)$. Then $u \in A$ is a best approximation to*

*$x \in X$ in the $n$-normed space $(X, \|.\|)$ if and only if $u$ is a $t$-best approximation to $x$ in the IFnNLS $(X, \mu, \nu, *, \circ)$ for each $t > 0$ and a linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of $X$.*

**Proof** Since $u$ is a best approximation to $x \in X$, for a linearly independent set $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ we have $\|y_1, y_2, \ldots, y_{n-1}, x - u\| = d^Y(x, A)$, where $d$ is the metric induced by the $n$-norm $\|.\|$. Then

$$\mu^Y(x - A, t) = \frac{t}{t + d^Y(x, A)} = \frac{t}{t + \|y_1, y_2, \ldots, y_{n-1}, x - u\|} \\ = \mu^Y(x - u, t),$$

and

$$\nu^Y(x - A, t) = \frac{d^Y(x, A)}{t + d^Y(x, A)} = \frac{\|y_1, y_2, \ldots, y_{n-1}, x - u\|}{t + \|y_1, y_2, \ldots, y_{n-1}, x - u\|} \\ = \nu^Y(x - u, t).$$

Hence the result.

**Remark 3.8.** We recall that a set $A$ is said to be countably compact, if every countable open cover has a finite subcover, or equivalently, if for every decreasing sequence $A_1 \supset A_2 \supset \ldots$ of non-empty closed subsets of $A$ we have $\cap_{n=1}^{\infty} A_n \neq \phi$.

**Theorem 3.9.** *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. If $A$ is a non-empty subset of $X, \lambda \in (0, 1)$ and $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ is a linearly independent subset of $X$ such that $A \cap B_x^Y[\lambda, t]$ is countably compact, then $A$ is $t$-proximal.*

**Proof** For every $n \in \mathbb{N}$,

$$0 < 1 - \mu^Y(x - A, t) + \frac{\mu^Y(x - A, t)}{n + 1} < 1,$$
$$\text{and } 0 < \nu^Y(x - A, t) - \frac{\nu^Y(x - A, t)}{n + 1} < 1.$$

Put $A_n^t = A \cap B_x^Y[1 - \mu^Y(x - A, t) + \frac{\mu^Y(x-A,t)}{n+1}, t] \cap B_x^Y[\nu^Y(x - A, t) - \frac{\nu^Y(x-A,t)}{n+1}, t]$ $(n = 1, 2, \ldots)$. We have $\ldots \supset A_n^t \supset A_{n+1}^t \supset \ldots$ and each $A_n^t$ is non-empty. Since for every $n \in \mathbb{N}$, $\mu^Y(x - A, t)(1 - \frac{1}{n+1}) < \mu^Y(x - A, t)$ and $1 - \nu^Y(x - A, t) + \frac{\nu^Y(x-A,t)}{n+1} > \nu^Y(x - A, t)$, there exists $a_n^t \in A$ such that $\mu^Y(x - A, t)(1 - \frac{1}{n+1}) < \mu^Y(x - a_n^t, t)$ and $1 - \nu^Y(x - A, t) + \frac{\nu^Y(x-A,t)}{n+1} > \nu^Y(x - a_n^t, t)$. Hence $a_n^t \in A_n^t$. Since each $A_n^t$ is countably compact and closed, it follows that there exists an $a_0 \in \cap_{n=1}^{\infty} A_n^t$. Then we have

$$\mu^Y(x - A, t) \geq \mu^Y(x - a_0, t) \geq \mu^Y(x - A, t)(1 - \frac{1}{n + 1}) \quad (n = 1, 2, \ldots)$$
$$\Rightarrow \mu^Y(x - A, t) = \mu^Y(x - a_0, t),$$

and

$$\nu^Y(x - A, t) \leq \nu^Y(x - a_0, t) \leq \nu^Y(x - A, t)(1 - \frac{1}{n + 1}) \quad (n = 1, 2, \ldots)$$
$$\Rightarrow \nu^Y(x - A, t) = \nu^Y(x - a_0, t),$$

whence $a_0 \in P_A^Y(x, t)$.

**Definition 3.10.** A non-empty subset $A$ of an IFnNLS $(X, \mu, \nu, *, \circ)$ is said to be $t$-approximatively compact if for each $x \in X, t > 0$, a linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of $X$ and each sequence $\{y_k\}$ in $A$ with $\mu^Y(x - y_k, t) \to \mu^Y(x - A, t)$ and $\nu^Y(x - y_k, t) \to \nu^Y(x - A, t)$ there exists a subsequence $\{y_{k_n}\}$ of $\{y_k\}$ converging to an element $u$ in $A$.

**Lemma 3.11.** *If $A$ is*

(i) *approximately compact in an $n$-normed space $(X, \|.\|)$, then for each $t > 0$, $A$ is $t$-approximatively compact in the induced IFnNLS $(X, \mu, \nu, *, \circ)$.*
(ii) *a compact subset of an IFnNLS, then $A$ is $t$-approximatively compact for each $t > 0$.*

**Theorem 3.12.** *For $t > 0$, let $A$ be a non-empty $t$-approximatively compact subset of an IFnNLS $(X, \mu, \nu, *, \circ)$, then $A$ is a $t$-proximal set.*

**Proof** For $x \in X, t > 0$ and a linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of $X$ there exists a sequence $\{y_k\} \subset A$ such that $\mu^Y(x - y_k, t) \to \mu^Y(x - A, t)$ and $\nu^Y(x - y_k, t) \to \nu^Y(x - A, t)$. Since $A$ is a $t$-approximatively compact set, there exists a subsequence $\{y_{k_n}\}$ of $\{y_k\}$ and $u \in A$ such that $y_{k_n} \overset{(\mu,\nu)^n}{\to} u$. Thus we have $\mu^Y(x - y_{k_n}, t) \to \mu^Y(x - u, t)$ and $\nu^Y(x - y_{k_n}, t) \to \nu^Y(x - u, t)$.Hence $\mu^Y(x - u, t) \geq \mu^Y(x - A, t)$ and $\nu^Y(x - u, t) \leq \nu^Y(x - A, t)$. Consequently $u$ is a $t$-best approximation to $x$ from $A$.

**Theorem 3.13.** *If for some $t > 0$, $A$ is a $t$-approximatively compact subset of an IFnNLS $(X, \mu, \nu, *, \circ)$, then $A$ is closed in $X$.*

**Proof** Let $x \in \overline{A}$. Then for a linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of $X$, $\mu^Y(x - A, t) = 1$ and $\nu^Y(x - A, t) = 0$. Since $A$ is $t$-approximatively compact, there exists $y \in A$ such that $\mu^Y(x - y, t) = \mu^Y(x - A, t) = 1$ and $\nu^Y(x - y, t) = \nu^Y(x - A, t) = 0$. Hence $x \in A$.

**Theorem 3.14.** *If $A$ is a $t$-approximatively compact subset of an IFnNLS $(X, \mu, \nu, *, \circ)$, then $A$ is a $t$-quasi-Chebyshev set.*

**Proof** Let $\{y_k\}$ be a sequence in $P_A^Y(x, t)$. Since $A$ is $t$-approximatively compact, there exists subsequence $\{y_{k_n}\}$ of $\{y_k\}$ and $u \in A$ such that $y_{k_n} \overset{(\mu,\nu)^n}{\to} u$. Then $\mu^Y(x - y_{k_n}, t) \to \mu^Y(x - u, t)$ and $\nu^Y(x - y_{k_n}, t) \to \nu^Y(x - u, t)$. On the other hand, $\mu^Y(x - y_{k_n}, t) \to \mu^Y(x - A, t)$ and $\nu^Y(x - y_{k_n}, t) \to \nu^Y(x - A, t)$. Therefore $\mu^Y(x - u, t) = \mu^Y(x - A, t)$ and $\nu^Y(x - u, t) = \nu^Y(x - A, t)$ and so $u \in P_A^Y(x, t)$. Hence $P_A^Y(x, t)$ is compact.

*Conclusion.* Most of the results in this paper run parallel to those of classical ones or related works in best approximation theory, but in the proofs a different approach has been adopted as the convergence of a sequence in an IFnNLS is defined in a different way (see Definition 2.3) than it has been defined in [37]. Results obtained here are more general than previous works done in this field and can give tools to deal with convergence related problems arising in science and engineering.

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