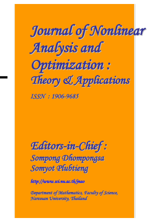

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EXISTENCE RESULTS FOR A QUASILINEAR BOUNDARY VALUE PROBLEM INVESTIGATED VIA DEGREE THEORY

G.A. AFROUZI*, A. HADJIAN, S. SHAKERI AND M. MIRZAPOUR

Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

ABSTRACT. In this article we prove the existence of at least one weak solution for the quasilinear problem

$$\begin{cases} -\Delta_p u(x) = \lambda |u(x)|^{p-2}u(x) + h(x,u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $p > 1$, $\Omega \subset \mathbb{R}^N$ is a non-empty bounded domain with Lipschitz boundary ($\Omega \in C^{0,1}$), λ is a positive parameter and $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a bounded Carathéodory function. The approach is fully based on the degree theory.

KEYWORDS : p-Laplacian; Principal eigenvalue; (S_+) condition; Topological degree.
AMS Subject Classification: 35J60; 35B30; 35B40.

1. INTRODUCTION

The aim of this paper is to establish the existence of at least one weak solution for the following quasilinear problem

$$\begin{cases} -\Delta_p u(x) = \lambda |u(x)|^{p-2}u(x) + h(x,u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $p > 1$, $\Omega \subset \mathbb{R}^N$ is a non-empty bounded domain with Lipschitz boundary ($\Omega \in C^{0,1}$), λ is a positive parameter and $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a bounded Carathéodory function.

On the Sobolev space $W_0^{1,p}(\Omega)$, we consider the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p dx \right)^{\frac{1}{p}}.$$

* *Corresponding author.*
Email address : afrouzi@umz.ac.ir (G.A. Afrouzi), a.hadjian@umz.ac.ir (A. Hadjian), s.shakeri@umz.ac.ir (S. Shakeri) and mirzapour@stu.umz.ac.ir (M. Mirzapour).
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By a (weak) solution of the problem (1.1), we mean any $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx - \lambda\int_\Omega |u(x)|^{p-2}u(x)v(x)dx - \int_\Omega h(x,u(x))v(x) = 0$$

for all $v \in W_0^{1,p}(\Omega)$. It is well known that the eigenvalue problem

$$\begin{cases} -\Delta_p u(x) = \lambda |u(x)|^{p-2}u(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

has a principal eigenvalue (i.e., the least one) $\lambda_1 > 0$ which is simple and characterized variationally by

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u(x)|^p\,dx}{\int_\Omega |u(x)|^p\,dx}.$$

Let X be a reflexive real Banach space and X^* its dual. Here and in the sequel we denote by $\langle f, u\rangle := f(u)$ the value of the linear form $f \in X^*$ for an element $u \in X$. If X is a Hilbert space, then according to the Riesz Representation Theorem, $\langle f, u\rangle = (u, f)$.

Definition 1.1. The operator $T : X \to X^*$ is said to satisfy the (S_+) condition, if the assumptions

$$u_n \rightharpoonup u_0 \quad \text{(weakly) in } X \qquad \text{and} \qquad \limsup_{n\to\infty}\langle T(u_n), u_n - u_0\rangle \le 0$$

imply

$$u_n \to u_0 \quad \text{(strongly) in } X.$$

It is clear that if $T : X \to X^*$ satisfies the (S_+) condition and $K : X \to X^*$ be a compact operator, then the sum $T + K : X \to X^*$ satisfies the (S_+) condition. We say that $T : X \to X^*$ is demicontinuous, if T maps strongly convergent sequences in X to weakly convergent sequences in X^*.

The main aim of the present paper is to prove the existence of at least one weak solution of (1.1) via degree theory. Various applications of degree theory for solutions of nonlinear boundary value problems are already available, see for instance [1, 2, 4–6, 8]. For other basic notations and definitions we refer to [3].

2. MAIN RESULTS

First we here recall for reader's convenience the following Theorem of [9] which is our main tool to prove the results.

Theorem 2.1 (Skrypnik [9]). *Let $T : X \to X^*$ be a bounded and demicontinuous operator satisfying the (S_+) condition. Let $\mathcal{D} \subset X$ be an open, bounded and nonempty set with the boundary $\partial\mathcal{D}$ such that $T(u) \neq 0$ for $u \in \partial\mathcal{D}$. Then there exists an integer*

$$\deg(T, \mathcal{D}, 0)$$

(called the degree of the mapping T) such that

(i) $\deg(T, \mathcal{D}, 0) \neq 0$ *implies that there exists an element $u_0 \in \mathcal{D}$ such that*

$$T(u_0) = 0.$$

(ii) *If $\mathcal{D}$ is symmetric with respect to the origin and T satisfies $T(u) = -T(-u)$ for any $u \in \partial\mathcal{D}$, then*

$$\deg(T, \mathcal{D}, 0)$$

is an odd number (and thus different from zero).

(iii) (*Homotopy invariance property*) *Let* T_λ *be a family of bounded and demicontinuous mappings which satisfy the* (S_+) *condition and which depend continuously on a real parameter* $\lambda \in [0,1]$, *and let* $T_\lambda(u) \neq 0$ *for any* $u \in \partial\mathcal{D}$ *and* $\lambda \in [0,1]$. *Then*

$$\deg(T_\lambda, \mathcal{D}, 0)$$

is constant with respect to $\lambda \in [0,1]$. *In particular, we have*

$$\deg(T_0, \mathcal{D}, 0) = \deg(T_1, \mathcal{D}, 0).$$

We introduce the operators $J, G, S : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ in the following way

$$\langle J(u), v\rangle := \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx,$$

$$\langle G(u), v\rangle := \int_\Omega |u(x)|^{p-2}u(x)v(x)dx,$$

$$\langle S(u), v\rangle := \int_\Omega h(x, u(x))v(x)dx$$

for any $u, v \in W_0^{1,p}(\Omega)$. First we sketch the properties of operators J, G and S.

Lemma 2.2. *The operators* J, G *and* S *are well defined. Also we have the following properties of* J, G *and* S.

(a) J, G *and* S *are bounded and continuous* (*and so demicontinuous*) *operators;*
(b) G *and* S *are compact operators;*
(c) J *satisfies the* (S_+) *condition;*
(d) J *is invertible and its inverse is continuous.*

Proof. The fact that J, G and S are well defined follows the standard procedure. The first two statements follows from the Hölder inequality, the boundedness of h and the compact embedding $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$. Let us prove the third statement. Indeed, let $u_n \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$ and

$$\limsup_{n\to\infty}\langle J(u_n), u_n - u_0\rangle \leq 0.$$

Then $\lim_{n\to\infty}\langle J(u_0), u_n - u_0\rangle = 0$, and so

$$\begin{aligned}
0 \quad &\geq \quad \limsup_{n\to\infty}\langle J(u_n) - J(u_0), u_n - u_0\rangle \\
&= \quad \limsup_{n\to\infty}\int_\Omega \Big(|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u_0(x)|^{p-2}\nabla u_0(x)\Big)\Big(\nabla u_n(x) - \nabla u_0(x)\Big)dx \\
&\geq \quad \limsup_{n\to\infty}\Bigg\{\int_\Omega |\nabla u_n(x)|^p dx - \left(\int_\Omega |\nabla u_n(x)|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |\nabla u_0(x)|^p dx\right)^{\frac{1}{p}} \\
&\qquad -\left(\int_\Omega |\nabla u_0(x)|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |\nabla u_n(x)|^p dx\right)^{\frac{1}{p}} - \int_\Omega |\nabla u_0(x)|^p dx\Bigg\} \\
&= \quad \limsup_{n\to\infty}\Big[\|u_n\|^{p-1} - \|u_0\|^{p-1}\Big]\Big[\|u_n\| - \|u_0\|\Big] \geq 0,
\end{aligned}$$

where the last inequality follows from the fact that $s \mapsto |s|^{p-1}$ is strictly increasing on $(0,\infty)$. Hence $\|u_n\| \to \|u_0\|$, and due to the uniform convexity of $W_0^{1,p}(\Omega)$ we have $u_n \to u_0$ in $W_0^{1,p}(\Omega)$. Thus J satisfies the (S_+) condition.
Finally, we prove the fourth statement. Indeed, the strict monotonicity of $s \mapsto |s|^{p-2}$ implies that

$$\langle J(u) - J(v), u - v\rangle > 0 \qquad \text{for} \qquad u \neq v.$$

Hence J is injective. To prove that J^{-1} is continuous we proceed via contradiction. Suppose there exists a sequence $\{f_n\}_{n=1}^{\infty}$, $f_n \to f$ in $(W_0^{1,p}(\Omega))^*$ and

$$\|J^{-1}(f_n) - J^{-1}(f)\| \geq \delta \qquad \text{for a} \quad \delta > 0.$$

Let $u_n := J^{-1}(f_n)$ and $u := J^{-1}(f)$. It follows that

$$\|f_n\|\|u_n\| \geq \langle f_n, u_n\rangle = \langle J(u_n), u_n\rangle = \|u_n\|^p, \qquad \text{i.e.,} \qquad \|u_n\|^{p-1} \leq \|f_n\|.$$

We may then assume $u_n \rightharpoonup \tilde{u}$ in $W_0^{1,p}(\Omega)$ due to the reflexivity of $W_0^{1,p}(\Omega)$. Hence

$$\langle J(u_n) - J(\tilde{u}), u_n - \tilde{u}\rangle = \langle J(u_n) - J(u), u_n - \tilde{u}\rangle + \langle J(u) - J(\tilde{u}), u_n - \tilde{u}\rangle \to 0$$

since $J(u_n) \to J(u)$ in $(W_0^{1,p}(\Omega))^*$. Then we have

$$0 = \lim_{n\to\infty} \langle J(u_n) - J(\tilde{u}), u_n - \tilde{u}\rangle \geq \lim_{n\to\infty} \left[\|u_n\|^{p-1} - \|\tilde{u}\|^{p-1}\right]\left[\|u_n\| - \|\tilde{u}\|\right] \geq 0,$$

i.e., $\|u_n\| \to \|\tilde{u}\|$. Hence $u_n \to \tilde{u}$ follows due to the fact that $W_0^{1,p}(\Omega)$ is a uniformly convex Banach space. Since J is continuous and injective, $\tilde{u} = u$, a contradiction. □

We state our main result as follows.

Theorem 2.3. *Let $\lambda < \lambda_1$ and let $h : \Omega \times \mathbb{R} \to \mathbb{R}$ be a bounded Carathéodory function. Then the problem (1.1) has at least one weak solution.*

Proof. We set

$$T = J - \lambda G - S,$$

such that J, G and S are as above. Then existence of a weak solution of (1.1) is equivalent to the existence of a solution of the operator equation

$$T(u) = 0. \tag{2.1}$$

Our plan is to use the degree argument to prove the existence of a solution of (2.1). By Lemma 2.2, the operator T is a bounded and demicontinuous operator satisfying the (S_+) condition.
The operator J satisfies

$$\langle J(u), u\rangle = \|u\|^p.$$

Moreover, J and G are odd mappings and $(p-1)$-homogeneous, i.e.,

$$J(tu) = t^{p-1}J(u) \quad \text{and} \quad G(tu) = t^{p-1}G(u) \quad \text{for any} \quad t > 0, \quad u \in W_0^{1,p}(\Omega).$$

Our sketch is the following. The existence of at least one solution of (2.1) would follow from

$$\deg(J - \lambda G - S, B(0; R), 0) \neq 0 \tag{2.2}$$

if we found a ball $B(0; R)$ for which (2.2) is valid. To prove (2.2) we use the homotopy invariance property of the degree (Theorem 2.1(iii)) and connect the operator $J - \lambda G - S$ with the operator $J - \lambda G$ on the boundary of a ball $B(0; R)$ with a sufficiently large radius $R > 0$. Once this is done we finally use

$$\deg(J - \lambda G, B(0; R), 0) \neq 0. \tag{2.3}$$

(The value of the degree in (2.3) is an odd number according to Theorem 2.1(ii)). So, to complete the proof, we have to find an admissible homotopy connecting $J - \lambda G - S$ and $J - \lambda G$. We define a homotopy

$$T_\tau(u) := J(u) - \lambda G(u) - \tau S(u), \qquad \tau \in [0, 1], \qquad u \in W_0^{1,p}(\Omega).$$

It is enough to prove that there exists $R > 0$ such that for all $u \in W_0^{1,p}(\Omega)$, $\|u\| = R$ and $\tau \in [0, 1]$ we have

$$T_\tau(u) \neq 0. \tag{2.4}$$

Assume, by contradiction, that no such $R > 0$ exists, i.e., we can find sequence $\{u_n\}_{n=1}^{\infty} \subset W_0^{1,p}(\Omega)$ and $\{\tau_n\}_{n=1}^{\infty} \subset [0,1]$ such that $\|u_n\| \to \infty$ and

$$J(u_n) - \lambda G(u_n) - \tau_n S(u_n) = 0. \tag{2.5}$$

We set $v_n := \frac{u_n}{\|u_n\|^{p-1}}$, divide (2.5) by $\|u_n\|^{p-1}$ and use that J and G are $(p-1)$-homogenous to get

$$J(v_n) - \lambda G(v_n) - \tau_n \frac{S(u_n)}{\|u_n\|^{p-1}} = 0. \tag{2.6}$$

Due to the reflexivity of $W_0^{1,p}(\Omega)$ and the compactness of the interval $[0,1]$, passing to suitable subsequence, we may assume that

$$v_{n_k} \rightharpoonup v \quad \text{in } W_0^{1,p}(\Omega) \qquad \text{and} \qquad \tau_{n_k} \to \tau \in [0,1].$$

Let $M := \sup_{x\in\Omega,\, s\in\mathbb{R}} |h(x,s)|$. We have

$$\int_\Omega \frac{|h(x,u_{n_k}(x))|}{\|u_{n_k}\|^{p-1}} |v(x)| dx \le M \int_\Omega \frac{|v(x)|}{\|u_{n_k}\|^{p-1}} dx \le M_1 \frac{\|v\|}{\|u_{n_k}\|^{p-1}} \to 0 \qquad \text{as} \quad k \to \infty,$$

where $M_1 > 0$ is a constant. To summarize, since G is compact, we have

$$\tau_{n_k} \frac{S(u_{n_k})}{\|u_{n_k}\|^{p-1}} \to 0, \tag{2.7}$$

$$\lambda G(v_{n_k}) \to \lambda G(v), \tag{2.8}$$

in $(W_0^{1,p}(\Omega))^*$ as $k \to \infty$.

So, putting together (2.6)-(2.8) we also obtain that

$$J(v_{n_k}) \to \lambda G(v)$$

in $(W_0^{1,p}(\Omega))^*$ as $k \to \infty$, i.e.,

$$v_{n_k} \to J^{-1}(\lambda G(v))$$

in $W_0^{1,p}(\Omega)$ as $k \to \infty$ (Remember that J is invertible and its inverse is continuous). Since at the same time $v_{n_k} \rightharpoonup v$ in $W_0^{1,p}(\Omega)$, we have

$$v_{n_k} \to v \qquad \text{in} \quad W_0^{1,p}(\Omega)$$

and

$$J(v) - \lambda G(v) = 0 \qquad \text{in} \quad (W_0^{1,p}(\Omega))^* \qquad \text{for a} \quad \tau \in [0,1]. \tag{2.9}$$

Since $\|v_{n_k}\| = 1$ for all $k = 1, 2, \dots$, we have $\|v\| = 1$. However, this contradicts the assumption $\lambda < \lambda_1$. It proves that (2.4) holds, i.e., the homotopy T_τ is admissible. This completes the proof. □

3. THE CASE $p = 2$ AND $\lambda = \lambda_1$

Let us assume that $p = 2$ and consider the eigenvalue problem

$$\begin{cases} -\Delta u(x) = \lambda u(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

It is known that the eigenvalues of (3.1) form an increasing sequence

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots, \qquad \lambda_n \to \infty.$$

In fact, it is also possible to prove that λ_1 has multiplicity 1 (i.e., $\lambda_1 < \lambda_2$) and the corresponding eigenfunction $\varphi_1 \in W_0^{1,2}(\Omega)$ is positive in Ω. Moreover, we have

$$\int_\Omega \nabla\varphi_1(x) \nabla v(x) dx = \lambda_1 \int_\Omega \varphi_1(x) v(x) dx, \tag{3.2}$$

for any $v \in W_0^{1,2}(\Omega)$.

Now, We formulate the following Theorem.

Theorem 3.1. *Let $h : \Omega \times \mathbb{R} \to \mathbb{R}$ be a bounded Carathéodory function and satisfy the following conditions:*

(i) $\lim_{s\to+\infty} h(x,s) = h(x,+\infty)$, $\lim_{s\to-\infty} h(x,s) = h(x,-\infty)$, *for a.a.* $x \in \Omega$;

(ii) $h(x,-\infty) < h(x,s) < h(x,+\infty)$, *for a.a.* $x \in \Omega$ *and for all* $s \in \mathbb{R}$.

Then, the problem

$$\begin{cases} -\Delta u(x) = \lambda_1 u(x) + h(x,u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

has at least one weak solution if and only if

$$\int_\Omega h(x,-\infty)\varphi_1(x)dx < 0 < \int_\Omega h(x,+\infty)\varphi_1(x)dx. \tag{3.4}$$

Proof. For the sufficiency part we will follow a scheme similar to the proof of Theorem 2.3, but now $J, G, S : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ and

$$(J(u),v) := \int_\Omega \nabla u(x)\nabla v(x)dx = (u,v),$$

$$(G(u),v) := \int_\Omega u(x)v(x)dx,$$

$$(S(u),v) := \int_\Omega h(x,u(x))v(x)dx$$

for any $u, v \in W_0^{1,2}(\Omega)$. For $\delta > 0$ so small that $\lambda_1 + \delta < \lambda_2$ we define a homotopy

$$T_\tau(u) := u - \lambda_1 G(u) - (1-\tau)\delta G(u) - \tau S(u), \qquad \tau \in [0,1], \qquad u \in W_0^{1,2}(\Omega).$$

Performing all steps as in the proof of Theorem 2.3 we arrive at an analogue of (2.9), namely,

$$v - \big[\lambda_1 + (1-\tau)\delta\big]G(v) = 0, \qquad \|v\| = 1, \qquad \text{for a} \quad \tau \in [0,1],$$

This is a contradiction if $\tau \neq 1$, since $\lambda_1 + (1-\tau)\delta$ is not an eigenvalue ($\lambda_1 < \lambda_1 + (1-\tau)\delta < \lambda_2$) and $v \neq 0$.

Let us assume $\tau = 1$, i.e., $\tau_{n_k} \to 1$. Now, however, we have no contradiction, since λ_1 is an eigenvalue and

$$v - \lambda_1 G(v) = 0$$

has a solution with $\|v\| = 1$. Another step is necessary to reach a contradiction and to prove that the homotopy T_τ is admissible. We have to revise the last step when passing to the limit in

$$v_n - \lambda_1 G(v_n) - (1-\tau_n)\delta G(v_n) - \tau_n \frac{S(u_n)}{\|u_n\|} = 0$$

and employ special properties of S. Namely,

$$u_{n_k} - \lambda_1 G(u_{n_k}) - (1-\tau_{n_k})\delta G(u_{n_k}) - \tau_{n_k} S(u_{n_k}) = 0$$

is equivalent to the integral identity

$$\int_\Omega \nabla u_{n_k}(x)\nabla w(x)dx = \big[\lambda_1 + (1-\tau_{n_k})\delta\big] \int_\Omega u_{n_k}(x)w(x)dx + \tau_{n_k} \int_\Omega h(x,u_{n_k}(x))w(x)dx \tag{3.5}$$

for all $w \in W_0^{1,2}(\Omega)$. Taking $w = \varphi_1$ in (3.5) and using the fact that

$$\int_\Omega \nabla u_{n_k}(x)\nabla\varphi_1(x)dx = \lambda_1 \int_\Omega u_{n_k}(x)\varphi_1(x)dx,$$

(see (3.2)), we obtain

$$(\tau_{n_k}-1)\delta\int_\Omega u_{n_k}(x)\varphi_1(x)dx=\tau_{n_k}\int_\Omega h(x,u_{n_k}(x))\varphi_1(x)dx. \tag{3.6}$$

As above, $v_{n_k}:=\frac{u_{n_k}}{\|u_{n_k}\|}\to v$ in $W_0^{1,2}(\Omega)$ and $v=\kappa\varphi_1$ with a $\kappa\neq 0$. Assume that $\kappa>0$. Since $v_{n_k}\to\kappa\varphi_1$ in $W_0^{1,2}(\Omega)$, by the compact embedding $W_0^{1,2}(\Omega)\hookrightarrow\hookrightarrow L^2(\Omega)$, we have $v_{n_k}\to\kappa\varphi_1$ in $L^2(\Omega)$. Hence (at least for a subsequence) $v_{n_k}(x)\to\kappa\varphi_1>0$ a.e. in Ω, i.e., $u_{n_k}(x)\to+\infty$ a.e. in Ω. Passing to the limit in (3.6) and using $\tau_{n_k}\to 1_-$ and the Lebesgue Dominated Convergence Theorem, we obtain

$$\int_\Omega h(x,+\infty)\varphi_1(x)dx=\lim_{k\to\infty}(\tau_{n_k}-1)\delta\int_\Omega u_{n_k}(x)\varphi_1(x)dx\le 0.$$

This contradicts the second inequality in (3.4). Similarly, if $\kappa<0$, then (at least for a subsequence) $u_{n_k}(x)\to-\infty$ a.e. in Ω. Passing to the limit in (3.6), we obtain

$$\int_\Omega h(x,-\infty)\varphi_1(x)dx=\lim_{k\to\infty}(\tau_{n_k}-1)\delta\int_\Omega u_{n_k}(x)\varphi_1(x)dx\ge 0.$$

This contradicts the first inequality in (3.4). This proves that T_τ is admissible, and so (3.4) is sufficient for the existence of a weak solution of (3.3).

To prove that (3.4) is also necessary we proceed as follows. Let u_0 be a weak solution of (3.3), i.e.,

$$\int_\Omega \nabla u_0(x)\nabla v(x)dx=\lambda_1\int_\Omega u_0(x)v(x)dx+\int_\Omega h(x,u_0(x))v(x)dx,$$

for any $v\in W_0^{1,2}(\Omega)$. Take $v=\varphi_1$, then

$$\int_\Omega \nabla u_0(x)\nabla \varphi_1(x)dx=\lambda_1\int_\Omega u_0(x)\varphi_1(x)dx+\int_\Omega h(x,u_0(x))\varphi_1(x)dx.$$

Using (3.2), we have

$$\int_\Omega h(x,u_0(x))\varphi_1(x)dx=0.$$

By assumption (ii),

$$h(x,-\infty)<h(x,u_0(x))<h(x,+\infty). \tag{3.7}$$

Multiply (3.7) by $\varphi_1(>0)$ and integrate. Then

$$\int_\Omega h(x,-\infty)\varphi_1(x)dx<0<\int_\Omega h(x,+\infty)\varphi_1(x)dx,$$

and we have the result. □

Similarly to the proof of Theorem 3.1, we can prove the following

Theorem 3.2. *Let $h:\Omega\times\mathbb{R}\to\mathbb{R}$ be a bounded Carathéodory function and satisfy the following conditions:*

(i) $\lim_{s\to+\infty}h(x,s)=h(x,+\infty)$, $\lim_{s\to-\infty}h(x,s)=h(x,-\infty)$, *for a.a.* $x\in\Omega$;

(ii) $h(x,+\infty)<h(x,s)<h(x,-\infty)$, *for a.a.* $x\in\Omega$ *and for all* $s\in\mathbb{R}$.

Then, the problem (3.3) *has at least one weak solution if and only if*

$$\int_\Omega h(x,+\infty)\varphi_1(x)dx<0<\int_\Omega h(x,-\infty)\varphi_1(x)dx. \tag{3.8}$$

Remark 3.3. It is possible to solve the problem (3.3) directly by means of the Leray-Schauder degree theory as well, since the operator J in the proof of Theorem 3.1 is just an identity on $W_0^{1,2}(\Omega)$.

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