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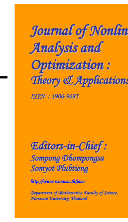

ITERATIVE ALGORITHM FOR A SYSTEM OF MULTI-VALUED VARIATIONAL INCLUSIONS INVOLVING (B, ϕ)-MONOTONE MAPPINGS IN BANACH SPACES

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ABSTRACT. In this paper, we introduce a new class of resolvent mappings for (B, ϕ)-monotone mappings in Banach space, which is a natural and important generalization of a class of resolvent mappings studied in [X.-P. Luo, N.-J. Huang; A new class of variational inclusions with B-monotone operators in Banach spaces, J. Comput. Appl. Math. 233 (2010), 1888-1896]. We study some properties of this new class of resolvent mappings and by making use of it, we discuss the existence and iterative approximation of solutions of a system of multi-valued variational inclusions. The method and results presented in this paper improve and generalize many known results in the literature.

KEYWORDS : System of multi-valued variational inclusions; (B, ϕ)-monotone mappings; Iterative algorithm and convergence analysis.
AMS Subject Classification: 47J22, 49J40.

1. INTRODUCTION

In 1968, Brézis [5] initiated the study of the existence theory of a class of variational inequalities later known as variational inclusions, using resolvent (proximal) mapping. Variational inclusions include variational inequalities as special cases. For applications of variational inclusions, see [4, 10]. In 1994, Hassouni and Moudafi [15] discussed iterative approximation of solutions for an important class of variational inclusions using resolvent mapping. Since then various resolvent mappings have been introduced and used to develop the iterative methods for studying the existence and iterative approximation of solutions of variational inclusions, see for example [1–3, 6–9, 11–14, 16, 18–26, 29–35].

Very recently Luo and Huang [25, 26] introduced and studied the classes of (H, ϕ)-η-monotone mappings and B-monotone mappings in Banach space, respectively, and discussed their properties. Using these classes of resolvent mappings for

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(H, ϕ)-η-monotone mappings and B-monotone mappings, they studied the convergence analysis of the iterative algorithms for some classes of variational inclusions.

In this paper, we introduce a class of resolvent mappings for (B, ϕ)-monotone mappings in Banach space, which is a natural and important generalization of the class of resolvent mappings studied in [26]. We study some properties of this new class of resolvent mappings and by making use of it, we discuss the existence and iterative approximation of solutions of a system of multi-valued variational inclusions. The method and results presented in this paper improve and generalize many known results in the literature.

2. PRELIMINARIES

Let X be a real Banach space with the topological dual space X^* and $\langle\cdot,\cdot\rangle$ denote the dual pair between X and its dual X^* and 2^X denote the family of all nonempty subsets of X. The *normalized duality* mapping $J : X \longrightarrow 2^{X^*}$ is defined by

$$J(x) = \{f \in X^*,\ \langle f, x\rangle = \|f\|\|x\|,\ \|f\| = \|x\|\},\ \forall x \in X.$$

The *modulus of smoothness* of X is the function $\rho_X : [0,\infty) \longrightarrow [0,\infty)$ defined by

$$\rho_X(t) = \sup\Big\{\frac{(\|x+y\| + \|x-y\|)}{2} - 1 :\ x, y \in X, \|x\| = 1, \|y\| = t\Big\}.$$

A Banach space X is called *uniformly smooth* if

$$\lim_{t\longrightarrow 0}\frac{\rho_X(t)}{t} = 0.$$

We denote by $CB(X)$ the family of all nonempty, closed and bounded subsets of X and $\mathcal{D}(\cdot,\cdot)$ is the Hausdorff metric on $CB(X)$ defined by

$$\mathcal{D}(A, B) = \max\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\},\ A, B \in CB(X).$$

Definition 2.1. [31] Let $A : X \longrightarrow X^*$ be a single-valued mapping. A is said to be:

(i) *monotone*, if

$$\langle A(x) - A(y), x - y\rangle \geq 0,\ \forall x, y \in X;$$

(ii) *strictly monotone*, if

$$\langle A(x) - A(y), x - y\rangle > 0,\ \forall x, y \in X,$$

and equal to 0 if and only if $x = y$;

(iii) γ-*strongly monotone*, if there exists a constant $\gamma > 0$ such that

$$\langle A(x) - A(y), x - y\rangle \geq \gamma\|x-y\|,\ \forall x, y \in X;$$

(iv) m-*relaxed monotone*, if there exists a constant $m > 0$ such that

$$\langle A(x) - A(y), x - y\rangle \geq -m\|x-y\|,\ \forall x, y \in X;$$

(v) δ-*Lipschitz continuous*, if there exists a constant $\delta > 0$ such that

$$\|A(x) - A(y)\| \leq \delta\|x-y\|,\ \forall x, y \in X.$$

Definition 2.2. [26] Let $B : X \longrightarrow X^*$, $\phi : X^* \longrightarrow X^*$, $f, g : X \longrightarrow X$ be single-valued mappings, and let $M : X \times X \longrightarrow 2^{X^*}$ be a multi-valued mapping. Then

(i) $M(f,\cdot)$ is said to be α-*strongly monotone* with respect to f, if there exists a constant $\alpha > 0$ such that

$$\langle u-v, x-y\rangle \geq \alpha\|x-y\|^2,\ \forall x,y,w\in X, u\in M(f(x),w), v\in M(f(y),w);$$

(ii) $M(\cdot,g)$ is said to be β-*relaxed monotone* with respect to g if there exists a constant $\beta > 0$ such that

$$\langle u-v, x-y\rangle \geq -\beta\|x-y\|^2,\ \forall x,y,w\in X, u\in M(w,g(x)), v\in M(w,g(y));$$

(iii) $M(\cdot,\cdot)$ is said to be $\alpha\beta$-*symmetric monotone* with respect to f and g if $M(f,\cdot)$ is α-strongly monotone with respect to f and $M(\cdot,g)$ is β-relaxed monotone with respect to g with $\alpha \geq \beta$ and $\alpha = \beta$ if and only $x = y$ $\forall x,y\in X$.

Lemma 2.3. *[28] Let X be a real Banach space and let $J : X \longrightarrow 2^{X^*}$ be the normalized duality mapping. Then for any given $x,y\in X$, the following inequality holds:*

$$\|x+y\|^2 \ \leq\ \|x\|^2 + 2\langle y, j(x+y)\rangle,\ \forall j(x+y)\in J(x+y).$$

Lemma 2.4. *[27] Let X be a complete metric space; let $C(X)$ be the family of all nonempty compact subsets of X and let $W : X \longrightarrow C(X)$ be a multi-valued mapping. Then for any given $x,y\in X,\ u\in W(x)$, there exists $v\in W(y)$ such that*

$$d(u,v) \leq \mathcal{D}(W(x),W(y)),$$

where $\mathcal{D}(\cdot,\cdot)$ is Hausdorff metric on $C(X)$.

Lemma 2.5. *[27] Let X be a complete metric space and let $W : X \longrightarrow CB(X)$ be a multi-valued mapping. Then for any $\epsilon > 0$ and for any given $x,y\in X,\ u\in W(x)$, there exists $v\in W(y)$ such that*

$$d(u,v) \leq (1+\epsilon)\mathcal{D}(W(x),W(y)),$$

where $\mathcal{D}(\cdot,\cdot)$ is Hausdorff metric on $CB(X)$.

3. (B,ϕ)-MONOTONE MAPPINGS

First, we define the notion of (B,ϕ)-monotone mappings.

Definition 3.1. Let $B : X \longrightarrow X^*,\ \phi : X^* \longrightarrow X^*,\ f,g : X \longrightarrow X$ be single-valued mappings, and let $M : X\times X \longrightarrow 2^{X^*}$ be a multi-valued mapping. Then M is said to be (B,ϕ)-monotone if $\phi\circ M$ be $\alpha\beta$-*symmetric monotone* with respect to f and g, and $(B+\phi\circ M(f,g))(X) = X^*$.

Remark 3.2. (i) If $\phi\circ M(f,g) = \lambda M(f,g)$, for $\lambda > 0$ and M be $\alpha\beta$-*symmetric monotone* with respect to f and g, then (B,ϕ)-monotone mapping reduces to the B-monotone mapping considered in [26].

(ii) If $\phi\circ M(f,g) = \lambda M$, for $\lambda > 0$ and M be relaxed monotone then (B,ϕ)-monotone mapping reduces to the A-monotone mapping considered in [11].

Theorem 3.3. *Let $f,g : X \longrightarrow X$ and $\phi : X^* \longrightarrow X^*$ be single-valued mappings; let $B : X \longrightarrow X^*$ be a strictly monotone mapping and let $M : X\times X \longrightarrow 2^{X^*}$ be a (B,ϕ)-monotone mapping. If $\langle u-v, x-y\rangle \geq 0$ holds for all $(y,v)\in$ Graph $(\phi\circ M(f,g))$, then $u\in \big(\phi\circ M(f,g)\big)(x)$, where Graph$(\phi\circ M(f,g)) = \{(x,x^*)\in X\times X^* :\ x^*\in (\phi\circ M(f,g))(x)\}$.*

Proof. Suppose that there exists (x_0, u_0) such that

$$\langle u_0 - v, x_0 - y\rangle \geq 0,\ \forall (y, v) \in \textit{Graph}\ (\phi \circ M(f, g)). \tag{3.1}$$

Since M is (B, ϕ)-monotone, we know that $(B+\phi\circ M(f,g))(X) = X^*$ and so there exists $(x_1, u_1) \in \textit{Graph}\ (\phi \circ M(f, g))$ such that

$$B(x_1) + u_1 = B(x_0) + u_0. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$0 \leq \langle u_0 - u_1, x_0 - x_1\rangle = -\langle B(x_0) - B(x_1), x_0 - x_1\rangle.$$

But the strictly monotonicity of B implies that $x_1 = x_0$. By (3.2) we also observe that $u_1 = u_0$. Hence $(x_0, u_0) \in \textit{Graph}\ (\phi \circ M(f, g))$, that is, $u_0 \in \big(\phi \circ M(f, g)\big)(x_0)$. This completes the proof. □

Theorem 3.4. *Let $f, g : X \longrightarrow X$ and $\phi : X^* \longrightarrow X^*$ be single-valued mappings; let $B : X \longrightarrow X^*$ be a strictly monotone mapping and let $M : X \times X \longrightarrow 2^{X^*}$ be a (B, ϕ)-monotone mapping. Then $(B + \phi \circ M(f, g))^{-1}$ be a single-valued mapping.*

Proof. For any $x^* \in X^*$, let $x, y \in (B + \phi \circ M(f, g))^{-1}(x^*)$, then it follows that

$$x^* - B(x) \in \phi \circ M(f(x), g(x))$$
$$x^* - B(y) \in \phi \circ M(f(y), g(y)).$$

Pick any given $w \in \phi \circ M(f(y), g(x))$, since $\phi \circ M$ be *$\alpha\beta$-symmetric monotone* with respect to f and g, we have

$$(\alpha - \beta)\|x - y\|^2 \leq \langle x^* - B(x) - w + w - (x^* - B(y)), x - y\rangle.$$

It follows from $\alpha \geq \beta$ and the strictly monotonicity of B that $x = y$. Thus $(B + \phi \circ M(f, g))^{-1}$ be a single-valued mapping. This completes the proof. □

Based on Theorems 3.3-3.4, we can define the following resolvent mapping $R^B_{M(\cdot,\cdot),\phi}$.

Definition 3.5. Let X be a reflexive Banach space with the dual space X^*. Let $f, g : X \longrightarrow X$, $\phi : X^* \longrightarrow X^*$ be single-valued mappings; let $B : X \longrightarrow X^*$ be a σ-strongly monotone mapping and let $M : X \times X \longrightarrow 2^{X^*}$ be a (B, ϕ)-monotone mapping. A resolvent mapping $R^B_{M(\cdot,\cdot),\phi} : X^* \longrightarrow X$ is defined by $R^B_{M(\cdot,\cdot),\phi}(x^*) = (B + \phi \circ M(f, g))^{-1}(x^*),\ \forall x^* \in X^*$.

Theorem 3.6. *Let X be a reflexive Banach space with the dual space X^*. Let $f, g : X \longrightarrow X$, $\phi : X^* \longrightarrow X^*$ be single-valued mappings; let $B : X \longrightarrow X^*$ be a σ-strongly monotone mapping and let $M : X \times X \longrightarrow 2^{X^*}$ be a (B, ϕ)-monotone mapping. Then the resolvent mapping $R^B_{M(\cdot,\cdot),\phi} : X^* \longrightarrow X$ is Lipschitz continuous with constant $\frac{1}{(\alpha-\beta+\sigma)}$, i.e.,*

$$\|R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\| \leq \frac{1}{(\alpha - \beta + \sigma)}\|x^* - y^*\|,\ \forall x^*, y^* \in X^*.$$

Proof. Let $x^*, y^* \in X^*$. It follows that

$$R^B_{M(\cdot,\cdot),\phi}(x^*) = (B + \phi \circ M(f, g))^{-1}(x^*)$$
$$R^B_{M(\cdot,\cdot),\phi}(y^*) = (B + \phi \circ M(f, g))^{-1}(y^*)$$

and so

$$x^* - B(R^B_{M(\cdot,\cdot),\phi}(x^*)) \in \phi \circ M\big(f(R^B_{M(\cdot,\cdot),\phi}(x^*)), g(R^B_{M(\cdot,\cdot),\phi}(x^*))\big)$$

$$y^* - B(R^B_{M(\cdot,\cdot),\phi}(y^*)) \in \phi \circ M\big(f(R^B_{M(\cdot,\cdot),\phi}(y^*)), g(R^B_{M(\cdot,\cdot),\phi}(y^*))\big).$$

Pick any given $w \in \phi \circ M(f(R^B_{M\left(\cdot,\cdot\right),\phi}(y^*)), g(R^B_{M(\cdot,\cdot),\phi}(x^*))\big)$. Since $\phi \circ M$ be *$\alpha\beta$-symmetric monotone* with respect to f and g, we have

$$\begin{aligned}
(\alpha-\beta)&\|R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\|^2 \\
&\le \langle (x^* - B(R^B_{M(\cdot,\cdot),\phi}(x^*))) - w + w \\
&\quad -(y^* - B(R^B_{M(\cdot,\cdot),\phi}(y^*))), R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\rangle, \\
&= \langle x^* - y^* - B(R^B_{M(\cdot,\cdot),\phi}(x^*)) \\
&\quad +B(R^B_{M(\cdot,\cdot),\phi}(y^*)), R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\rangle.
\end{aligned}$$

Since B be a σ- strongly monotone mapping, then

$$\begin{aligned}
\|x^* - y^*\| &\|R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\| \\
&\ge \langle x^* - y^*, R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\rangle \\
&\ge (\alpha-\beta)\|R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\| \\
&\quad +\langle B(R^B_{M(\cdot,\cdot),\phi}(x^*) - B(R^B_{M(\cdot,\cdot),\phi}(y^*), R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\rangle \\
&\ge (\alpha-\beta+\sigma)\|R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\|^2
\end{aligned}$$

and so

$$\|R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\| \le \frac{1}{(\alpha-\beta+\sigma)}\|x^* - y^*\|,\ \forall x^*, y^* \in X^*.$$

This completes the proof. □

4. SYSTEM OF MULTI-VALUED VARIATIONAL INCLUSIONS

Throughout the rest of this paper, unless otherwise stated for each $i = 1, 2$, we assume that X_i be a real Banach space with norm $\|\cdot\|_i$ and denote the duality pairing between X_i and X_i^* by $\langle\cdot,\cdot\rangle_i$. Let $A_i : X_i \longrightarrow X_i^*,\ p_i, f_i, g_i : X_i \longrightarrow X_i,\ F_i : X_i \times X_i \longrightarrow X_i$ be single-valued mappings and $M_i : X_i \times X_i \longrightarrow 2^{X_i^*},\ W_i : X_1 \longrightarrow CB(X_1)$ and let $V_i : X_2 \longrightarrow CB(X_2)$ be multi-valued mappings. We shall study the following system of multi-valued variational inclusions (in short, SMVI). For given $\theta_i \in X_i^*$, the zero element, find $(x_1, x_2) \in (X_1, X_2),\ w_i \in W_i(x_1),\ v_i \in V_i(x_2)$ such that

$$\begin{cases} \theta_1 \in A_1(x_1 - p_1(x_1)) + F_1(w_1, v_1) + M_1(f_1(x_1), g_1(x_1)) \\ \theta_2 \in A_2(x_2 - p_2(x_2)) + F_2(w_2, v_2) + M_2(f_2(x_2), g_2(x_2)). \end{cases} \tag{4.1}$$

We remark that by giving suitable choices of mappings $A_i, p_i, F_i, M_i, f_i, g_i, W_i, V_i$ and of spaces $X_i,\ X_i^*,\ (i = 1, 2)$, we can observe that SMVI (4.1) reduces to many new and previously known systems of variational inclusions, systems of variational inequalities, variational inclusions and variational inequalities in Banach spaces as well as in Hilbert spaces, see for example, Kazmi and Khan [18], Kazmi and Khan [19], Kazmi and Bhat [17], Luo and Haung [25], Wang and Ding [30].

Theorem 4.1. *For each $i = 1, 2$, let $\phi_i : X_i^* \longrightarrow X_i^*$ be a single-valued mapping satisfying $\phi_i(x_i + y_i) = \phi_i(x_i) + \phi_i(y_i)$ and $ker(\phi_i) = \{\theta_i\}$, where $ker(\phi_i) = \{x_i \in X_i^* : \phi_i(x_i) = \theta_i\}$. Let $A_i : X_i \longrightarrow X_i^*,\ p_i, f_i, g_i : X_i \longrightarrow X_i,\ F_i : X_i \times X_i \longrightarrow X_i$ be single-valued mappings and let $M_i : X_i \times X_i \longrightarrow 2^{X_i^*},\ W_i : X_1 \longrightarrow CB(X_1)$ and $V_i : X_2 \longrightarrow CB(X_2)$ be multi-valued mappings. Let $B_i : X_i \longrightarrow X_i^*$ be a σ_i-strongly monotone mapping. Then $(x_1, x_2, w_1, w_2, v_1, v_2)$ is a solution of SMVI (4.1) if and only if*

$$x_i = R^{B_i}_{M_i(\cdot,\cdot),\phi_i}[B_i(x_i) - \phi_i \circ (A_i(x_i - p_i(x_i)) + F_i(w_i, v_i))] \tag{4.2}$$

where $w_i \in W_i(x_1),\ v_i \in V_i(x_2)$ and $R^{B_i}_{M_i(\cdot,\cdot),\phi_i} = (B_i + \phi_i \circ M_i(f_i, g_i))^{-1}$.

Proof. By definition of $R^{B_i}_{M_i(\cdot,\cdot),\phi_i}$, we know that (4.2) holds if and only if

$$B_i(x_i) - \phi_i \circ (A_i(x_i - p_i(x_i)) + F_i(w_i, v_i)) \in (B_i + \phi_i \circ M_i(f_i, g_i))(x_i)$$

which is equivalent to

$$-\phi_i \circ (A_i(x_i - p_i(x_i)) + F_i(w_i, v_i)) \in \phi_i \circ M_i(f_i(x_i), g_i(x_i)).$$

It follows from $\phi_i(x_i + y_i) = \phi_i(x_i) + \phi_i(y_i)$ that (4.2) holds if and only if

$$\theta_i \in \phi_i \circ [A_i(x_i - p_i(x_i)) + F_i(w_i, v_i) + M_i(f_i(x_i), g_i(x_i))].$$

Since $ker(\phi_i) = \{\theta_i\}$, (4.2) holds if and only if

$$\theta_i \in A_i(x_i - p_i(x_i)) + F_i(w_i, v_i) + M_i(f_i(x_i), g_i(x_i)).$$

□

Based on Theorem 4.1, we construct the following iterative algorithm for solving SMVI (4.1):

Iterative Algorithm 4.1. For each $i = 1, 2$, given $x_i^0 \in X_i,\ w_i^0 \in W_i(x_1^0),\ v_i^0 \in V_i(x_2^0)$, compute the sequences $\{x_i^n\}, \{w_i^n\}, \{v_i^n\}$ defined by the iterative schemes:

$$x_i^{n+1} = R^{B_i}_{M_i(\cdot,\cdot),\phi_i}\Big[B_i(x_i^n) - \phi_i \circ [A_i(x_i^n - p_i(x_i^n)) + F_i(w_i^n, v_i^n)]\Big],$$

$$w_i^n \in W_i(x_1^n),\ \|w_i^{n+1} - w_i^n\| \le (1 + n^{-1})\mathcal{D}_1\Big(W_i(x_1^{n+1}), W_i(x_1^n)\Big),$$

$$v_i^n \in V_i(x_2^n),\ \|v_i^{n+1} - v_i^n\| \le (1 + n^{-1})\mathcal{D}_2\Big(V_i(x_1^{n+1}), V_i(x_1^n)\Big),$$

for all $n = 0, 1, 2, ...,$.

Now we give the sufficient conditions which guarantee the convergence of the iterative sequences generated by Iterative Algorithm 4.1.

Theorem 4.2. *For each $i = 1, 2$, let X_i be a reflexive Banach space with dual space X_i^*. Let $\phi_i : X_i^* \longrightarrow X_i^*$ be a λ_i-Lipschitz continuous satisfying $\phi_i(x_i + y_i) = \phi_i(x_i) + \phi_i(y_i)$ and $ker(\phi_i) = \{\theta_i\}$. Let $f_i, g_i : X_i \longrightarrow X_i$ be single-valued mappings; let $B_i : X_i \longrightarrow X_i^*$ be a σ_i-strongly monotone and δ_i-Lipschitz continuous mapping and let $M_i : X_i \times X_i \longrightarrow 2^{X_i^*}$ be a (B_i, ϕ_i)-monotone mapping. Let $A_i : X_i \longrightarrow X_i^*$ be γ_i-Lipschitz continuous mapping; let $p_i : X_i \longrightarrow X_i$ be a m_i-strongly accretive and ξ_i-Lipschitz continuous mapping; let $W_i : X_1 \longrightarrow CB(X_1)$ be a $\mathcal{D}_1$-Lipschitz continuous mapping with respect to $\lambda_{W_i} > 0$ and let $V_i : X_2 \longrightarrow CB(X_2)$ be a $\mathcal{D}_2$-Lipschitz continuous mapping with respect to $\lambda_{V_i} > 0$ respectively. Let the mapping*

$F_i : X_i \times X_i \longrightarrow X_i$ be a $\bar{\alpha}_i$-Lipschitz continuous in the first argument and $\bar{\beta}_i$-Lipschitz continuous in the second argument. Suppose that the following conditions are satisfied:

$$k_1 = \frac{1}{(\alpha_1 - \beta_1 + \sigma_1)}\Big[\delta_1 + \lambda_1\bar{\alpha}_1\lambda_{W_1} + \lambda_1\gamma_1\sqrt{1 - 2m_1 + 2\xi_1(2+\xi_1)} + \lambda_1\bar{\beta}_1\lambda_{V_1}\Big] < 1,$$
$$k_2 = \frac{1}{(\alpha_2 - \beta_2 + \sigma_2)}\Big[\delta_2 + \lambda_2\bar{\beta}_2\lambda_{V_2} + \lambda_2\gamma_2\sqrt{1 - 2m_2 + 2\xi_2(2+\xi_2)} + \lambda_2\bar{\alpha}_2\lambda_{W_1}\Big] < 1. \tag{4.3}$$

Then, for each $i = 1, 2$, the iterative sequences $\{x_i^n\}, \{w_i^n\}, \{v_i^n\}$ generated by Iterative Algorithm 4.1, converge strongly to $x_1, x_2, w_1, w_2, v_1, v_2$, respectively and $(x_1, x_2, w_1, w_2, v_1, v_2)$ is a solution of SMVI (4.1).

Proof. By Iterative Algorithm 4.1 and Theorem 3.6, we have

$$\begin{aligned}\Big\|x_1^{n+1} - x_1^n\Big\|_1 &= \Big\|R^{B_1}_{M_1(\cdot,\cdot),\phi_1}\Big[B_1(x_1^n) - \phi_1\circ[A_1(x_1^n - p_1(x_1^n)) + F_1(w_1^n, v_1^n)]\Big] \\ &\quad - R^{B_1}_{M_1(\cdot,\cdot),\phi_1}\Big[B_1(x_1^{n-1}) - \phi_1\circ[A_1(x_1^{n-1} - p_1(x_1^{n-1})) + F_1(w_1^{n-1}, v_1^{n-1})]\Big]\Big\|_1 \\ &\le \frac{1}{(\alpha_1 - \beta_1 + \sigma_1)}\Big\|\Big[B_1(x_1^n) - \phi_1\circ[A_1(x_1^n - p_1(x_1^n)) + F_1(w_1^n, v_1^n)] - B_1(x_1^{n-1}) \\ &\quad + \phi_1\circ[A_1(x_1^{n-1} - p_1(x_1^{n-1})) + F_1(w_1^{n-1}, v_1^{n-1})]\Big]\Big\|_1 \\ &\le \frac{1}{(\alpha_1 - \beta_1 + \sigma_1)}\Big(\|B_1(x_1^n) - B_1(x_1^{n-1})\|_1 + \lambda_1\|A_1(x_1^n - p_1(x_1^n)) - A_1(x_1^{n-1} - p_1(x_1^{n-1}))\|_1 \\ &\quad + \lambda_1\|F_1(w_1^n, v_1^n) - F_1(w_1^{n-1}, v_1^{n-1})\|_1\Big).\end{aligned} \tag{4.4}$$

By Lipschitz continuity of $\phi_1, A_1, B_1, F_1, W_1, V_1$, we have

$$\|B_1(x_1^n) - B_1(x_1^{n-1})\|_1 \le \delta_1\|x_1^{n+1} - x_1^n\|_1, \tag{4.5}$$

and

$$\begin{aligned}&\|F_1(w_1^n, v_1^n) - F_1(w_1^{n-1}, v_1^{n-1})\|_1 \\ &\le \|F_1(w_1^n, v_1^n) - F_1(w_1^{n-1}, v_1^n)\|_1 + \|F_1(w_1^{n-1}, v_1^n) - F_1(w_1^{n-1}, v_1^{n-1})\|_2 \\ &\le \bar{\alpha}_1\|w_1^n - w_1^{n-1}\|_1 + \bar{\beta}_1\|v_1^n - v_1^{n-1}\|_2 \\ &\le \bar{\alpha}_1(1+n^{-1})\mathcal{D}_1\Big(W_1(x_1^n), W_1(x_1^{n-1})\Big) + \bar{\beta}_1(1+n^{-1})\mathcal{D}_2\Big(V_1(x_2^n), V_1(x_2^{n-1})\Big) \\ &\le \bar{\alpha}_1(1+n^{-1})\lambda_{W_1}\|x_1^n - x_1^{n-1}\|_1 + \bar{\beta}_1(1+n^{-1})\lambda_{V_1}\|x_2^n - x_2^{n-1}\|_2 \\ &\le (1+n^{-1})\Big[\bar{\alpha}_1\lambda_{W_1}\|x_1^n - x_1^{n-1}\|_1 + \bar{\beta}_1\lambda_{V_1}\|x_2^n - x_2^{n-1}\|_2\Big],\end{aligned} \tag{4.6}$$

$$\|A_1(x_1^n - p_1(x_1^n)) - A_1(x_1^{n-1} - p_1(x_1^{n-1}))\|_1 \le \gamma_1\|x_1^n - x_1^{n-1} - (p_1(x_1^n) - p_1(x_1^{n-1}))\|_1. \tag{4.7}$$

Since p_1 is a m_1-strongly accretive and ξ_1-Lipschitz continuous mapping, then we have

$$\begin{aligned}&\|x_1^n - x_1^{n-1} - (p_1(x_1^n) - p_1(x_1^{n-1}))\|_1^2 \\ &\le \|x_1^n - x_1^{n-1}\|_1^2 - 2\Big\langle p_1(x_1^n) - p_1(x_1^{n-1}), j\Big((x_1^n - x_1^{n-1}) - (p_1(x_1^n) - p_1(x_1^{n-1}))\Big)\Big\rangle \\ &\le \|x_1^n - x_1^{n-1}\|_1^2 - 2\Big\langle p_1(x_1^n) - p_1(x_1^{n-1}), j(x_1^n - x_1^{n-1})\Big\rangle\end{aligned}$$

$$+2\Big\langle p_1(x_1^n)-p_1(x_1^{n-1}),-j\Big((x_1^n-x_1^{n-1})-(p_1(x_1^n)-p_1(x_1^{n-1}))\Big)+j((x_1^n-x_1^{n-1})\Big\rangle$$
$$\leq \|x_1^n-x_1^{n-1}\|_1^2-2m_1\|x_1^n-x_1^{n-1}\|_1^2+2\|p_1(x_1^n)-p_1(x_1^{n-1}\|_1$$
$$\Big[\|x_1^n-x_1^{n-1}\|_1+\|p_1(x_1^n)-p_1(x_1^{n-1}\|_1+\|x_1^n-x_1^{n-1}\|_1\Big]$$
$$\leq \|x_1^n-x_1^{n-1}\|_1^2-2m_1\|x_1^n-x_1^{n-1}\|_1^2+2\xi_1(2+\xi_1)\|x_1^n-x_1^{n-1}\|_1^2$$
$$\leq \Big(1-2m_1+2\xi_1(2+\xi_1)\Big)\|x_1^n-x_1^{n-1}\|_1^2$$
$$\|x_1^n-x_1^{n-1}-(p_1(x_1^n)-p_1(x_1^{n-1}))\|_1\leq\sqrt{1-2m_1+2\xi_1(2+\xi_1)}\|x_1^n-x_1^{n-1}\|_1. \quad (4.8)$$

It follows from (4.4)-(4.8), we have

$$\|x_1^{n+1}-x_1^n\Big\|_1\leq\frac{1}{(\alpha_1-\beta_1+\sigma_1)}\Big[\delta_1\|x_1^n-x_1^{n-1}\|_1+\lambda_1(1+n^{-1})\Big(\bar{\alpha}_1\lambda_{W_1}\|x_1^n-x_1^{n-1}\|_1$$
$$+\bar{\beta}_1\lambda_{V_1}\|x_2^n-x_2^{n-1}\|_2\Big)+\lambda_1\gamma_1\sqrt{1-2m_1+2\xi_1(2+\xi_1)}\|x_1^n-x_1^{n-1}\|_1\Big].$$

Hence we have

$$\|x_1^{n+1}-x_1^n\Big\|_1\leq\frac{1}{(\alpha_1-\beta_1+\sigma_1)}\Big[\Big(\delta_1+\lambda_1\bar{\alpha}_1\lambda_{W_1}(1+n^{-1})+\lambda_1\gamma_1\sqrt{1-2m_1+2\xi_1(2+\xi_1)}\Big)$$
$$\times\|x_1^n-x_1^{n-1}\|_1+\lambda_1\bar{\beta}_1\lambda_{V_1}(1+n^{-1})\|x_2^n-x_2^{n-1}\|_2\Big)\Big]. \quad (4.9)$$

Next

$$\Big\|x_2^{n+1}-x_2^n\Big\|_2=\Big\|R^{B_2}_{M_2(\cdot,\cdot),\phi_2}\Big[B_2(x_2^n)-\phi_2\circ[A_2(x_2^n-p_2(x_2^n))+F_2(w_2^n,v_2^n)]\Big]$$
$$-R^{B_2}_{M_2(\cdot,\cdot),\phi_2}\Big[B_2(x_2^{n-1})-\phi_2\circ[A_2(x_2^{n-1}-p_2(x_2^{n-1}))+F_2(w_2^{n-1},v_2^{n-1})]\Big]\Big\|_2$$
$$\leq\frac{1}{(\alpha_2-\beta_2+\sigma_2)}\Big\|\Big[B_2(x_2^n)-\phi_2\circ[A_2(x_2^n-p_2(x_2^n))+F_2(w_2^n,v_2^n)]-B_2(x_2^{n-1})$$
$$+\phi_2\circ[A_2(x_2^{n-1}-p_2(x_2^{n-1}))+F_2(w_2^{n-1},v_2^{n-1})]\Big]\Big\|_2$$
$$\leq\frac{1}{(\alpha_2-\beta_2+\sigma_2)}\Big(\|B_2(x_2^n)-B_2(x_2^{n-1})\|_2+\lambda_2\|A_2(x_2^n-p_2(x_2^n))-A_2(x_2^{n-1}-p_2(x_2^{n-1}))\|_2$$
$$+\lambda_2\|F_2(w_2^n,v_2^n)-F_2(w_2^{n-1},v_2^{n-1})\|_2\Big). \quad (4.10)$$

By Lipschitz continuity of $\phi_2, A_2, B_2, F_2, W_2, V_2$, we have

$$\|B_2(x_2^n)-B_2(x_2^{n-1})\|_2\leq\delta_2\|x_1^{n+1}-x_1^n\|_2, \quad (4.11)$$

and

$$\|F_2(w_2^n,v_2^n)-F_2(w_2^{n-1},v_2^{n-1})\|_2$$
$$\leq\|F_2(w_2^n,v_2^n)-F_2(w_1^{n-1},v_2^n)\|_1+\|F_2(w_2^{n-1},v_2^n)-F_2(w_2^{n-1},v_2^{n-1})\|_2$$
$$\leq\bar{\alpha}_2\|w_2^n-w_2^{n-1}\|_1+\bar{\beta}_2\|v_2^n-v_2^{n-1}\|_2$$
$$\leq\bar{\alpha}_2(1+n^{-1})\mathcal{D}_1\Big(W_2(x_1^n),W_2(x_1^{n-1})\Big)+\bar{\beta}_2(1+n^{-1})\mathcal{D}_2\Big(V_2(x_2^n),V_2(x_2^{n-1})\Big)$$
$$\leq\bar{\alpha}_2(1+n^{-1})\lambda_{W_2}\|x_1^n-x_1^{n-1}\|_1+\bar{\beta}_2(1+n^{-1})\lambda_{V_2}\|x_2^n-x_2^{n-1}\|_2$$
$$\leq(1+n^{-1})\Big[\bar{\alpha}_2\lambda_{W_2}\|x_1^n-x_1^{n-1}\|_1+\bar{\beta}_2\lambda_{V_2}\|x_2^n-x_2^{n-1}\|_2\Big] \quad (4.12)$$
$$\|A_2(x_2^n-p_2(x_2^n))-A_2(x_2^{n-1}-p_2(x_2^{n-1}))\|_2\leq\gamma_2\|x_2^n-x_2^{n-1}-(p_2(x_2^n)-p_2(x_2^{n-1}))\|_2. \quad (4.13)$$

Since p_2 is a m_2-strongly accretive and ξ_2-Lipschitz continuous mapping, then we have

$$\|x_2^n-x_2^{n-1}-(p_2(x_2^n)-p_2(x_2^{n-1}))\|_2 \le \sqrt{1-2m_2+2\xi_2(2+\xi_2)}\|x_2^n-x_2^{n-1}\|_2. \quad (4.14)$$

It follows from (4.10)-(4.14), we have

$$\|x_2^{n+1}-x_2^n\|_2 \le \frac{1}{(\alpha_2-\beta_2+\sigma_2)}\Big[\delta_2\|x_2^n-x_2^{n-1}\|_2+\lambda_2(1+n^{-1})\Big(\bar{\alpha}_2\lambda_{W_2}\|x_1^n-x_1^{n-1}\|_1 + \bar{\beta}_2\lambda_{V_2}\|x_2^n-x_2^{n-1}\|_2\Big)+\lambda_2\gamma_2\sqrt{1-2m_2+2\xi_2(2+\xi_2)}\|x_2^n-x_2^{n-1}\|_2\Big]$$

$$\|x_2^{n+1}-x_2^n\|_2 \le \frac{1}{(\alpha_1-\beta_1+\sigma_1)}\Big[\Big(\delta_2+\lambda_2\bar{\beta}_2\lambda_{V_2}(1+n^{-1})+\lambda_2\gamma_2\sqrt{1-2m_2+2\xi_2(2+\xi_2)}\Big) \times\|x_2^n-x_2^{n-1}\|_2+\lambda_2\bar{\alpha}_2\lambda_{W_2}(1+n^{-1})\|x_1^n-x_1^{n-1}\|_1\Big)\Big]. \quad (4.15)$$

From (4.9) and (4.15), we have

$$\begin{aligned}\|(x_1^{n+1},x_2^{n+1})-(x_1^n,x_2^n)\|_* &= \|x_1^{n+1}-x_1^n\|_1+\|x_2^{n+1}-x_2^n\|_2 \\ &\le k_1^n\|x_1^n-x_1^{n-1}\|_1+k_2^n\|x_2^n-x_2^{n-1}\|_2 \\ &\le \max\{k_1^n,k_2^n\}\Big(\|x_1^n-x_1^{n-1}\|_1+\|x_2^n-x_2^{n-1}\|_2\Big) \\ &\le \max\{k_1^n,k_2^n\}\Big[\|(x_1^n,x_2^n)-(x_1^{n-1},x_2^{n-1})\|_*\Big],\end{aligned} \quad (4.16)$$

where $X^*=X_1\times X_2$ is a reflexive Banach space with norm $\|\cdot\|_*=\|\cdot\|_1+\|\cdot\|_2$.

Letting $n\longrightarrow\infty$, we obtain $\max\{k_1^n,k_2^n\}\longrightarrow\max\{k_1,k_2\}$, where

$$k_1=m_1+\frac{1}{(\alpha_1-\beta_1+\sigma_1)}\lambda_1\bar{\beta}_1\lambda_{V_1};\ k_2=m_2+\frac{1}{(\alpha_2-\beta_2+\sigma_2)}\lambda_2\bar{\alpha}_2\lambda_{W_2}; \quad (4.17)$$

$$m_1=\frac{1}{(\alpha_1-\beta_1+\sigma_1)}\Big(\delta_1+\lambda_1\bar{\alpha}_1\lambda_{W_1}+\lambda_1\gamma_1\sqrt{1-2m_1+2\xi_1(2+\xi_1)}\Big); \quad (4.18)$$

$$m_2=\frac{1}{(\alpha_2-\beta_2+\sigma_2)}\Big(\delta_2+\lambda_2\bar{\beta}_2\lambda_{V_2}+\lambda_2\gamma_2\sqrt{1-2m_2+2\xi_2(2+\xi_2)}\Big). \quad (4.19)$$

By (4.3), it follows that $0<\max\{k_1,k_2\}<1$ and $\{(x_1^n,x_2^n)\}$ is a Cauchy sequence. Thus there exists $(x_1^*,x_2^*)\in X^*$ such that $(x_1^n,x_2^n)\longrightarrow(x_1^*,x_2^*)$ as $n\longrightarrow\infty$. Now we claim that $w_i^n\longrightarrow w_i\in W_i(x_1)$. In fact it follows from the Lipschitz continuity of W_i and Iterative Algorithm 4.1

$$\begin{aligned}\|w_i^{n+1}-w_i^n\| &\le (1+n^{-1})\mathcal{D}_1\Big(W_i(x_1^{n+1}),W_i(x_1^n)\Big) \\ &\le (1+n^{-1})\lambda_{W_i}\|x_1^{n+1}-x_1^n\|.\end{aligned} \quad (4.20)$$

Since $\{x_i^n\}$ is a Cauchy sequence, it follows that $\{w_i^n\}$ is also a Cauchy sequence. In a similar way, one can show that $\{v_i^n\}$ is a Cauchy sequence. Thus there exist $w_i\in X_1,\ v_i\in X_2$ such that $w_i^n\longrightarrow w_i,\ v_i^n\longrightarrow v_i$ as $n\longrightarrow\infty$. Further

$$\begin{aligned}d_i(w_i,W_i(x_1)) &\le \|w_i-w_i^n\|+d_i(w_i^n,W_i(x_1)) \\ &\le \|w_i-w_i^n\|+\mathcal{D}_1\Big(W_i(x_1^n),W_i(x_1)\Big) \\ &\le \|w_i-w_i^n\|+\lambda_{W_i}\|x_1^n-x_1\|\longrightarrow 0 \text{ as } n\longrightarrow\infty.\end{aligned}$$

Since $W_i(x_1) \in CB(X_1)$. It follows that $w_i \in W_i(x_1)$. Similarly we can show that $v_i \in V_i(x_2)$. By the continuity of $\phi_i, A_i, B_i, F_i, W_i, V_i, R^{B_i}_{M_i(\cdot,\cdot),\phi_i}$ and Iterative Algorithm 4.1, we have

$$x_i = R^{B_i}_{M_i(\cdot,\cdot),\phi_i}\Big[B_i(x_i) - \phi_i \circ [A_i(x_i - p_i(x_i)) + F_i(w_i, v_i)]\Big].$$

By Theorem 4.1, $(x_1, x_2, w_1, w_2, v_1, v_2)$ is a solution of problem SMVI (4.1). This is completes the proof. □

Remark 4.3. (i) If $\phi \circ M(f, g) = \lambda M(f, g)$, for $\lambda > 0$ and M be $\alpha\beta$-*symmetric monotone* with respect to f and g, then Theorem 3.3-3.6 reduce to Theorem 3.3-3.4 given in [26].

(ii) If $\phi \circ M(f, g) = \lambda M$, for $\lambda > 0$ and M be relaxed monotone then Theorem 3.3-3.6 reduce to Theorem 3.3-3.4 given in [11].

(iii) For each $i \in \{1, 2\}$, if $g_i = I_i$, identity operator on X_i; $\phi \circ M_i(f_i, g_i) = \lambda M_i(f_i, I_i)$, for $\lambda > 0$ and M be $\alpha_i\beta_i$-*symmetric monotone* with respect to f_i and I_i, then Theorem 4.2 is a generalization of Theorem 5.1 given in [11].

(iv) The method presented in this paper can be used to extend the results given in [30, 34, 35].

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