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CONVERGENCE THEOREMS OF HYBRID METHODS FOR GENERALIZED MIXED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS OF AN INFINITE FAMILY OF LIPSCHITZIAN QUASI-NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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ABSTRACT. We use a hybrid iterative method to find a common element of the set of fixed points of an infinite family of Lipschitzian quasi-nonexpansive mappings, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem in real Hilbert spaces. We also show that our main strong convergence theorem for finding that common element can be deduced for nonexpansive mappings and applied for strict pseudo-contraction mappings. Our results extend the work by Cho et al. (2009) [4].

KEYWORDS : Generalized mixed equilibrium problem; Variational inequality; Equilibrium problem; Fixed point; Quasi-nonexpansive mappings; Strict-pseudo contraction.
AMS Subject Classification: 47H05, 47H10.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. Let $B : C \to H$ be a nonlinear mapping, $\varphi : C \to \mathbb{R} \bigcup \{+\infty\}$ be a function and $f : C \times C \to \mathbb{R}$ be a bifunction. Peng and Yao [10] considered the following *generalized mixed equilibrium problem*:

$$\text{Finding } u \in C \text{ such that } f(u, y) + \varphi(y) + \langle Bu, y - u \rangle \geq \varphi(u), \quad \forall y \in C. \tag{1.1}$$

In this paper, we denote the set of solutions of (1.1) by *GMEP* (f, φ, B). It is obvious that if u is a solution of (1.1), it implies that $u \in$ dom $\varphi = \{u \in C : \varphi(u) < +\infty\}$.

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If $B = 0$ in (1.1), we obtain the following *mixed equilibrium problem* [3]:

$$\text{Finding } u \in C \text{ such that } f(u, y) + \varphi(y) \geq \varphi(u), \quad \forall y \in C. \tag{1.2}$$

We denote the set of solutions of (1.2) by *MEP* (f, φ).

If $\varphi = 0$ in (1.1), we obtain the following *generalized equilibrium problem* [16]:

$$\text{Finding } u \in C \text{ such that } f(u, y) + \langle Bu, y - u\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

We denote the set of solutions of (1.3) by *GEP* (f, B).

If $\varphi = 0$ and $B = 0$ in (1.1), we obtain the following *equilibrium problem* [2]:

$$\text{Finding } u \in C \text{ such that } f(u, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

We denote the set of solutions of (1.4) by *EP* (f).

If $f(x, y) = 0$ for all $x, y \in C$ in (1.1), we obtain the following *generalized variational inequality problem*:

$$\text{Finding } u \in C \text{ such that } \varphi(y) + \langle Bu, y - u\rangle \geq \varphi(u), \quad \forall y \in C. \tag{1.5}$$

We denote the set of solutions of (1.5) by *GVI* (C, φ, B).

If $\varphi = 0$ and $f(x, y) = 0$ for all $x, y \in C$ in (1.1), we obtain the following *variational inequality problem* (see also [1, 5]):

$$\text{Finding } u \in C \text{ such that } \langle Bu, y - u\rangle \geq 0, \quad \forall y \in C. \tag{1.6}$$

We denote the set of solutions of (1.6) by *VI* (C, B).

If $B = 0$ and $f(x, y) = 0$ for all $x, y \in C$ in (1.1), we obtain the following *minimization problem*:

$$\text{Finding } u \in C \text{ such that } \varphi(y) \geq \varphi(u), \quad \forall y \in C. \tag{1.7}$$

We denote the set of solutions of (1.7) by *MP*(C, φ).

In 1994, Blum and Oettli showed that the formulation of (1.4) covered monotone inclusive problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems and Nash equilibria in noncooperative games. Several problems in physics, optimization and economics can be reduced to find solutions of (1.4). The existence of equilibrium problems has been discovered by many authors (see, for example, [1, 6, 8, 15] and the references therein). Also, some solution methods have been studied by some authors (see, for example, [6, 15, 13] and the references therein).

In 2003, Takahashi and Toyoda [16] introduced the method for finding an element of $F(S) \cap VI(C, A)$ in real Hilbert spaces, where $C \subset H$ is closed and convex, $S : C \to H$ is a nonexpansive mapping and $A : C \to H$ is an inverse-strongly monotone mapping. Their iteration is the following:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), n \geq 0,$$

where $x_0 \in C$, $\{\alpha_n\}$ is a sequence in (0,1), $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$ and P_C is the metric projection from H onto C. They proved that, if $F(S) \cap VI(C, A) \neq \emptyset$, $\{x_n\}$ converges weakly to a point $z \in F(S) \cap VI(C, A)$, where $z = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)} x_n$.

Later, Takahashi and Takahashi [15] studied the contraction method for finding $F(S) \cap EP(f)$ in real Hilbert spaces, where $C \subset H$ is closed and convex, $S : C \to H$

is a nonexpansive mapping, f is a bifunction from $C \times C$ to $\mathbb{R}$ with some specific conditions. Their algorithm is the following:

$$\begin{aligned} & x_1 \in H, \\ & f(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ & x_{n+1} = \alpha_n f_1(x_n) + (1-\alpha_n)Sy_n \quad \forall n \geq 1, \end{aligned}$$

where $\{\alpha_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ and some appropriate conditions. They proved that, if $F(S) \cap EP(f) \neq \emptyset$, $\{x_n\}$ and $\{y_n\}$ converge strongly to a point $z \in F(S) \cap EP(f)$, where $z = P_{F(S)\cap EP(f)} f(z)$.

Recently, Cho et al. [4] introduced a hybrid projection method for finding $F := F(S) \cap VI(C,B) \cap GEP(f,A)$ in real Hilbert spaces, where $C \subset H$ is closed and convex, $S : C \to C$ is a k-strict pseudo-contraction with a fixed point, f is a bifunction from $C \times C$ to $\mathbb{R}$ with some specific conditions, $A : C \to H$ is an α-inverse-strongly monotone mapping and $B : C \to H$ is an β-inverse-strongly monotone mapping. Their iterative scheme is the following:

$$\begin{aligned} & x_1 \in C, \\ & C_1 = C, \\ & f(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ & z_n = P_C(u_n - \lambda_n Bu_n), \\ & y_n = \alpha_n x_n + (1-\alpha_n)S_k z_n, \\ & C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\ & x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{aligned}$$

where $S_k x = kx + (1-k)Sx$ for all $x \in C$, $\{\alpha_n\} \subset [0,1)$, $\{\lambda_n\} \subset (0, 2\beta)$ and $\{r_n\} \subset (0, 2\alpha)$ and some appropriate conditions. They proved that, if $F \neq \emptyset$, $\{x_n\}$ converges strongly to a point $\overline{x} = P_F x_1$, where P_F is the metric projection of H onto F.

In this paper, motivated by the above result, we prove a strong convergent theorem of a hybrid projection iterative method defined by (3.1) for finding a common element of the set of fixed points of an infinite family of Lipschitzian quasi-nonexpansive mappings, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem in the framework of real Hilbert spaces. Our main result can be deduced for nonexpansive mappings applied for strict pseudo-contraction mappings. It is clear that our result generalizes the work by [4].

2. PRELIMINARIES

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. Let $B : C \to H$. $x_n \to x$ means $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ implies $\{x_n\}$ converges weakly to x. We denote the set of fixed points of T by $F(T)$, i.e. $F(T) = \{x \in C : Tx = x\}$.

Recall the following definitions:

(1) A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(2) A mapping $T : C \to C$ is said to be *quasi-nonexpansive* if

$$\|Tx - p\| \leq \|x - p\|, \quad \forall p \in F(P).$$

(3) A mapping $T : C \to C$ is said to be *Lipschitzian* if there is a positive constant L such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

(4) A mapping $T : C \to C$ is said to be *strictly pseudo-contractive* with the coefficient $k \in [0, 1)$ if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

(5) B is said to be *monotone* if

$$\langle Bx - By, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

(6) B is said to be α*-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

(7) B is said to be α*-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2, \quad \forall x, y \in C.$$

(8) A set-valued mapping $T : H \to 2^H$ is said to be *monotone* if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$.

(9) A monotone mapping $T : H \to 2^H$ is said to be *maximal* if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping.

In the other words, a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$. Let $B : C \to H$ be a monotone mapping and $N_c v$ be the normal cone to C at $v \in C$, i.e., $N_c v = \{w \in H : \langle v - u, w\rangle \geq 0, \quad \forall u \in C\}$. Define a mapping T on C by

$$Tv = \begin{cases} Bv + N_c v & \text{if } v \in C \\ \emptyset & \text{if } v \notin C. \end{cases} \tag{2.1}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $\langle Bv, u - v\rangle \geq 0$ for all $u \in C$ (see [14]).

Let B be a β-inverse-strongly monotone mapping of C into H. It is easy to show that B is $\frac{1}{\beta}$-Lipschitz. For $\lambda \in (0, 2\beta]$, it is known that $I - \lambda B$ is a nonexpansive mapping of C into H.

Let C be a nonempty closed convex subset of H. Therefore, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the *metric projection* of H on C. We know that for $x \in H$ and $z \in C$, $z = P_C x$ is equivalent to $\langle x - z, z - y\rangle \geq 0$ for all $y \in C$. It is also known that a Hilbert space H satisfies the *Opial condition*, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ and $x \neq y$.

Let A be a monotone mapping from C into H. In the context of the variational inequality problem, it is known that

$$u \in \text{VI}(C, A) \Rightarrow u = P_C(u - \lambda Au), \text{ for all } \lambda > 0,$$

and

$$u = P_C(u - \lambda Au), \text{ for some } \lambda > 0 \Rightarrow u \in \text{VI}(C, A).$$

Let C be a nonempty closed subset of a Hilbert space H. Let $\{T_n\}$ and Γ be two families of nonlinear mappings of C into itself such that $F(\Gamma) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, where $F(\Gamma) = \bigcap_{T\in\Gamma} F(T)$. $\{T_n\}$ is said to satisfy the *NST-condition* ([9]) with Γ if for each bounded sequence $\{z_n\} \subset C$,

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \|z_n - T z_n\| = 0 \text{ for all } T \in \Gamma.$$

In the case $\Gamma \in \{T\}$, i.e., Γ consists of one mapping T, $\{T_n\}$ is said to satisfy the NST-condition with T.

For solving the generalized mixed equilibrium problem, we may assume the following conditions for the bifunction f, the function φ and the set C:

(A1) $f(x,x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;
(A3) for each $x \in C, y \mapsto f(x,y)$ is convex and lower semi-continuous;
(A4) for each $x \in C, y \mapsto f(x,y)$ is weakly upper semicontinuous;
(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq C$ and $y_x \in C \bigcap \text{dom}(\varphi)$ such that for any $z \in C \backslash D_x$,

$$f(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z);$$

(B2) C is a bounded set.

The following lemmas are useful for proving some convergence results in next two sections.

Lemma 2.1. *([10, 11, 12]). Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\varphi : C \to \mathbb{R} \bigcup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z,y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z), \quad \forall y \in C\}$$

for all $x \in H$. Then the following conclusions hold:

(1) T_r is single-valued;
(2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(3) $F(T_r) =$ MEP (F, φ);
(4) MEP (F, φ) is closed and convex.

Lemma 2.2. *([7]). Let $A : C \to H$ be α-inverse-strongly monotone. If $r \in (0, 2\alpha)$, then we have $I - rA$ is nonexpansive.*

Lemma 2.3. *([17]). Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a k-strict pseudo-contraction. Define a mapping $S : C \to C$ by $Sx = \alpha x + (1-\alpha)Tx$ for all $x \in C$ and $\alpha \in [k, 1)$. Then S is a nonexpansive mapping such that $F(S) = F(T)$.*

We would like to mention the following remark since our result is very interesting. It shows that a monotone mapping maps all points in a generalized mixed equlibrium problem to the same point.

Remark 2.4. Let C be a closed convex subset of a real Hilbert space H, $f : C\times C \to \mathbb{R}$ be a bifunction satisfying *(A2)* and $\varphi : C \to \mathbb{R}\cup\{+\infty\}$ be a function. Let A be a monotone mapping of C into H. Then $Au = Av$ for all $u, v \in GMEP(f, \varphi, A)$.

Proof. Let $u, v \in GMEP(f, \varphi, A)$. We then get

$$f(u, y) + \varphi(y) + \langle Au, y - u\rangle \geq \varphi(u), \quad \forall y \in C \tag{2.2}$$

and

$$f(v, y) + \varphi(y) + \langle Av, y - v\rangle \geq \varphi(v), \quad \forall y \in C. \tag{2.3}$$

By letting $y = v$ in (2.2) and $y = u$ in (2.3), we get

$$f(u, v) + \varphi(v) + \langle Au, v - u\rangle \geq \varphi(u) \tag{2.4}$$

and

$$f(v, u) + \varphi(u) + \langle Av, u - v\rangle \geq \varphi(v). \tag{2.5}$$

By (2.4), (2.5) and the condition (A2), we have

$$\langle Av - Au, u - v\rangle \geq f(u, v) + f(v, u) + \langle Au, v - u\rangle + \langle Av, u - v\rangle \geq 0. \tag{2.6}$$

From A is a α-inverse-strongly monotone mapping,

$$0 \leq \alpha\|Au - Av\|^2 \leq \langle Au - Av, u - v\rangle \leq 0.$$

That is $Au = Av$. □

By letting $f = 0$ and $\varphi = 0$ in Lemma 2.4, we obtain the following remark.

Remark 2.5. Let C be a closed convex subset of a real Hilbert space H and A be a monotone mapping of C into H. Then $Au = Av$ for all $u, v \in VI(C, A)$.

3. MAIN RESULT

In this section, we show a strong convergent theorem of hybrid methods for finding a common element of the set of fixed points of an infinite family of Lipschitzian quasi-nonexpansive mappings, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem in the framework of real Hilbert spaces under some appropriate conditions.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H and B be a β-inverse-strongly monotone mapping of C into H, respectively. Let $\{S_n\}$ and $\mathcal{S}$ be families of Lipschitzian quasi-nonexpansive mappings of C into itself such that $\lim_{n\to\infty}\|S_nx-S_ny\| \leq L_n\|x-y\|$ for all $x, y \in C$, $\sup_n L_n = L$, $\bigcap_{n=1}^{\infty} F(S_n) = F(\mathcal{S})$ and $F = F(\mathcal{S})\cap VI(C, B)\cap GMEP(f, \varphi, A) \neq \emptyset$. Suppose that $\{S_n\}$ satisfies the NST-condition with $\mathcal{S}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the algorithm:*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ f(u_n, y) + \varphi(y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \varphi(u_n), \quad \forall y \in C, \\ z_n = P_C(u_n - \lambda_n Bu_n), \\ y_n = \alpha_n x_n + (1 - \alpha_n)S_n z_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\} \subset [0,1)$, $\{\lambda_n\} \subset (0,2\beta)$, $\{r_n\} \subset (0,2\alpha)$,

$$0 \le \alpha_n \le a < 1, \quad 0 < b \le \lambda_n \le c < 2\beta, \quad \text{and} \quad 0 < d \le r_n \le e < 2\alpha,$$

for some $a,b,c,d,e \in \mathbb{R}$. *Then the sequence* $\{x_n\}$ *defined by the algorithm (3.1) converges strongly to a point* $\overline{x} = P_F x_1$, *where* P_F *is the metric projection of* H *onto* F.

Proof. We divide our proof into 5 steps.

Step 1: We show that $F \subset C_n$ and C_n is closed and convex for all $n \ge 1$.

From the assumption, $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for some $m \ge 1$. Next, we show that C_{m+1} is closed and convex. For any $w \in C_m$, we see that

$$\|y_m - w\| \le \|x_m - w\|$$

is equivalent to

$$\|x_m\|^2 - \|y_m\|^2 - 2\langle w, x_m - y_m\rangle \ge 0.$$

Therefore, C_{m+1} is closed and convex.

Since A is α-inverse-strongly monotone and B is β-inverse-strongly monotone, by Lemma 2.2, we get that $I - r_nA$ and $I - \lambda_nB$ are nonexpansive.

By nonexpansiveness of T_{r_n} and $I - r_nA$, we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp)\|^2 \\
&\le \|(x_n - r_nAx_n) - (p - r_nAp)\|^2 \\
&= \|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\
&= \|x_n - p\|^2 - 2r_n\langle x_n - p, Ax_n - Ap\rangle + r_n^2\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2 - 2r_n\alpha\|Ax_n - Ap\|^2 + r_n^2\|Ax_n - Ap\|^2 \\
&= \|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2. \qquad (3.2)
\end{aligned}$$

We are now ready to show that $F \subset C_n$ for each $n \ge 1$. From the assumption, we have that $F \subset CC_1$. Suppose $F \subset C_m$ for some $m \ge 1$. For any $w \in F \subset C_m$, by nonexpansiveness of $I - \lambda_mB$, we have

$$\begin{aligned}
\|y_m - w\| &= \|\alpha_m x_m + (1-\alpha_m)S_m z_m - w\| \\
&\le \alpha_m\|x_m - w\| + (1-\alpha_m)\|z_m - w\| \\
&= \alpha_m\|x_m - w\| + (1-\alpha_m)\|P_C(I - \lambda_mB)u_m - P_C(I - \lambda_mB)w\| \\
&\le \alpha_m\|x_m - w\| + (1-\alpha_m)\|u_m - w\| \\
&\le \alpha_m\|x_m - w\| + (1-\alpha_m)\|x_m - w\| \\
&= \|x_m - w\|.
\end{aligned}$$

That is $w \in C_{m+1}$. By mathematical induction, we conclude that $F \subset C_n$ for each $n \ge 1$.

Step 2: We show that $\{x_n\}$ is bounded.

Since $x_n = P_{C_n}x_1$ and $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, we get

$$\begin{aligned}
0 &\le \langle x_1 - x_n, x_n - x_{n+1}\rangle \qquad (3.3) \\
&= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1}\rangle \\
&\le -\|x_1 - x_n\|^2 + \|x_1 - x_n\|\|x_1 - x_{n+1}\|.
\end{aligned}$$

Thus
$$\|x_1 - x_n\| \leq \|x_1 - x_{n+1}\|. \tag{3.4}$$
Since $x_n = P_{C_n}x_1$, for any $w \in F \subset C_n$, we have
$$\|x_1 - x_n\| \leq \|x_1 - w\|. \tag{3.5}$$
In particular, we obtain
$$\|x_1 - x_n\| \leq \|x_1 - P_F x_1\|. \tag{3.6}$$
By (3.4) and (3.6), we get that $\lim_{n\to\infty} \|x_n - x_1\|$ exists. It implies that $\{x_n\}$ is bounded.

Step 3: We show that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ for all $S \in \mathcal{S}$.

By using (3.3), we obtain that
$$\begin{aligned}\|x_n - x_{n+1}\|^2 &= \|x_n - x_1 + x_1 - x_{n+1}\| \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1}\rangle \\ &\quad +\|x_1 - x_{n+1}\|^2 \\ &\leq \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2\end{aligned}$$
and so
$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.7}$$
Since $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1}$, we have
$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$$
and then
$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \leq 2\|x_n - x_{n+1}\|.$$
By (3.7), we get
$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.8}$$
On the other hand, we have
$$\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1-\alpha_n)S_n z_n\| = (1-\alpha_n)\|x_n - S_n z_n\|.$$
It follows from (3.8) and the assumption $0 \leq \alpha_n \leq a < 1$ that
$$\lim_{n\to\infty} \|x_n - S_n z_n\| = 0. \tag{3.9}$$
For any $w \in F$, we have
$$\begin{aligned}\|y_n - w\|^2 &= \|\alpha_n x_n + (1-\alpha_n)S_n z_n - w\|^2 \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|S_n z_n - w\|^2 \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|z_n - w\|^2. \end{aligned}\tag{3.10}$$
From (3.2), we obtain
$$\begin{aligned}\|y_n - w\|^2 &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|P_C(I - \lambda_n B)u_n - w\|^2 \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|(I - \lambda_n B)u_n - (I - \lambda_n B)w\|^2 \\ &= \alpha_n\|x_n - w\|^2 + (1-\alpha_n)(\|u_n - w\|^2 + \lambda_n^2\|Bu_n - Bw\|^2 \\ &\quad -2\lambda_n\langle u_n - w, Bu_n - Bw\rangle) \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)(\|u_n - w\|^2 + \lambda_n(\lambda_n - 2\beta)\|Bu_n - Bw\|^2) \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)(\|x_n - w\|^2 + \lambda_n(\lambda_n - 2\beta)\|Bu_n - Bw\|^2)\end{aligned}$$

$$\leq \quad \|x_n - w\|^2 + (1-\alpha_n)\lambda_n(\lambda_n - 2\beta)\|Bu_n - Bw\|^2.$$

We then have

$$\begin{aligned}(1-a)b(2\beta - c)\|Bu_n - Bw\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 \\ &= (\|x_n - w\| - \|y_n - w\|)(\|x_n - w\| + \|y_n - w\|) \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|).\end{aligned}$$

By (3.8), we obtain that

$$\lim_{n\to\infty} \|Bu_n - Bw\| = 0. \tag{3.11}$$

On the other hand, since P_C is firmly nonexpansive and $I - \lambda_n B$ is nonexpansive, we have

$$\begin{aligned}\|z_n - w\|^2 &= \|P_C(I-\lambda_n B)u_n - P_C(I-\lambda_n B)w\|^2 \\ &\leq \langle (I-\lambda_n B)u_n - (I-\lambda_n B)w, z_n - w\rangle \\ &= \frac{1}{2}\{\|(I-\lambda_n B)u_n - (I-\lambda_n B)w\|^2 + \|z_n - w\|^2 \\ &\quad -\|(I-\lambda_n B)u_n - (I-\lambda_n B)w - (z_n - w)\|^2\} \\ &\leq \frac{1}{2}\{\|u_n - w\|^2 + \|z_n - w\|^2 \\ &\quad -\|u_n - z_n - \lambda_n(Bu_n - Bw)\|^2\} \qquad (3.12) \\ &= \frac{1}{2}\{\|u_n - w\|^2 + \|z_n - w\|^2 - \|u_n - z_n\|^2 \\ &\quad +2\lambda_n\langle u_n - z_n, Bu_n - Bw\rangle - \lambda_n^2\|Bu_n - Bw\|^2\}.\end{aligned}$$

From (3.2), it implies that

$$\begin{aligned}\|z_n - w\|^2 &\leq \|u_n - w\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\langle u_n - z_n, Bu_n - Bw\rangle \\ &\quad -\lambda_n^2\|Bu_n - Bw\|^2 \\ &\leq \|x_n - w\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|Bu_n - Bw\|. \qquad (3.13)\end{aligned}$$

By (3.10) and (3.13), we get

$$\begin{aligned}(1-\alpha_n)\|u_n - z_n\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 \\ &\quad +2(1-\alpha_n)\lambda_n\|u_n - z_n\|\|Bu_n - Bw\| \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|) \\ &\quad +2(1-\alpha_n)\lambda_n\|u_n - z_n\|\|Bu_n - Bw\|.\end{aligned}$$

By (3.8), (3.11) and the assumption $0 \leq \alpha_n \leq a < 1$, we get

$$\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{3.14}$$

Also, by (3.10) and (3.12), we obtain that

$$\begin{aligned}\|y_n - w\|^2 &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|u_n - w\|^2 \qquad (3.15) \\ &= \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(w - r_nAw)\|^2 \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|(x_n - r_nAx_n) - (w - r_nAw)\|^2 \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)(\|x_n - w\|^2 - 2r_n\langle x_n - w, Ax_n - Aw\rangle \\ &\quad +r_n^2\|Ax_n - Aw)\|^2) \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)(\|x_n - w\|^2 \\ &\quad +r_n(r_n - 2\alpha)\|Ax_n - Aw\|^2) \\ &\leq \|x_n - w\|^2 + (1-\alpha_n)r_n(r_n - 2\alpha)\|Ax_n - Aw\|^2.\end{aligned}$$

From the assumptions $0 \le \alpha_n \le a < 1$ and $0 < d \le r_n \le e < 2\alpha$, we have

$$\begin{aligned}(1-a)d(2\alpha - e)\|Ax_n - Aw\|^2 &\le \|x_n - w\|^2 - \|y_n - w\|^2 \\ &\le \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|).\end{aligned}$$

By (3.8), we obtain

$$\lim_{n\to\infty} \|Ax_n - Aw\| = 0. \tag{3.16}$$

On the other hand, by using Lemma 2.1, we have T_{r_n} is firmly nonexpansive. Then we get

$$\begin{aligned}\|u_n - w\|^2 &= \|T_{r_n}(I - r_nA)x_n - T_{r_n}(I - r_nA)w\|^2 \\ &\le \langle (I - r_nA)x_n - (I - r_nA)w, u_n - w\rangle \\ &= \frac{1}{2}(\|(I - r_nA)x_n - (I - r_nA)w\|^2 + \|u_n - w\|^2 \\ &\quad -\|(I - r_nA)x_n - (I - r_nA)w - (u_n - w)\|^2) \\ &\le \frac{1}{2}(\|x_n - w\|^2 + \|u_n - w\|^2 - \|(x_n - u_n) - r_n(Ax_n - Aw)\|^2) \\ &= \frac{1}{2}(\|x_n - w\|^2 + \|u_n - w\|^2 - \|x_n - u_n\|^2 \\ &\quad +2r_n\langle x_n - u_n, Ax_n - Aw\rangle - r_n^2\|Ax_n - Aw\|^2)\end{aligned}$$

and so

$$\begin{aligned}\|u_n - w\|^2 &\le \|x_n - w\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Ax_n - Aw\rangle \\ &\quad -r_n^2\|Ax_n - Aw\|^2 \\ &\le \|x_n - w\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Aw\|. \end{aligned} \tag{3.17}$$

By (3.15) and (3.17), we get

$$\begin{aligned}\|y_n - w\|^2 &\le \|x_n - w\|^2 - (1 - \alpha_n)\|x_n - u_n\|^2 \\ &\quad +2(1 - \alpha_n)r_n\|x_n - u_n\|\|Ax_n - Aw\|,\end{aligned}$$

which implies that

$$\begin{aligned}(1 - \alpha_n)\|x_n - u_n\|^2 &\le \|x_n - w\|^2 - \|y_n - w\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Aw\| \\ &\le \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|) \\ &\quad +2r_n\|x_n - u_n\|\|Ax_n - Aw\|.\end{aligned}$$

From the assumptions $0 \le \alpha_n \le a < 1$, $0 < d \le r_n \le e < 2\alpha$, (3.8) and (3.16), we obtain

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.18}$$

On the other hand, we have

$$\begin{aligned}\|x_n - S_nx_n\| &\le \|S_nx_n - S_nz_n\| + \|S_nz_n - x_n\| \\ &\le L\|x_n - z_n\| + \|S_nz_n - x_n\| \\ &\le L\|x_n - u_n\| + L\|u_n - z_n\| + \|S_nz_n - x_n\|.\end{aligned}$$

Using (3.9), (3.14) and (3.18), we conclude that

$$\lim_{n\to\infty} \|x_n - S_nx_n\| = 0. \tag{3.19}$$

From the assumption $\{S_n\}$ satisfies the NST-condition with $\mathcal{S}$, we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0, \quad \forall S \in \mathcal{S}. \tag{3.20}$$

Since $\{x_n\}$ is bounded, we assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to ξ.

Step 4: We show that $\xi \in F = F(\mathcal{S}) \cap VI(C, B) \cap GMEP(f, \varphi, A)$.

First, we show that $\xi \in F(\mathcal{S})$. Suppose that $\xi \neq S\xi$ for some $S \in \mathcal{S}$. From Opial's condition and (3.20), we obtain

$$\begin{aligned} \liminf_{i\to\infty} \|x_{n_i} - \xi\| &< \liminf_{i\to\infty} \|x_{n_i} - S\xi\| \\ &= \liminf_{i\to\infty} \|x_{n_i} - Sx_{n_i} + Sx_{n_i} - S\xi\| \\ &\leq \liminf_{i\to\infty} \|x_{n_i} - \xi\|, \end{aligned}$$

which give us a contradiction. Hence, $\xi \in F(\mathcal{S})$.

Now, we prove that $\xi \in VI(C, B)$. Let T be the maximal monotone mapping defined by (2.1):

$$Tx = \begin{cases} Bx + N_c x & \text{if } x \in C \\ \emptyset & \text{if } x \notin C. \end{cases}$$

For any given $(x, y) \in G(T)$, we get $y - Bx \in N_C x$. By $z_n \in C$ and the definition of N_C, we have

$$\langle x - z_n, y - Bx\rangle \geq 0. \tag{3.21}$$

On the other hand, since $z_n = P_C(I - \lambda_n B)u_n$, we obtain

$$\langle x - z_n, z_n - (I - \lambda_n B)u_n\rangle \geq 0$$

and then

$$\langle x - z_n, \frac{z_n - u_n}{\lambda_n} + Bu_n\rangle \geq 0. \tag{3.22}$$

Since $u_n - z_n \to 0$ as $n \to \infty$, we have that

$$Bu_n - Bz_n \to 0 \text{ as } n \to \infty. \tag{3.23}$$

From (3.21), (3.22) and the β-inverse monotonicity of B, we obtain

$$\begin{aligned} \langle x - z_{n_i}, y\rangle &\geq \langle x - z_{n_i}, Bx\rangle \\ &\geq \langle x - z_{n_i}, Bx\rangle - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} + Bu_{n_i}\rangle \\ &= \langle x - z_{n_i}, Bx - Bz_{n_i}\rangle + \langle x - z_{n_i}, Bz_{n_i} - Bu_{n_i}\rangle \\ &\quad -\langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\ &\geq \langle x - z_{n_i}, Bz_{n_i} - Bu_{n_i}\rangle - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\ &= \langle x, Bz_{n_i} - Bu_{n_i}\rangle - \langle z_{n_i}, Bz_{n_i} - Bu_{n_i}\rangle - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\ &\geq \langle x, Bz_{n_i} - Bu_{n_i}\rangle - \|z_{n_i}\|\|Bz_{n_i} - Bu_{n_i}\| \\ &\quad -\frac{1}{\lambda_{n_i}}\|x - z_{n_i}\|\|z_{n_i} - u_{n_i}\|. \end{aligned} \tag{3.24}$$

By (3.14) and (3.18), we get

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Since $x_{n_i} \rightharpoonup \xi$, we obtain $z_{n_i} \rightharpoonup \xi$. From (3.14), (3.23) and (3.24), we obtain

$$\langle x - \xi, y\rangle = \lim_{n_i\to\infty} \langle x - z_{n_i}, y\rangle \geq 0.$$

Since T is maximal monotone, we obtain that $0 \in T\xi$. It follows that $\xi \in VI(C, B)$.

Next, we show that $\xi \in GMEP(f, \varphi, A)$. For any $y \in C$,

$$f(u_n, y) + \varphi(y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \varphi(u_n).$$

From the condition (A2), we get that

$$\varphi(y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq f(y, u_n) + \varphi(u_n).$$

Replacing n by n_i, we obtain

$$\varphi(y) + \langle Ax_{n_i}, y - u_{n_i}\rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq f(y, u_{n_i}) + \varphi(u_{n_i}). \tag{3.25}$$

For any t with $0 < t \leq 1$ and $y \in C$, put $\rho_t = ty + (1 - t)\xi$. Since $y \in C$ and $\xi \in C$, we obtain $\rho_t \in C$. It follows from (3.25) and the monotonicity of A that

$$\begin{aligned}
\langle \rho_t - u_{n_i}, A\rho_t\rangle &\geq \langle \rho_t - u_{n_i}, A\rho_t\rangle - \langle Ax_{n_i}, \rho_t - u_{n_i}\rangle - \langle \rho_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \\
&\quad + f(\rho_t, u_{n_i}) + \varphi(u_{n_i}) - \varphi(\rho_t) \\
&= \langle \rho_t - u_{n_i}, A\rho_t - Au_{n_i}\rangle + \langle \rho_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle \\
&\quad - \langle \rho_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + f(\rho_t, u_{n_i}) + \varphi(u_{n_i}) - \varphi(\rho_t) \\
&\geq \langle \rho_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle - \langle \rho_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \\
&\quad + f(\rho_t, u_{n_i}) + \varphi(u_{n_i}) - \varphi(\rho_t).
\end{aligned} \tag{3.26}$$

Since A is Lipschitzian, by (3.18), we get $Au_{n_i} - Ax_{n_i} \to 0$ as $i \to \infty$. From (A3) and (3.26), we arrive at

$$\langle \rho_t - \xi, A\rho_t\rangle \geq f(\rho_t, \xi) + \varphi(\xi) - \varphi(\rho_t). \tag{3.27}$$

From (A1), (A3), (3.27) and convexity of φ, we have that

$$\begin{aligned}
0 &= f(\rho_t, \rho_t) \leq tf(\rho_t, y) + (1 - t)f(\rho_t, \xi) \\
&\leq tf(\rho_t, y) + (1 - t))\big(\langle \rho_t - \xi, A\rho_t\rangle + \varphi(\rho_t) - \varphi(\xi)\big) \\
&\leq tf(\rho_t, y) + (1 - t)t\big(\langle y - \xi, A\rho_t\rangle + \varphi(y) - \varphi(\xi)\big),
\end{aligned}$$

which implies that

$$f(\rho_t, y) + (1 - t)(\langle y - \xi, A\rho_t\rangle + \varphi(y) - \varphi(\xi)) \geq 0.$$

Letting $t \to 0$, by (A4), we arrive at

$$f(\xi, y) + \langle y - \xi, A\xi\rangle + \varphi(y) - \varphi(\xi) \geq 0.$$

This shows that $\xi \in GMEP(f, \varphi, A)$.

Step 5: We show that $x_n \to P_F x_1$.

Let $\overline{x} = P_F x_1$ Since $\overline{x} = P_F x_1 \subset C_{n+1}$ and $x_{n+1} = P_{C_{n+1}} x_1$, we get

$$\|x_1 - x_{n+1}\| \leq \|x_1 - \overline{x}\|.$$

On the other hand, we have

$$\|x_1 - \overline{x}\| \quad \leq \quad \|x_1 - \xi\|$$

$$
\begin{aligned}
&\leq \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\
&\leq \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \\
&\leq \|x_1 - \overline{x}\|.
\end{aligned}
$$

Therefore, we get

$$\|x_1 - \xi\| = \lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - \overline{x}\|.$$

This implies that $\overline{x} = \xi$. Since H has the Kadec-Klee property and $x_1 - x_{n_i} \rightharpoonup x_1 - \overline{x}$, it follows that $x_{n_i} \to \overline{x}$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we conclude that $x_n \to \overline{x}$ as $n \to \infty$. The proof is now complete. □

4. DEDUCED THEOREMS AND APPLICATIONS

Theorem 3.1 can be reduced to many different results. By putting $S_n = S$ for all $n \geq 1$ in Theorem 3.1, we obtain the following theorem:

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H and B be a β-inverse-strongly monotone mapping of C into H, respectively. Let $S : C \to C$ be a L-Lipschitzian quasi-nonexpansive mapping such that $F = F(S) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$
\begin{cases}
x_1 \in C, \\
C_1 = C, \\
f(u_n, y) + \varphi(y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \varphi(u_n), \forall y \in C, \\
z_n = P_C(u_n - \lambda_n B u_n), \\
y_n = \alpha_n x_n + (1 - \alpha_n) S z_n, \\
C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\
x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1,
\end{cases}
\tag{4.1}
$$

where $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (4.1) converges strongly to a point $\overline{x} = P_F x_1$, where P_F is the metric projection of H onto F.

When $\{S_n\}$ and $\mathcal{S}$ are families of nonexpansive mappings, we get the following theorem:

Theorem 4.2. *Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H and B be a β-inverse-strongly monotone mapping of C into H, respectively. Let $\{S_n\}$ and $\mathcal{S}$ be families of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) = F(\mathcal{S})$ and $F = F(\mathcal{S}) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Suppose that $\{S_n\}$ satisfies the NST-condition with $\mathcal{S}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the algorithm (3.1), where $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,*

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (3.1) converges strongly to a point $\overline{x} = P_F x_1$, where P_F is the metric projection of H onto F.

Now we show how to apply Theorem 4.2 for families of strict pseudo-contraction mappings.

Theorem 4.3. *Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H and B be a β-inverse-strongly monotone mapping of C into H, respectively. Let $\{R_n\}$ and $\mathcal{R}$ be families of $k-$strict pseudo-contraction mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(R_n) = F(\mathcal{R})$ and $F = F(\mathcal{R}) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Define a mapping $S_n : C \to C$ by $S_n x = kx + (1-k)R_n x$ for all $x \in C$. Suppose that $\{R_n\}$ satisfies the NST-condition with $\mathcal{R}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the algorithm (3.1), where $\{\alpha_n\} \subset [0, 1), \{\lambda_n\} \subset (0, 2\beta), \{r_n\} \subset (0, 2\alpha)$,*

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \textit{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (3.1) converges strongly to a point $\overline{x} = P_F x_1$, where P_F is the metric projection of H onto F.

Proof. By Lemma 2.3, we obtain that S_n is nonexpansive for all positive integer n. We also get that $\{S_n\}$ satisfies the NST-condition with $\mathcal{S} = \{kI + (1-k)T : T \in \mathcal{R}\}$. The proof is now complete because of the direct result of Theorem 4.2. □

By putting $S_n = S$ for all $n \geq 1$ in Theorem 4.3, we obtain the following corallary.

Corollary 4.4. *Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H and B be a β-inverse-strongly monotone mapping of C into H, respectively. Let $R : C \to C$ be a $k-$strict pseudo-contraction such that $F = F(R) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Define a mapping $S : C \to C$ by $Sx = kx + (1-k)Rx$ for all $x \in C$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the following algorithm (4.1), where $\{\alpha_n\} \subset [0, 1), \{\lambda_n\} \subset (0, 2\beta), \{r_n\} \subset (0, 2\alpha)$,*

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \textit{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (4.1) converges strongly to a point $\overline{x} = P_F x_1$, where P_F is the metric projection of H onto F.

Remark 4.5. By letting $\varphi = 0$ in Corollary 4.4, we obtain Theorem 2.1 of [4].

Remark 4.6. Since Theorems 3.1, 4.1, 4.2, 4.3 and Corollary 4.4 are for finding a common element of the set of fixed points, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem, we can reduce each theorem or corollary by letting $B = 0$, $A = 0$, $\varphi = 0$ or $f(x, y) = 0$.

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