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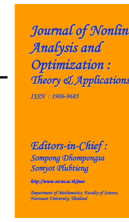

STUDY OF HARMONIC MULTIVALENT MEROMORPHIC FUNCTIONS BY USING GENERALIZED HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. In this paper we have studied a class of complex valued multivalent meromorphic harmonic and orientation preserving functions by using the generalized hypergeometric functions in the punctured unit disk and we have obtained coefficient estimates, distortion theorem . Other interesting properties are also investigated.

KEYWORDS : Multivalent functions; Meromorphic functions; Harmonic functions; Distortion theorem; Starlike functions.
AMS Subject Classification: 30C45.

1. INTRODUCTION

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. In any simply connected domain D , we can write
$f = h + \overline{g}$, where h and g are analytic in D.We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for $f$to be univalent and orientation preserving in D that $|h'(z)| > |g'(z)|$ in D (see [4]).

W. Hengartner and G. Schober [2], considered harmonic sense preserving univalent mappings defined on $\overline{\mathcal{N}U} = \{z : |z| > 1\}$ that map ∞ to ∞ and represented by
$f(z) = h(z) + \overline{g(z)} + A \log |z|$ where $h(z) = \alpha z + \sum_{n=0}^{\infty} a_n z^{-n}, g(z) = \underline{z} + \sum_{n=1}^{\infty} b_n z^{-n}$ are analytic in $\overline{\mathcal{N}U}$ and $|\alpha| > |\underline{!} \geq 0, A \in \mathbb{C}$, further $\frac{\overline{f}_{\overline{z}}}{f_z}$ is analytic and $\left|\frac{\overline{f}_{\overline{z}}}{f_z}\right| < 1$. Jahangiri [6] , O Ahuja and Jahangiri [1] and Murugusundaramoorthy [7] have studied classes of meromorphic harmonic functions.

Let us denoted the family $\Sigma_p(H)$ consisting of all harmonic sense-preserving multivalent meromorphic mapping in $\mathcal{N}U^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\}$

$$f(z) = h(z) + \overline{g(z)} \tag{1-1}$$

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where

$$h(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n+p-1} z^{(n+p-1)}, g(z) = \sum_{n=1}^{\infty} b_{n+p-1} z^{(n+p-1)},\ |b_p| < 1. \qquad (1\text{-}2)$$

Also , we denote by $\overline{\Sigma_p(H)}$ the subfamily of $\Sigma_p(H)$ consisting of harmonic functions $f = h + \overline{g(z)}$ of the form

$$f(z) = z^{-p} - \sum_{n=1}^{\infty} a_{n+p-1} z^{(n+p-1)} + \overline{\sum_{n=1}^{\infty} b_{n+p-1} z^{(n+p-1)}},\ (|b_p| < 1). \qquad (1\text{-}3)$$

The β- Convolution of$\phi(z)$ and $\psi(z)$ where

$$\phi(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n+p-1} z^{(n+p-1)} + \sum_{n=1}^{\infty} b_{n+p-1} z^{(n+p-1)} \qquad (1\text{-}4)$$

and

$$\psi(z) = z^{-p} + \sum_{n=1}^{\infty} c_{n+p-1} z^{(n+p-1)} + \sum_{n=1}^{\infty} d_{n+p-1} z^{(n+p-1)} \qquad (1\text{-}5)$$

is defined by

$$(\phi \otimes_\beta \psi)(z) = z^{-p} + \sum_{n=1}^{\infty} \frac{a_{n+p-1} c_{n+p-1}}{(n+p-1)^\beta} \overline{z^{(n+p-1)}} + \sum_{n=1}^{\infty} \frac{b_{n+p-1} d_{n+p-1}}{(n+p-1)^\beta} \overline{z^{(n+p-1)}}. \qquad (1\text{-}6)$$

The 0-convolution of ϕ and ψ is the familiar Hadamard product,also the 1-convolution of ϕ and ψ is named integral convolution.
For real or complex numbers $\alpha_1, \alpha_2, ..., \alpha_q$ and $\beta_1, \beta_2 ...., \beta_s (\beta_j \neq 0, -1, -2, -3, ...; j = 1, 2, ..., s)$, we define the generalized hypergeometric function ${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ by

$${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n .....(\alpha_q)_n}{(\beta_1)_n ......(\beta_s)_n} \frac{z^n}{n!} \qquad (1\text{-}7)$$

$$(q \leq s+1; q, s \in N_0 = N \cup \{0\}; z \in \mathcal{N}U),$$

where $(x)_k$ is the pochhammer symbol, defined , in term of the Gamma function Γ, by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}.$$

Corresponding to a function

$$\mathcal{H}_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z^{-p}\ {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z), \qquad (1\text{-}8)$$

we consider a linear operator $H_{p,\mu}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)$ defined by the convolution

$$H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) * H_{p,\mu}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \frac{1}{z^p(1-z)^{\mu+p}}.(\mu > -p)$$

Let $H_{p,q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) : \Sigma_p(H) \to \Sigma_p(H)$defined by

$$H^\mu_{p,q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) f(z) = H_{p,\mu}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * f(z). \qquad (1\text{-}9)$$

$$(\alpha_j, \beta_j \neq 0, -1, -2, -3, ...; i = 1, ..., q; j = 1, 2, ..., s, \mu > -p; f \in \Sigma_p(H); z \in \mathcal{N}U^*)$$

For notational simplicity , we use shorter notation

$$H^\mu_{p,q,s}(\alpha_1) = H_p(\alpha_1, ..., \alpha_q; \beta_1, .., \beta_s)$$

Thus, from (1-9) we deduce that after simple calculations , we obtain

$$z(H^\mu_{p,q,s}(\alpha_1) f(z)) = z^{-p} + \sum_{n=1}^{\infty} \frac{(\mu+p)_{p+n} (\beta_1)_{n+p} .....(\beta_s)_{n+p}}{(\alpha_1)_{p+n} ......(\alpha_q)_{n+p}} a_{n+p-1} z^{n+p-1}. \qquad (1\text{-}10)$$

We note that the linear operator $H^{\mu}_{p,q,s}$ is closely related to the Choi- Saigo-Srivastava operator [3].In view of relationship 1-10 for harmonic function $f = h + \overline{(g)}$ given by 1-1 , we define the operator

$$H^{\mu}_{p,q,s}f(z) = H^{\mu}_{p,q,s}h(z) + \overline{H^{\mu}_{p,q,s}g(z)}.$$

A function $f \in \Sigma_p(H)$ is said to be in the subclass $JH(\alpha)$ of meromorphic harmonic α-starlike function in $\mathcal{N}U^*$ if its satisfies the condition

$$Re\left\{-\frac{z(H^{\mu}_{p,q,s}h(z))' - \overline{z(H^{\mu}_{p,q,s}g(z))'}}{H^{\mu}_{p,q,s}h(z) + \overline{H^{\mu}_{p,q,s}g(z)}}\right\} > \alpha$$

where $(0 \geq \alpha < p)$ and $H^{\mu}_{p,q,s}f(z)$ is given by 1-10. We also let $\overline{JH(\alpha)} = JH(\alpha) \cap \overline{\Sigma_p(H)}$.
In this paper we obtain coefficient conditions for the classes $\overline{JH(\alpha)}$and $JH(\alpha)$

2. Coefficient Estimates

Theorem 2.1 : Let $f = h + \overline{g}$ where g and h are given by (1-2) if

$$\sum_{n=1}^{\infty}(1-\alpha-n-p)|a_{n+p-1}|T^{\mu}_p(n) + (n+p-1-\alpha)|b_{n+p-1}|T^{\mu}_p(n) \leq 1, \tag{2-1}$$

then $f \in JH(\alpha)$, where $T^{\mu}_p(n) = \frac{(\mu+p)_{p+n}(\beta_1)_{n+p}\cdots\cdots(\beta_s)_{n+p}}{(\alpha_1)_{p+n}\cdots\cdots(\alpha_q)_{n+p}}$
Proof :Suppose the condition (2-1) holds true, we show that

$$Re\left\{-\frac{z(H^{\mu}_{p,q,s}h(z))' - \overline{z(H^{\mu}_{p,q,s}g(z))'}}{H^{\mu}_{p,q,s}h(z) + \overline{H^{\mu}_{p,q,s}g(z)}}\right\} = Re\left\{\frac{A(z)}{B(z)}\right\} > \alpha,$$

where
$A(z) = -z(H^{\mu}_{p,q,s}h(z))' - \overline{z(H^{\mu}_{p,q,s}g(z))'} = -(-pz^{-p-1} + \sum_{n=1}^{\infty} T^{\mu}_p(n)(n+p-1)a_{n+p-1}z^{n+p-2}) + z(\sum_{n=1}^{\infty} T^{\mu}_p(n)(n+p-1)\overline{b_{n+p-1}z^{n+p-2}})$
and
$B(z) = H^{\mu}_{p,q,s}h(z) + \overline{H^{\mu}_{p,q,s}g(z)} = z^{-p} + \sum_{n=1}^{\infty} T^{\mu}_p(n)a_{n+p-1}z^{n+p-1}$
$+ \sum_{n=1}^{\infty} T^{\mu}_p(n)\overline{b_{n+p-1}z^{n+p-1}}$.

Using the fact that $Re w > \alpha$ if and only if $|1-\alpha+w| > |1+\alpha-w|$, its suffices to show that

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| > 0. \tag{2-2}$$

Substituting for $A(z)$ and $B(z)$ in (2-2), and performing elementary calculations, we find that

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)|$$

$> 2|z|^{-p} - 2\sum_{n=1}^{\infty}[(1-\alpha-n-p)|a_{n+p-1}| + (n+p-1-\alpha)|b_{n+p-1}|]T^{\mu}_p(n)|z|^{n+p-1}$
$> 2[1 - \sum_{n=1}^{\infty}[(1-\alpha-n-p)|a_{n+p-1}| + (n+p-1-\alpha)|b_{n+p-1}|]T^{\mu}_p(n) \geq 0,$
which implies that $f \in JH(\alpha)$.
Theorem 2.2 : For $0 \leq \alpha < p$, $f = h + \overline{g} \in \overline{JH(\alpha)}$ if and only if

$$\sum_{n=1}^{\infty}(1-\alpha-n-p)|a_{n+p-1}|T^{\mu}_p(n) + (n+p-1-\alpha)|b_{n+p-1}|T^{\mu}_p(n) \leq 1, \tag{2-3}$$

where $T^{\mu}_p(n) = \frac{(\mu+p)_{p+n}(\beta_1)_{n+p}\cdots\cdots(\beta_s)_{n+p}}{(\alpha_1)_{p+n}\cdots\cdots(\alpha_q)_{n+p}}$

Proof : Since $\overline{JH(\alpha)} \subset JH(\alpha)$, we need to prove the "only if" part. To this end, for functions f of the form (1-3) , we have

$$Re\left\{-\frac{z(H^{\mu}_{p,q,s}h(z))' - \overline{z(H^{\mu}_{p,q,s}g(z))'}}{H^{\mu}_{p,q,s}h(z) + \overline{H^{\mu}_{p,q,s}g(z)}}\right\} > \alpha$$

implies that

$$Re\left\{\frac{z^{-p} - \sum_{n=1}^{\infty} T^{\mu}_{p}(n)(1-\alpha-n-p)a_{n+p-1}z^{n+p-1} - \sum_{n=1}^{\infty} T^{\mu}_{p}(n)\overline{b_{n+p-1}z^{n+p-1}}}{z^{-p} - \sum_{n=1}^{\infty} T^{\mu}_{p}(n)a_{n+p-1}z^{n+p-1} + \sum_{n=1}^{\infty} T^{\mu}_{p}(n)\overline{b_{n+p-1}z^{n+p-1}}}\right\} > 0.$$

The above required condition must hold for all values of z in $\mathcal{N}U^*$. Upon choosing the values of z on the positive real axis where $0 = r < 1$, we must have

$$Re\left\{\frac{1 - \sum_{n=1}^{\infty} T^{\mu}_{p}(n)(1-\alpha-n-p)a_{n+p-1}r^{n+2p-1} - \sum_{n=1}^{\infty} T^{\mu}_{p}(n)\overline{b_{n+p-1}r^{n+2p-1}}}{1 - \sum_{n=1}^{\infty} T^{\mu}_{p}(n)a_{n+p-1}r^{n+2p-1} + \sum_{n=1}^{\infty} T^{\mu}_{p}(n)\overline{b_{n+p-1}r^{n+2p-1}}}\right\} > 0. \tag{2-4}$$

If the condition (2-3) does not hold , then the numerator in (2-4) is negative for r sufficiently close to 1 . Here , there exist $z_0 = r_0$ in (0,1) for which the quotient of (2-4) is negative .This contradicts the required condition for $f(z) \in \overline{JH(\alpha)}$.
The following result gives the distortion bounds

Theorem 2.3 :Let the function f be in the class $\overline{JH(\alpha)}$. Then , for $0 < |z| = r < 1$, we have

$$r^{-p} - \frac{r^p}{(p-\alpha)T^{\mu}_{p}(1)} \leq |f(z)| \leq r^{-p} + \frac{r^p}{(p-\alpha)T^{\mu}_{p}(1)}.$$

Proof :In view of Theorem 2.2 ,for $0 < |z| = r < 1$,

$$|f(z)| = |z^{-p} + \sum_{n=1}^{\infty} a_{n+p-1}z^{n+p-1} - \overline{\sum_{n=1}^{\infty} b_{n+p-1}z^{n+p-1}}|$$

$$\leq r^{-p} + r^p \sum_{n=1}^{\infty}(a_{n+p-1} + b_{n+p-1})$$

$$\leq r^{-p} + \frac{r^p}{(p-\alpha)T^{\mu}_{p}(1)} \sum_{n=1}^{\infty}(p-\alpha)T^{\mu}_{p}(1)(a_{n+p-1} + b_{n+p-1})$$

$$\leq r^{-p} + \frac{r^p}{(p-\alpha)T^{\mu}_{p}(1)} \times \sum_{n=1}^{\infty} T^{\mu}_{p}(n)[(1-\alpha-n-p)a_{n+p-1} + (n+p-1-\alpha)b_{n+p-1}]$$

$$\leq r^{-p} + \frac{r^p}{(p-\alpha)T^{\mu}_{p}(1)}$$

and

$$|f(z)| = |z^{-p} + \sum_{n=1}^{\infty} a_{n+p-1}z^{n+p-1} - \overline{\sum_{n=1}^{\infty} b_{n+p-1}z^{n+p-1}}|$$

$$\geq r^{-p} - r^p \sum_{n=1}^{\infty}(a_{n+p-1} + b_{n+p-1})$$

$$\geq r^{-p} - \frac{r^p}{(p-\alpha)T^{\mu}_{p}(1)} \sum_{n=1}^{\infty}(p-\alpha)T^{\mu}_{p}(1)(a_{n+p-1} + b_{n+p-1})$$

$$\geq r^{-p} - \frac{r^p}{(p-\alpha)T^{\mu}_{p}(1)} \times \sum_{n=1}^{\infty} T^{\mu}_{p}(n)[(1-\alpha-n-p)a_{n+p-1} + (n+p-1-\alpha)b_{n+p-1}]$$

$$\geq r^{-p} - \frac{r^p}{(p-\alpha)T_p^\mu(1)}$$

Theorem 2.4 :Let $\phi(z)$ and $\psi(z)$ have the form (1-5) and (1-6), respectively be in $\overline{JH(\alpha)}$ .Then the β- Convolution of$\phi(z)$ and $\psi(z)$ belongs to $\overline{JH(\alpha)}$ where $\beta \geq \max\left\{(\log(n+p-1))^{-1}\log\frac{1}{1-\alpha-n-p}, (\log(n+p-1))^{-1}\log\frac{1}{n+p-1-\alpha}\right\}$.

Proof : Since $\phi(z), \psi(z) \in \overline{JH(\alpha)}$ we can write

$$\sum_{n=1}^{\infty}[(1-\alpha-n-p)a_{n+p-1} + (n+p-1-\alpha)b_{n+p-1}]T_p^\mu(n) \leq 1 \tag{2-5}$$

$$\sum_{n=1}^{\infty}[(1-\alpha-n-p)c_{n+p-1} + (n+p-1-\alpha)d_{n+p-1}]T_p^\mu(n) \leq 1,$$

also we have $c_{n+p-1} \leq \frac{1}{1-\alpha-n-p}, d_{n+p-1} \leq \frac{1}{n+p-1-\alpha}$,we must show

$$\sum_{n=1}^{\infty}[\frac{(1-\alpha-n-p)a_{n+p-1}c_{n+p-1}T_p^\mu(n)}{(n+p-1)^\beta} + \frac{(n+p-1-\alpha)b_{n+p-1}d_{n+p-1}T_p^\mu(n)}{(n+p-1)^\beta} \leq 1. \tag{2-6}$$

For this purpose we have

$$\sum_{n=1}^{\infty}[\frac{(1-\alpha-n-p)a_{n+p-1}c_{n+p-1}}{(n+p-1)^\beta} + \frac{(n+p-1-\alpha)b_{n+p-1}d_{n+p-1}}{(n+p-1)^\beta}]T_p^\mu(n)$$
$$\leq \sum_{n=1}^{\infty}\frac{a_{n+p-1}+b_{n+p-1}}{(n+p-1)^\beta},$$

thus in view of (2-5) the (2-6) holds true if both the following hold true

$$(n+p-1)^\beta \geq \frac{1}{1-\alpha-n-p}, (n+p-1)^\beta \geq \frac{1}{n+p-1-\alpha}.$$

Equivalently if

$$\beta \geq \max\left\{(\log(n+p-1))^{-1}\log\frac{1}{1-\alpha-n-p}, (\log(n+p-1))^{-1}\log\frac{1}{n+p-1-\alpha}\right\}.$$

Theorem 2.5: Let $0 \leq \alpha \leq \gamma < 1$ and $\phi \in \overline{JH(\alpha)}, \psi \in \overline{JH(\gamma)}$. Then

$$\phi * \psi \in \overline{JH(\gamma)} \subset \overline{JH(\alpha)}$$

Proof :Its clear that$\overline{JH(\gamma)} \subset \overline{JH(\alpha)}$, also we have

$$\sum_{n=1}^{\infty}[(1-\gamma-n-p)a_{n+p-1} + (n=p-1-\gamma)b_{n+p-1}]T_p^\mu(n) \leq 1,$$

$$\sum_{n=1}^{\infty}[(1-\alpha-n-p)c_{n+p-1} + (n+p-1-\alpha)d_{n+p-1}]T_p^\mu(n) \leq 1,$$

and consequently we can write $c_{n+p-1} \leq \frac{1}{1-\alpha-n-p}, d_{n+p-1} \leq \frac{1}{n+p-1-\alpha}$, therefore we obtain

$$\sum_{n=1}^{\infty}[(1-\gamma-n-p)a_{n+p-1}c_{n+p-1} + (n+p-1-\gamma)b_{n+p-1}d_{n+p-1}]T_p^\mu(n)$$

$$\leq \sum_{n=1}^{\infty}[\frac{(1-\gamma-n-p)}{(1-\alpha-n-p)}a_{n+p-1}]T_p^{\mu}(n)$$
$$+\sum_{n=1}^{\infty}[\frac{(n+p-1-\gamma)}{(n+p-1-\alpha)}b_{n+p-1}]T_p^{\mu}(n) \leq 1$$

and this shows that

$$\phi * \psi \in \overline{JH(\gamma)}$$

Theorem 2.6: Let $f_j(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n+p-1,j} z^{n+p-1} - \sum_{n=1}^{\infty} b_{n+p-1,j}\overline{z^{n+p-1}}$ belong to $\overline{JH(\alpha)}$,for $j = 1, 2, ...$ Then $F(z) = \sum_{j=1}^{\infty} \sigma_j f_j(z)$belongs to $\overline{JH(\alpha)}$,where $\sum_{j=1}^{\infty} \sigma_j = 1, si_j \geq 0$.

Proof :We have $F(z) = z^{-p}+\sum_{n=1}^{\infty}(\sum_{j=1}^{\infty} \sigma_j a_{n+p-1,j})z^{n+p-1}-\sum_{n=1}^{\infty}(\sum_{j=1}^{\infty} \sigma_j b_{n+p-1,j})\overline{z^{n+p-1}}$, therefore

$$\sum_{n=1}^{\infty} T_p^{\mu}(n)(1-\alpha-n-p)(\sum_{j=1}^{\infty}\sigma_j a_{n+p-1}) + T_p^{\mu}(n)(n+p-1-\alpha)(\sum_{j=1}^{\infty}\sigma_j b_{n+p-1})$$
$$= \sum_{j=1}^{\infty}\sigma_j \sum_{n=1}^{\infty} T_p^{\mu}(n)(1-\alpha-n-p)a_{n+p-1} + T_p^{\mu}(n)(n+p-1-\alpha)b_{n+p-1}$$
$$\leq \sum_{j=1}^{\infty}\sigma_j = 1.$$

Then $F(z) \in \overline{JH(\alpha)}$.

Corollary 2.7: $\overline{JH(\alpha)}$ is convex set.

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