

STUDY ON EXISTENCE THEORY FOR INTEGRAL EQUATIONS OF MIXED TYPE

GHASEM ALIZADEH AFROUZI^{1,*}, SOHRABALI YOUSEFI², R.A. TALARPOSHTI¹

¹Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

² Department of Mathematics, Shahid Beheshti University , Tehran, Iran

ABSTRACT. This article presents an existence of monotonic solution for the nonlinear mixed Volterra-Fredholm integral equation acting on $L_1[0, 1]$. By using the techniques of the De Blasi measure of weak noncompactness, we employ the basic fixed point theorems such as Darbo's theorem to obtain the mentioned aims in Banach spaces.

KEYWORDS : Measure of weak noncompactness; Nonlinear mixed integral equation; Darbo's theorem; Banach algebra.

AMS Subject Classification: 35B30, 35B40.

1. INTRODUCTION

In the present paper, we try to prove the existence of monotonic solution of the following mixed Volterra-Fredholm integral equation

$$u(x, t) = f(x, t) + \int_0^t \int_0^1 g(x, t, \epsilon, \tau) h(\epsilon, \tau, u(\epsilon, \tau)) d\epsilon d\tau \quad (x, t) \in [0, t] \times [0, 1], \quad (1.1)$$

in $L_1[0, 1]$.

Here $u(x, t)$, is an unknown function, and the functions $f(x, t)$, $g(x, t, \epsilon, \tau)$ and $h(\epsilon, \tau, u(\epsilon, \tau))$ are analytic on $D = [0, 1] \times [0, T]$ and $t \in [0, 1]$.

This equation arises in the theory of parabolic boundary value problems, the mathematical modeling of the spatio-temporal development of an epidemic and varies physical and biological problems.

The aim of this paper is to use Theorem 2.1 and the fixed point theorem with respect to a measure of weak noncompactness in the Banach algebras for solving Eq.1.1.

* Corresponding author.

Email address : afrouzi@umz.ac.ir(G.A. Afrouzi).

Article history : Received 31 August 2011. Accepted 5 September 2011.

2. BACKGROUND CONCEPTS

Let X be a Banach space and θ its zero vector. Denote by B_r the closed ball in X centered at θ and with radius r . Let H be a nonempty bounded subset of X , De-Blasi measure of weak noncompactness β is defined in [1] as

$$\beta(H) = \inf\{r > 0 : \text{there exists a weakly compact subset } W \text{ of } X \text{ such that } H \subset W + B_r\}.$$

β has the following useful properties (see [2] for a proof):

- i) $\beta(H) = 0$ if and only if H is relatively weakly compact;
- ii) $\beta(H_1) \leq \beta(H_2)$ if $H_1 \subseteq H_2$;
- iii) $\beta(H) = \beta(\overline{\text{co}}(H))$;
- iv) $\beta(rH) = |r| \beta(H)$, $r \in \mathbb{R}$;
- v) If (H_n) is a decreasing sequence of nonempty, bounded, closed and convex of X with $\beta(H_n) \rightarrow 0$, then $H = \bigcap H_n$ is nonempty (obviously, it is closed, convex and relatively weakly compact by i) and ii)).

The Appell and De Pascal formula [3]

$$\beta(H) = \lim_{\epsilon \rightarrow 0} \left\{ \sup_{\psi \in H} \left\{ \sup_{D \subset [0,1] \times [0,1], \text{meas}(D) \leq \epsilon} \int_D |\psi(t)| dt \right\} \right\},$$

where $\text{meas}(D)$ is the Lebesgue measure of D , so easy in calculation.

Another useful measure is the Hausdorff measure χ of noncompactness is defined as [4]

$$\chi(H) = \inf\{r > 0 : \text{there exists a compact subset } W \text{ of } X \text{ such that } H \subset W + B_r\}.$$

Both β and χ have several properties, also these two measure are connected in the following theorem.

Theorem 2.1. [5] Let H be an arbitrary nonempty and bounded subset of $L_1[0, 1]$. If H is compact in measure then $\beta(H) = \chi(H)$.

Consider the following integral operator

$$Gu(x, t) = \int_0^t \int_0^1 g(x, t, \epsilon, \tau) u(\epsilon, \tau) d\tau d\epsilon, \quad (x, t) \in [0, 1] \times [0, 1]$$

generated by the kernel $g : [0, 1] \times [0, 1] \times [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ which is assumed to be measurable with respect to all its variables. Then we have the following proposition [6].

Proposition 2.2. [6] If the linear integral operator G transforms the space $L_1[0, 1]$ into itself, then G is continuous on $L_1[0, 1]$.

3. MAIN RESULT

In this section we discuss the solvability of the nonlinear integral equation of mixed type

$$u(x, t) = f(x, t) + \int_0^t \int_0^1 g(x, t, \epsilon, \tau) h(\epsilon, \tau, u(\epsilon, \tau)) d\epsilon d\tau, \quad (x, t) \in [0, 1] \times [0, 1].$$

For simplicity we suppose the operator

$$(Tu)(x, t) = f(x, t) + \int_0^t \int_0^1 g(x, t, \epsilon, \tau) h(\epsilon, \tau, u(\epsilon, \tau)) d\epsilon d\tau, \quad (x, t) \in [0, 1] \times [0, 1]. \quad (3.1)$$

In what follows, we formulate the assumptions under which Eq.3.1 will be investigated. Namely we assume the following hypotheses:

$H_1)$ $f : [0, 1] \times [0, 1] \rightarrow R$ is a continuous function.

$H_2)$ $h : [0, 1] \times [0, 1] \times R \rightarrow R$ is continuous and satisfies in sublinear condition, so that there exist the nonnegative constants c and d such that

$$h(\epsilon, \tau, u(\epsilon, \tau)) < c + d | u(\epsilon, \tau) | .$$

$H_3)$ $g : [0, 1] \times [0, 1] \times [0, 1] \times [0, 1] \rightarrow R$ is continuous function and measurable and the integral operator G maps L_1 into itself.

$H_4)$ There exists a positive constant, satisfies $d \| G \| < 1$.

Theorem 3.1. *Let the assumptions (H1) – (H4) be satisfied. Eq.(1) has at least one solution in $L_1[0, 1]$.*

Proof. In view the assumptions and Proposition 2.2 ,we deduce that the operator T maps continuously the space L_1 into itself. Further applying our assumptions we derive the following estimate:

$$\begin{aligned} \| (Tu)(x, t) \| &= \int_0^1 \int_0^1 |f(x, t) + \int_0^t \int_0^1 g(x, t, \epsilon, \tau) h(\epsilon, \tau, u(\epsilon, \tau)) d\epsilon d\tau| dx dt \\ &\leq \int_0^1 \int_0^1 f(x, t) dx dt + \int_0^1 \int_0^1 \int_0^1 \int_0^1 |g(x, t, \epsilon, \tau)| |h(\epsilon, \tau, u(\epsilon, \tau))| d\epsilon d\tau dx dt \\ &\leq \| f \| + \int_0^1 \int_0^1 \int_0^1 \int_0^1 |g(x, t, \epsilon, \tau)| [c + d | u(\epsilon, \tau) |] d\epsilon d\tau dx dt \end{aligned}$$

In the assumption (H3), g is continuous function in L_1 . Then it is bounded, namely $|g| \leq M$ (M is positive constant). We have

$$\| (Tu)(x, t) \| \leq \| f \| + Mc + d \| G \| \| u(\epsilon, \tau) \|$$

Hence the operator T maps the ball B_r into itself where

$$r = \frac{\| f \| + Mc}{1 - d \| G \|} > 0.$$

Next, let Q_r be a subset of B_r consisting of all functions that are a.e nonnegative and nondecreasing on $[0, 1]$, then as in [7] we deduce that Q_r is nonempty, closed, convex, bounded and compact in measure. Moreover, by using the assumptions and Proposition 2.2 we deduce that the operator T transforms Q_r continuously into Q_r .

Finally, we show that the operator T is contraction to the De-Blasi measure of weak non-compactness. For this, let $U \subset Q_r$, $\epsilon > 0$, then for a measurable subset $D \subset [0, 1] \times [0, 1]$ with $meas.(D) \leq \epsilon$ and for arbitrary $u \in H$ we have

$$\int_D |(Hu)(x, t)| dt \leq \| f \|_{L_1(D)} + \int_D \int_0^1 \int_0^1 |g(x, t, \epsilon, \tau)| [c + d | u(\epsilon, \tau) |] d\epsilon d\tau ds$$

$$\begin{aligned} &\leq \|f\|_{L_1(D)} + cM \text{meas}(D) + d \int_D \int_0^1 \int_0^1 |g(x, t, \epsilon, \tau)| |u(\epsilon, \tau)| d\epsilon d\tau ds \\ &\leq \|f\|_{L_1(D)} + cM \text{meas}(D) + d \|G\|_D \|u(\epsilon, \tau)\|_{L_1(D)}, \end{aligned}$$

where $\|G\|_D$ is the norm of the operator $G : L_1(D) \rightarrow L_1(D)$.

Since

$$\|f\|_{L_1(D)} = \lim_{\epsilon \rightarrow 0} \left\{ \sup \left\{ \int_D f(x, t) ds : D \subset [0, 1] \times [0, 1], \text{meas}(D) \leq \epsilon \right\} \right\} = 0.$$

Thus as $\epsilon \rightarrow 0$ and take the supremum for all $u \in U$, $D \subset [0, 1] \times [0, 1]$, we see that $\beta(TU) \leq d \|G\| \beta(U)$.

Since $U \subset Q_r$ and Q_r is compact in measure, then we have

$$\chi(TU) \leq d \|G\| \chi(U).$$

Using all properties of Q_r and applying Darbo fixed point [8] we deduce that our operator T has a fixed point which is the solution of Eq.3.1 and the proof is complete. \square

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