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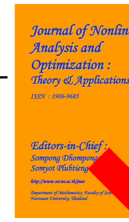

THE STABILITY OF GAUSS MODEL HAVING ONE-PREY AND TWO-PREDATORS

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ABSTRACT. Scientists are interesting to find conditi[illegible] to conti[illegible]usly use of living resources at the same time. In the present pa[illegible]me Ga[illegible]r-prey models in which tree ecologically interacting species has [illegible]n cons[illegible]ed and the behavior of their solutions in the stability aspect have been invest[illegible]ated. T[illegible] aim is to present a mathematical analysis for the above models as gl[illegible]l a[illegible] lo[illegible] stabil[illegible]nally, stability of some examples of Gauss model with two preys a[illegible] on preda[illegible]r are d[illegible]cussed.

KEYWORDS : Equilib[illegible]oint; [illegible]ss Model; Predator-Prey System; locally asymptotically Stability.
AMS Subject Classificatio[illegible] 3A30, 9[illegible]8, 34D23, 65H17.

1. INTRODUCTION

[illegible] probl[illegible] predat[illegible]-prey is well-known and an old problem in mathematical biolo[illegible] Gaus[illegible] one [illegible]scientist that was studied predator-prey problem and he was o[illegible]ained ma[illegible] results to interpret and analyze this problem. In (1934) Gauss an[illegible] (1936) Gauss and Smaragdov was studied generalization of the following [illegible]del as a m[illegible]l for predator-prey interactions:

$$\begin{cases} \frac{dx}{dt} = ax - yp(x) \\ \\ \frac{dy}{dt} = y(-\gamma + cp(x)) \end{cases} \tag{1.1}$$

Above model states that the prey growth is enhanced by its own presence and its increase growth is limited at predators present, but the predator growth is decreased by its own presence and its growth rate is enhanced at preys present.
More general form of this model as an intermediate model of predator-prey interactions is as follow:

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$$\begin{cases} \frac{dx}{dt} = xg(x) - yp(x) \\ \\ \frac{dy}{dt} = y(-\gamma + q(x)) \end{cases} \tag{1.2}$$

Here the function $g(x)$ is the specific growth rate of the prey in the absence of any predators and represents the relative increase of preys in unit of time. The function $p(x)$ is efficiency of predator on particular prey and expresses the number of prey consumed by a predator in a unit of time. The function $q(x)$ is the predator response function with respect to that particular prey. The second statement [illegible] (1.2) describes the growth rate of the predator population and the function $q(x) - \gamma$ gives the total increase of the predator population. It is clear that in the absence of prey the predator population declines.
Some of properties of $p(x)$, $q(x)$ and $g(x)$ will be studied in continuance. First

$$g(0) = \alpha \geq 0 \text{ , } g(x) \text{ is continuous and differentiable}$$
$$\text{for } x \geq 0 \text{ , } g_x \leqslant 0$$

When the environment has a carrying capacity, there [illegible]

$$k > 0 \text{ ; } g(k) = 0$$

This last assumption is biologically realistic.
The term $p(x)$ will have the following properties:

$$p(0) = 0 \text{ , } p(x) \text{ is continuous and differentiable}$$
$$\text{for } x \geq 0 \text{ , } \frac{dp(x)}{dx} > 0$$

As a consequence, we have:

$$\lim_{x \to \infty} p(x) = p_\infty \text{ , } 0 < p_\infty \leq \infty$$

For definiteness we let

$$\frac{dp(x)}{dx}(0) > \beta$$

In Gauss model (1.1) $q(x) = cp(x)$. It will be helpful that we think $q(x)$ is in manner of $p(x)$. Essentially, $q(x)$ will have properties similar to $p(x)$, namely,

$$q(0) = 0 \text{ , } q(x) \text{ is continuous and differentiable}$$
$$\text{for } x \geq 0 \text{ , } \frac{dq(x)}{dx} > 0$$
$$\lim_{x \to \infty} q(x) = q_\infty \text{ , } 0 < q_\infty \leq \infty$$
$$\frac{dq(x)}{dx}(0) > \delta$$

System (1.2) always has two equilibrium points $(0, 0)$ and $(0, k)$.The prey isocline is [illegible].

2. THE PREDATOR-PREY GAUSS MODEL WITH TWO PREY AND ONE PREDATOR

Let x and y are density of preys species and z is density of predator species. Following system represents Gauss model with having two preys and one predator:

$$\begin{cases} \frac{dx}{dt} = a_1 x - zp_1(x) \\ \\ \frac{dy}{dt} = a_2 y - zp_2(y) \\ \\ \frac{dz}{dt} = -c_1 z + c_2 z p_1(x) + c_3 z p_2(x) \end{cases} \tag{2.1}$$

In above system two preys species live in an ecosystem independently and each species is bait of special predator z and all of coefficients a_1, a_2, c_1, c_2 and c_3 are positive constant. In this system preys enhance in absences of predator species and this increasing is limited by terms $-zp_1(x)$ and $-zp_2(x)$ respectively. In absence of preys density of predators populations decrease and preys have positive efficiency on predator population.
For example, let two species rabbit and rat live in an ecosystem and each of two species is baits of fox species. More than assumptions of Gauss model, assume that p_1 and p_2 has properties of $p(x)$ in Gauss model.
In system (2.1) following properties are holds:
i. Equilibrium point $(0, 0, 0)$ is stable point for system (2.1), in other hand when density of populations is zero density of population species will not change in different time.
ii. If population density of one of preys species is zero, then system (2.1) convert to system (1.1).
iii. If population density of predator species is zero, then system (2.1) convert to system with having two species that live in an ecosystem independently.
iv. If population density of two species are zero, then system (2.1) convert to equation of growth rate.
v. Orbit of solutions system (2.1) is $intR_3^+ = int\{x_i | x_i \geqslant 0, i = 1, 2, 3\}$.
The terms $p_1(x)$ and $p_2(y)$ has properties described as follow:

$$p_1(0) = 0 \ , \ p_1(x) \text{ is continuous and differentiable}$$
$$\text{for } x \geq 0 \ , \ \frac{dp_1(x)}{dx} > 0$$
$$p_2(0) = 0 \ , \ p_2(y) \text{ is continuous and differentiable}$$
$$\text{for } y \geq 0 \ , \ \frac{dp_2(y)}{dy} > 0$$

3. LOCAL STABILITY

We using the linearization method to study the stability of system (2.1). For this means, we account the jacobian matrix, which may be found as the follow:

$$J|_{(x,y,z)} = \begin{pmatrix} a_1 - z\frac{dp_1(x)}{dx} & 0 & -p_1(x) \\ 0 & a_2 - z\frac{dp_2(y)}{dy} & -p_2(y) \\ c_2 z\frac{dp_1(x)}{dx} & c_3 z\frac{dp_2(y)}{dy} & -c_1 + c_2 p_1(x) + c_3 p_2(y) \end{pmatrix}$$

Now let $(\overline{x}, \overline{y}, \overline{z})$ be equilibrium point of system (2.1). Then

$$A = J|_{(\overline{x},\overline{y},\overline{z})} = \begin{pmatrix} a_1 - \overline{z}\frac{dp_1(\overline{x})}{dx} & 0 & -p_1(\overline{x}) \\ 0 & a_2 - \overline{z}\frac{dp_2(\overline{y})}{dy} & -p_2(\overline{y}) \\ c_2\overline{z}\frac{dp_1(\overline{x})}{dx} & c_3 z\frac{dp_2(\overline{y})}{dy} & -c_1 + c_2 p_1(\overline{x}) + c_3 p_2(\overline{y}) \end{pmatrix}$$

So if $trA < 0$ and $detA > 0$, then system (2.1) is locally asymptotically stable.
Let $p_1(\bar{x}), p_2(\overline{y}) > 0$

$$A_1 = a_1 - \overline{z}\frac{dp_1(\overline{x})}{dx}$$
$$A_2 = a_2 - \overline{z}\frac{dp_2(\overline{y})}{dy}$$
$$A_3 = -c_1 + c_2 p_1(\overline{x}) + c_3 p_2(\overline{y})$$
$$A_4 = c_2\overline{z}\frac{dp_1(\overline{x})}{dx}$$
$$A_5 = c_3 z\frac{dp_2(\overline{y})}{dy}$$

So

$$A = J|_{(\overline{x},\overline{y},\overline{z})} = \begin{pmatrix} A_1 & 0 & -p_1(\overline{x}) \\ 0 & A_2 & -p_2(\overline{y}) \\ A_4 & A_5 & A_3 \end{pmatrix}$$

Therefore

$$detA = A_1A_2A_3 - p_2A_1A_5 + p_1A_2A_4$$

Now if $A_1, A_3 < 0$ and $A_2 > 0$ Then $detA > 0$, if $A_2 < A_1 + A_3$ too, then $trA < o$ and so system (2.1) is locally asymptotically stable.
So following proposition is proved.

Proposition 3.1. *Let $p_1(\overline{x}), p_2(\overline{y}) > 0$, the system (2.1) is locally asymptotically stable in equilibrium point $(\overline{x},\overline{y},\overline{z})$ provided $A_1, A_3 < 0$ and $A_2 > 0$ and $A_2 < A_1 + A_3$* .

4. GLOBAL STABILITY

In this section, we will prove the global stability of the system (2.1) by constructing a suitable Lyapunove function.

Theorem 4.1. *The system (2.1) is globally asymptotically stable in equilibrium point $(\overline{x},\overline{y},\overline{z})$ provided $x < \overline{x}$, $y < \overline{y}$ and $z > \overline{z}$.*

Proof:
Let us consider a suitable Lyapunove function

$$v(x, y, z) = hx + ky + z$$

where $h = c_2$ and $k = c_3$. Obviously v is positive definite. Now the time derivative of v along the solution of (2.1) is given by:

$$\frac{dv}{dt} = h\frac{dx}{dt} + k\frac{dy}{dt} + \frac{dz}{dt}$$

Now by substituting $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$ from system (2.1) $\frac{dv}{dt}$ is given by:

$$\frac{dv}{dt} = h\frac{dx}{dt} + k\frac{dy}{dt} + \frac{dz}{dt}$$

$\frac{dv}{dt} = h[a_1x - zp_1(x) - a_1\overline{x} + \overline{z}p_1(\overline{x})] + k[a_2y - zp_2(y) - a_2\overline{y} + \overline{z}p_2(\overline{y})] + [-c_1z + c_2zp_1(x) + c_3zp_2($[illegible]$) + c_1\overline{z}$ [illegible] $zp_1(x)$ [illegible] $zp_2(y)] = ha_1(x - \overline{x}) - h(zp_1(x) - \overline{z}p_1(\overline{x})) + ka_2(y - \overline{y}) - h(zp_2(y) - \overline{z}p_2(\overline{y})) - c_1(z - \overline{z}) - c_2(zp_1(x) - \overline{z}p_1(\overline{x})) + c_3(zp_2(y) - \overline{z}p_2(\overline{y}))$
As regarding to $h = c_2$ and $k = c_3$

$$\frac{dv}{dt} = c_2a_1(x - \overline{x}) + c_3a_2(y - \overline{y}) - c_1(z - \overline{z})$$

Therefor [illegible] if $x < \overline{x}$, $y < \overline{y}$ and $z > \overline{z}$.

4.1. Analysis of Example 1. Consider following system:

$$\begin{cases} \frac{dx}{dt} = a_1x - b_1xz \\ \\ \frac{dy}{dt} = a_2y - b_2yz \\ \\ \frac{dz}{dt} = -c_1z + b_1c_2zx + b_2c_3zy \end{cases} \qquad (4.1)$$

In the above system all of coefficients $a_1, a_2, b_1, b_2, c_1, c_2$ and c_3 are positive constant. In system (4.1) efficiency of the predator species on preys species and so efficiency of the preys species on predator species are linear.
Points $(0, 0, 0), (0, \frac{c_1}{b_2c_3}, \frac{a_2}{b_2})$ and $(\frac{c_1}{b_1c_2}, 0, \frac{a_1}{b_1})$ are equilibrium points of system (4.1), in which we analyzing stability of this point by using Jacobian matrix. Now assume that $b_1 = b_2 = b$ and $a_1 = a_1 = a$, thus intersection points of two lines $bz = a$ and $c_2bx + c_3by$ are equilibrium points of system (4.1) too. In order to analyzing

this points, which we denote these by $(\overline{x}, \overline{y}, \overline{z})$ using the Lyapunov function. First Jacobian matrix of system (4.1) is found out as follow:

$$J|_{(x,y,z)} = \begin{pmatrix} a_1 - b_1 z & 0 & -b_1 x \\ 0 & a_2 - b_2 z & -b_2 y \\ b_1 c_2 z & b_2 c_3 z & -c_1 + b_2 c_2 x + b_3 c_3 y \end{pmatrix}$$

So

$$J|_{(0,0,0)} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & -c_1 \end{pmatrix}$$

Thus equilibrium point $(0,0,0)$ is a saddle point for system(4.1).

$$J|_{(0, \frac{c_1}{b_2 c_3}, \frac{a_2}{b_2})} = \begin{pmatrix} a_1 - \frac{b_1 a_2}{b_2} & 0 & 0 \\ 0 & 0 & [illegible] \\ \frac{a_2 b_1 c_2}{b_2} & a_2 c_3 & 0 \end{pmatrix}$$

So $a_1 - \frac{b_1 a_2}{b_2}$ and $\pm i\sqrt{a_2 c_1}$ are eigenvalues of ab[illegible]e matrix, thus [illegible]ibrium point $(0, \frac{c_1}{b_2 c_3}, \frac{a_2}{b_2})$ is hyperbolic point for system (4.1).
If $\frac{a_1}{a_2} < \frac{b_1}{b_2}$ system (4.1) is stable in equilibrium p[illegible]t $(\frac{c_1}{b_1 c_2}, 0, [illegible])$. Because

$$J|_{(\frac{c_1}{b_1 c_2}, 0, \frac{a_1}{b_1})} = \begin{pmatrix} 0 & [illegible]0 & -a_1 \\ 0 & a_2 - [illegible] b_2 & 0 \\ a_1 [illegible] & \frac{a_1 b [illegible] 3}{[illegible]} & 0 \end{pmatrix}$$

So zero and $a_2 - \frac{a [illegible]}{[illegible]}$ are eig[illegible]alues of a[illegible]e matrix, thus system (4.1)in equilibrium point $(\frac{c_1}{b_1 c_2}, 0, \frac{a [illegible]}{b [illegible]})$ [illegible] stable [illegible] $\frac{a_1}{[illegible]} < \frac{b_1}{b_2}$.
so the following proposi[illegible] has be[illegible]proved:

Proposition [illegible] *[illegible]wing [illegible]ents for system (4.1) are held:*
i. Equilibriu[illegible] point $([illegible]0, 0)$ is a saddle point for system(4.1).
ii. Equilibri[illegible] po[illegible] $(0, \frac{[illegible]}{b_2 [illegible]}, \frac{a_2}{b_2})$ is hyperbolic point for system (4.1).
iii. [illegible] $< \frac{b [illegible]}{b_2}$ [illegible]stem (4[illegible]s stable in equilibrium point $(\frac{c_1}{b_1 c_2}, 0, \frac{a_1}{b_1})$.

Th[illegible] *The system (4.1) is stable in equilibrium point $(\overline{x}, \overline{y}, \overline{z})$.*

Proof:
[illegible]nsider Lyapunov function
$v([illegible]) = c_2 \int_{\overline{x}}^{x} \frac{s-\overline{x}}{s} ds + c_3 \int_{\overline{y}}^{y} \frac{t-\overline{y}}{t} dt + \int_{\overline{z}}^{z} \frac{v-\overline{z}}{v} dv$
Now by differentiate of above Lyapunov function with respect to variable t $\frac{dv}{dt}$ is found out as follow:
$\frac{dv}{dt} = c_2 \frac{x-\overline{x}}{x} \frac{dx}{dt} + c_3 \frac{y-\overline{y}}{y} \frac{dy}{dt} + \frac{z-\overline{z}}{z} \frac{dz}{dt}$
Now by instituted $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$ from system (4.1) $\frac{dv}{dt}$ is found out as follow:
$\frac{dv}{dt} = c_2(x-\overline{x})(a_1 - b_1 z) + c_3(y - \overline{y})(a_2 - b_2 z) + (z - \overline{z})(-c_1 + b_1 c_2 x + b_2 c_3 y)$
As regarding, $b_1 = b_2 = b$ and $a_1 = a_1 = a$ and $bz = a$ and $c_2 bx + c_3$

$$\frac{dv}{dt} = 0$$

and proof is completed.

4.2. **Analysis of Example 2.** Consider following system of system (2.1):

$$\begin{cases} \frac{dx}{dt} = x(1 - \frac{z}{1+x}) \\ \\ \frac{dy}{dt} = y(1 - \frac{z}{1+y}) \\ \\ \frac{dz}{dt} = z(1 - \frac{z}{v_1 x} - \frac{z}{v_2 y}) \end{cases} \tag{4.2}$$

It is clear that x, y and $z \neq 0$. To obtaining equilibrium points of system (4.2) let first equation of above system is zero, in other hand $1 + x = z$. Now let twi[illegible] equation of above system is zero, in other hand $1+y = z$. two terms above conclu[illegible] $x = y$. Now constitute x and y in third equation of system (4.2) and simplify z [illegible] given by:

$$z = \frac{v_1 v_2}{v_1+v_2} = h$$

Therefor equilibrium point(s) of system (4.2)is given by $(x, [illegible] hx)$.
Jacobian of system(4.2) is as follow:

$$J|_{(x,y,z)} = \begin{pmatrix} 1 - \frac{z}{(1+x)^2} & 0 & -\frac{x}{1+x} \\ 0 & 1 - \frac{z}{(1+y)^2} & -\frac{y}{1+y} \\ \frac{z^2}{v_1 x^2} & \frac{z^2}{v_2 y^2} & 1 - 2\frac{z}{v_1 x} - 2\frac{z}{v_2 y} \end{pmatrix}$$

Now by substituting equilibrium point(s) and simplifying, jacobian matrix is given by:

$$A = J|_{(x,x,hx)} = \begin{pmatrix} 1 - \frac{hx}{(1+x)^2} & 0 & -\frac{x}{1+x} \\ 0 & 1 - \frac{hx}{(1+x)^2} & -\frac{x}{1+x} \\ \frac{h^2}{v_1} & \frac{h^2}{v_2} & -1 \end{pmatrix}$$

$$trA = 1 - [illegible]\frac{x}{([illegible])^2}$$

And by simplifying

$$detA = h[illegible] + \frac{2[illegible]}{(1+x)} [illegible] 1 - h^2[\frac{x^2}{(1+x)^4} + \frac{x^2}{(1+x)^3}]$$

If $trA < 0$ and $detA > 0$ the system (4.2) is locally asymptotically stable. Let $1 < \frac{hx}{(1+x)^2}$ and $1 > [illegible]$ since [illegible]ptions of proposition(3.1) is true, thus the system (4.2) is locally asymptotically stable.
Therefor following proposition is proved.

Proposition 4.2. *[illegible] $1 < \frac{[illegible]}{([illegible]x)^2}$ and $1 > \frac{hx}{(1+x)^2}$, thus the system (4.2) is locally asymptotically stable.*

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