Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 2, No. 2, (2011), 337-353
ISSN : 1906-9685
http://www.math.sci.nu.ac.th

APPROXIMATION METHOD FOR GENERALIZED MIXED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS FOR A COUNTABLE FAMILY OF NONEXPANSIVE MAPPINGS

SUTEE CHAIYASIL, SUTHEP SUANTAI*

Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand

Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand

ABSTRACT. In this research, we introduce an iterative scheme by using the concept of K-mapping for finding a common element of the set of fixed points of an infinite family of nonexpansive mappings and the set of solution of a generalized mixed equilibrium problem in a Hilbert space. Then, we prove strong convergence of the purposed iterative algorithm to a common element of the two sets. Moreover, we also give a numerical result of the studied method.

KEYWORDS : Generalized mixed equilibrium problems; Nonexpansive mapping; K-mapping; Strong convergence; Fixed point.

1. INTRODUCTION

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $A : C \to H$ be a nonlinear mapping, $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a function and $F : C \times C \to \mathbb{R}$ be a bifunction. A mapping T of H into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$, $f : C \to C$ be a contraction if $\|fx - fy\| \leq a\|x - y\|$ where $a \in (0, 1)$. The set of fixed points of T (i.e. $F(T) = \{x \in H : Tx = x\}$) denoted by $F(T)$. Goebel and Kirk [2] showed that $F(T)$ is always closed convex and also nonempty provided T has a bounded trajectory.

A bounded linear operator A on H is called *strongly positive* with coefficient $\bar{\gamma}$ if there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2 \ \forall x \in H.$$

For a bifunction $F : C \times C \longrightarrow \mathbb{R}$, equilibrium problem for F is to find $x \in C$ such that

$$F(x, y) \geq 0 \ \forall y \in C. \tag{1.1}$$

* *Corresponding author.*
Email address : su_t_na@hotmail.com (S. Chaiyasil) and scmti005@chiangmai.ac.th (S. Suantai).
Article history : Received 28 August 2011. Accepted 30 August 2011.

The set of solutions of (1.1) is denoted by $EP(F)$. Many problems in physics, optimization, and economics are seeking some elements of $EP(F)$, see [4], [5]. Several iterative methods have been proposed to solve the equilibrium problem, see for instance, [1], [4], [5], [6], [10], [12].

In 2005, Combettes and Hirstoaga [5] introduced an iterative scheme of finding the best approximation to the initial data when EP(F) is nonempty and proved a strong convergence theorem.

A mapping A of C into H is called *inverse-strongly monotone*, see [3], if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2 \ \forall x,y \in C.$$

The variational inequality problem is to find a point $u \in C$ such that

$$\langle v-u, Au\rangle \ \forall v \in C. \tag{1.2}$$

The set of solutions of the variational inequality is denoted by $VI(C,A)$.

In 2008, Ceng and Yao [10] considered the following mixed equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x,y)+\varphi(y) \geq \varphi(x),\ \forall y \in C, \tag{1.3}$$

where $\varphi : C \to R$ is a function.
The set of solutions of (1.3) is denoted by $MEP(F,\varphi)$.

In 2008, Peng and Yao [6] considered the following generalized mixed equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x,y)+\langle Ax, y-x\rangle+\varphi(y) \geq \varphi(x),\ \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $GMEP(F,\varphi,A)$. It is easy to see that x is a solution of problem (1.4) implies that $x \in dom\varphi = \{x \in C : \varphi(x) < +\infty\}$.

In the case of $A \equiv 0$, $GMEP(F,\varphi,A) = MEP(F,\varphi)$. In the case of $F \equiv \varphi \equiv 0$, then $GMEP(F,\varphi,A) = V(C,A)$. In the case of $A \equiv \varphi \equiv 0$, then $GMEP(F,\varphi,A) = EP(F)$.

In 2008, Peng and Yao [6] introduced an iterative scheme for finding a common element of the set of solutions of problem (1.4), the set of fixed points of a nonexpansive mappings and the set of solutions of a variational inequality for a monotone, Lipschitz continuous mapping and obtained a strong convergence theorem.

In 2009, Peng and Yao [12] introduced iterative schemes by using the concept of *W-mapping* for finding a common element of the set of solutions of $GMEP(F,\varphi,A)$ and the set of common fixed point of infinitely family of nonexpansive mappings of C into itself.

The concept of W-mapping was first introduced by Shimoji and Takahashi [7]. They defined the mapping W_n as follows:

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n)U_{n,n+1},\\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I,\\
U_{n,n-2} &= \lambda_{n-2} T_{n-2} U_{n,n-1} + (1-\lambda_{n-2})I,\\
&\ \vdots\\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,
\end{aligned}$$

$$\begin{aligned} &\vdots \\ U_{n,2} &= \lambda_2 T_2 U_{n,2} + (1-\lambda_2)I, \\ W_n &= U_{n,1} = \lambda_1 T_1 U_{n,1} + (1-\lambda_1)I, \end{aligned} \tag{1.5}$$

where $\{\lambda_n\} \subseteq [0,1]$ and $\{T_n\}_{i=1}^{\infty}$ is a sequence of nonexpansive mappings of C into itself. This mapping is called the $W-$*mapping* generated by $T_n, T_{n-1}, \dots, T_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$. They proved that if X is strictly convex Banach space, then $F(W_n) = \bigcap_{i=1}^{\infty} F(T_i)$ where $0 < \lambda_i \leq d < 1$ for every $i \in \mathbb{N}$.

Recently, A. Kangtunyakarn and S. Suantai [13] introduced the concept of K-mapping and employed this mapping for finding a common element of the set the of the solution of an equilibrium problem and the set of common fixed points of a finite family of nonexpansive mapping, they defined $K_n : C \to C$ as follows:

$$\begin{aligned} U_{n,0} &= I, \\ U_{n,1} &= \lambda_1 T_1 U_{n,0} + (1-\lambda_1)U_{n,0}, \\ U_{n,2} &= \lambda_2 T_2 U_{n,1} + (1-\lambda_2)U_{n,1}, \\ U_{n,3} &= \lambda_3 T_3 U_{n,2} + (1-\lambda_3)U_{n,2}, \\ &\vdots \\ U_{n,k} &= \lambda_k T_k U_{n,k-1} + (1-\lambda_k)U_{n,k-1}, \\ U_{n,k+1} &= \lambda_{k+1} T_{k+1} U_{n,k} + (1-\lambda_{k+1})U_{n,k}, \\ &\vdots \\ U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n-2} + (1-\lambda_{n-1})U_{n,n-2}, \\ K_n &= U_{n,n} = \lambda_n T_n U_{n,n-1} + (1-\lambda_n)U_{n,n-1}, \end{aligned}$$

such a mapping K_n is called the *K-mapping* generated by $T_1, T_2, \dots, T_n$ and $\lambda_1, \lambda_2, \dots \lambda_n$.
In this research, we introduce K-mapping defined as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \alpha_n T_n U_{n,n+1} + (1-\alpha_n)U_{n,n+1}, \\ U_{n,n-1} &= \alpha_{n-1} T_{n-1} U_{n,n} + (1-\alpha_{n-1})U_{n,n}, \\ U_{n,n-2} &= \alpha_{n-2} T_{n-2} U_{n,n-1} + (1-\alpha_{n-2})U_{n,n-1}, \\ &\vdots \\ U_{n,k} &= \alpha_k T_k U_{n,k+1} + (1-\alpha_k)U_{n,k+1}, \\ U_{n,k-1} &= \alpha_{k-1} T_{k-1} U_{n,k} + (1-\alpha_{k-1})U_{n,k}, \\ &\vdots \\ U_{n,2} &= \alpha_2 T_2 U_{n,3} + (1-\alpha_2)U_{n,3}, \\ K_n &= U_{n,1} = \alpha_1 T_1 U_{n,2} + (1-\alpha_1)U_{n,2}. \end{aligned}$$

Such a mapping K_n is called K-mapping generated by $T_n, T_{n-1}, \dots, T_1$ and $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$.

In 2007, S. Takahashi and W. Takahashi [15] modified the following iterative solution for finding a common element of the set of fixed point of a nonexpansive mapping and the solution of equilibrium problem by:
Let H be Hilbert space, C be a nonempty closed convex subset of H, f be a

contraction of H into itself, S be a a nonexpansive mapping from C to H, $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_1 \in H$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n, \ \forall n \in \mathbb{N}. \end{cases} \tag{1.6}$$

In 2008, S. Takahashi and W. Takahashi [14] modified the following iterative solution for finding a common element of the set of fixed point of a nonexpansive mapping and the solution of equilibrium problems by:
Let $u \in C$ and $x_1 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S[\alpha_n u + (1-\alpha_n)z_n], \ \forall n \in \mathbb{N}. \end{cases} \tag{1.7}$$

Motivated by this two works, we introduce an iterative scheme for finding a common element if the set of common fixed point of a countable family of nonexpansive mappings and the set of solution of generalized mixed equilibrium problems.

2. PRELIMINARIES

In this section, we give some useful lemmas that will be used for the main result in the next section.

Let C be closed convex subset of a Hilbert space H, let P_C be the metric projection of H onto C i.e., for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following characterizes the projection P_C.

Lemma 2.1. *(See [9]) Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality*
$\langle x - y, y - z\rangle \geq 0 \ \forall z \in C.$

Lemma 2.2. *(See [11]) Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1-\alpha_n)s_n + \delta_n, \ \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty,$

(2) $\displaystyle\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\displaystyle\sum_{n=1}^{\infty} |\delta_n| < \infty.$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3. *In a real Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, \ x + y\rangle \ \forall x, y \in H.$$

Theorem 2.4. *(See [16]) A Banach space X is said to satisfy Opial's condition if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

For solving the generalized mixed equilibrium problem, let us give the following assumptions for the bifunction F, the function φ and the set C:

$(A1)$ $F(x,x) = 0\ \forall x \in C$,

$(A2)$ F is monotone, i.e. $F(x,y) + F(y,x) \leq 0\ \forall x, y \in C$,

$(A3)$ for each $y \in C,\ x \mapsto F(x,y)$ is weakly upper semicontinuous,

$(A4)$ for $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous,

$(B1)$ $\forall x \in H$, and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y \in C \bigcap dom(\varphi)$ such that for any $z \in C \backslash D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(x),$$

$(B2)$ C is bounded set.

Lemma 2.5. *(See [6]) Let C be a nonempty closed convex subset of a Hilbert space H, $F : C \times C \to \mathbb{R}$ be a function such that satisfy $(A1)-(A4)$ and $\varphi : C \to \mathbb{R} \bigcup\{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either $(B1)$ or $(B2)$ holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$T_r(x) = \{z \in C : F(z,y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z),\ \forall y \in C\}$

for all $x \in H$. Then the following conclusions hold:

(1) for each $x \in H$, $T_r(x) \neq \emptyset$,

(2) T_r is is single-valued,

(3) T_r is firmly nonexpansive, i.e.

$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle\ \forall x, y \in H,$

(4) $F(T_r) = MEP(F, \varphi)$,

(5) $MEP(F, \varphi)$ is closed and convex.

Lemma 2.6. *In a strictly convex Banach space E, if*

$$\|x\| = \|y\| = \|\lambda x + (1 - \lambda)y\|$$

for all $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$.

By using the same argument as in [13] (Lemma 2.7 and Lemma 2.8), we obtain that the two following lemmas.

Lemma 2.7. *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Let K_n be the K-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$ for each $n \in \mathbb{N}$. Then for every $x \in C$ $\lim_{n\to\infty} K_n x$ exists.*

Let $K : C \to C$ be defined by $Kx = \lim_{n\to\infty} K_n x$. Such a mapping K is called K-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$.

Lemma 2.8. *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Let K be the K-mapping generated by $T_n, T_{n-1}, \ldots$ and $\lambda_n, \lambda_{n-1}, \ldots$ for each $n \in \mathbb{N}$. Then $F(K) = \bigcap_{i=1}^{\infty} F(T_i)$.*

Lemma 2.9. *Let C be a closed convex subset of Hilbert space, let $\{T_i\}_{i=1}^{\infty}$ be a infinite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i < 1$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$ for* *every* $i = 1, 2, \ldots$. *Let* K_n *be the K-mapping generated by* $T_n, T_{n-1}, \ldots, T_1$ *and* $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$ *for each* $n \in \mathbb{N}$. *Then*

$$\sum_{n=1}^{\infty} \sup\{\|K_{n+1}x - K_n x\| : x \in B\} < \infty \tag{2.1}$$

for every bounded subset B *of* C.

Proof. Let B be a bounded subset of C. Then for $n \in \mathbb{N}$, $x \in B$, we have

$$\begin{aligned}
\|K_{n+1}x - K_n x\| &= \|U_{n+1,n+1}x - U_{n,n}x\| \\
&= \|\lambda_1 T_1 U_{n+1,2}x + (1-\lambda_1)U_{n+1,2}x - (\lambda_1 T_1 U_{n,2}x + (1-\lambda_1)U_{n,2}x)\| \\
&= \|\lambda_1 (T_1 U_{n+1,2}x_2 - T_1 U_{n,2}x) + (1-\lambda_1)(U_{n+1,2}x - U_{n,2}x)\| \\
&\leq \lambda_1 \|U_{n+1,2}x - U_{n,2}x\| + (1-\lambda_1)\|U_{n+1,2}x - U_{n,2}x\| \\
&= \|U_{n+1,2}x - U_{n,2}x\| \\
&\quad \vdots \\
&= \|U_{n+1,n+1}x - U_{n,n+1}x\| \\
&= \|\lambda_{n+1} T_{n+1}x + (1-\lambda_{n+1})Ix - Ix\| \\
&= \lambda_{n+1}\|T_{n+1}x - x\| \\
&\leq \lambda_{n+1} M.
\end{aligned} \tag{2.2}$$

where $M = \sup_{n\in\mathbb{N}} \sup\{\|T_{n+1}x - x\| : x \in B\}$.
This implies $\sup\{\|K_{n+1}x - K_n x\| \, x \in B\} \leq \lambda_{n+1}M$. By the assumption $\sum_{i=1}^{\infty} \lambda_i < \infty$, we obtain

$$\sum_{n=1}^{\infty} \sup\{\|K_{n+1}x - K_n x\| : x \in B\} < \infty.$$

□

Lemma 2.10. *Let* C *be a nonempty closed convex subset of a strictly convex Banach space. Let* $\{T_i\}_{i=1}^{\infty}$ *be a infinite family of nonexpansive mappings of* C *into itself with* $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ *and let* $\lambda_1, \lambda_2, \ldots$ *be real numbers such that* $0 < \lambda_i < 1$ *for every* $i = 1, 2, \ldots$ *with* $\sum_{i=1}^{\infty} \lambda_i < \infty$. *Let* K_n *be the* $K-$*mapping generated by* $T_n, T_{n-1}, \ldots, T_1$ *and* $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$ *for each* $n \in \mathbb{N}$ *and* $\{x_n\} \subset C$ *a bounded sequence. Then*

$$\|Kx_n - K_n x_n\| \to 0 \text{ as } n \to \infty. \tag{2.3}$$

Proof. By Lemma 2.7, we have $Kx_n = \lim_{m\to\infty} K_m x_n$. For $m \geq n$, by (2.2) we have

$$\begin{aligned}
\|K_m x_n - K_n x_n\| &= \|K_m x_n - K_{m-1}x_n\| + \|K_{m-1}x_n - K_{m-2}x_n\| \\
&\quad + \ldots + \|K_{n+1}x_n - K_n x_n\| \\
&\leq (\lambda_m + \lambda_{m-1} + \ldots + \lambda_{n+1})M \\
&\leq \sum_{i=n+1}^{m} \lambda_i M,
\end{aligned}$$

where $M = \sup\{\|T_k x_n - x_n\| : k \in \mathbb{N}, n \in \mathbb{N}\}$. It follows that

$$\|Kx_n - K_n x_n\| = \lim_{m\to\infty} \|K_m x_n - K_n x_n\| \leq \sum_{i=n+1}^{\infty} \lambda_i M.$$

This implies $\lim_{n\to\infty} \|Kx_n - K_n x_n\| = 0$. □

3. MAIN RESULT

In this section, we modify iterative scheme (1.6) by using concept of K-mapping and prove strong convergence of the sequences $\{z_n\}$ and $\{x_n\}$ under some control conditions.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space, $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the condition (A1)-(A4) and (B1) or (B2) and let $\varphi : C \to \mathbb{R} \bigcup \{+\infty\}$ be a proper lower semi-continuous and convex function. Let A be an α-inverse strongly monotone mapping of C into H, $f : C \to C$ be a contraction map with coefficient $0 < a < 1$, $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself with $\Omega = \bigcap_{i=1}^{\infty} F(T_i) \bigcap GMEP(F, A, \varphi) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Let K_n be the K-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$ for each $n \in \mathbb{N}$. Let $x_1 \in C$ and $\{z_n\}$ and $\{x_n\}$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0 \; \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)K_n z_n, \; \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where

$$\{\alpha_n\} \subset [0, 1] \text{ and } \{r_n\} \subset [0, 2\alpha] \text{ satisfy the following conditions :}$$

$$(i) \; 0 < a \leq r_n \leq b < 2\alpha,$$

$$(ii) \; \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty,$$

$$(iii) \; \lim_{n\to\infty} \alpha_n = 0, \; \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

Then $\{x_n\}$ and $\{z_n\}$ converge strongly to $z_0 \in \Omega$,
where $z_0 = P_\Omega f(z_0)$.

Proof. First, we show that $(I - r_nA)$ is nonexpansive. Let $x, y \in C$. Since A is $\alpha-$inverse strongly monotone and $r_n < 2\alpha \; \forall n \in \mathbb{N}$, we have

$$\begin{aligned} \|(I - r_nA)x - (I - r_nA)y\|^2 &= \|x - y - r_n(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\alpha r_n\|Ax - Ay\|^2 + r_n^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 + r_n(r_n - 2\alpha)\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2, \end{aligned} \tag{3.2}$$

Thus $(I - r_nA)$ is nonexpansive.
Since

$$F(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0 \; \forall y \in C,$$

we obtain

$$F(z_n, y) + \varphi(y) - \varphi(z_n) + \frac{1}{r_n}\langle y - z_n, z_n - (I - r_nA)x_n\rangle \geq 0 \; \forall y \in C.$$

By Lemma 2.5, we have $z_n = T_{r_n}(x_n - r_nAx_n) \; \forall n \in \mathbb{N}$.
Let $z \in \bigcap_{i=1}^{\infty} F(T_i) \bigcap GMEP(F, A, \varphi)$. Then $F(z, y) + \langle Az, y - z\rangle + \varphi(y) - \varphi(z) \geq 0$

for all $y \in C$, so

$$F(z,y) + \frac{1}{r_n}\langle y-z, z-z+r_nAz\rangle + \varphi(y) - \varphi(z) \geq 0 \text{ for all } y \in C.$$

Again by Lemma 2.5, we have $z = T_{r_n}(z - r_nAz)$. Since $I - r_nA$ and T_{r_n} are nonexpansive, we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n(f(x_n) - z) + (1-\alpha_n)(K_nz_n - z)\| \\
&= \|\alpha_n f(x_n) + \alpha_n f(z) - \alpha_n f(z) - \alpha_n z + (1-\alpha_n)(K_nz_n - z)\| \\
&\leq \alpha_n\|f(x_n) - f(z)\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|K_nz_n - z\| \\
&\leq \alpha_n(a\|x_n - z\| + \|f(z) - z\|) + (1-\alpha_n)\|z_n - z\| \\
&= \alpha_n(a\|x_n - z\| + \|f(z) - z\|) \\
&\quad +(1-\alpha_n)\|T_{r_n}(I - r_nA)x_n - T_{r_n}(I - r_nA)z\| \\
&\leq \alpha_n(a\|x_n - z\| + \|f(z) - z\|) + (1-\alpha_n)\|x_n - z\| \\
&= \alpha_n a\|x_n - z\| + (1-\alpha_n)\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&= (1 - \alpha_n + \alpha_n a)\|x_n - z\| + \alpha_n(1-a)\frac{\|f(z) - z\|}{(1-a)} \\
&\leq \max\{\|x_n - z\|, \frac{\|f(z) - z\|}{(1-a)}\}. \qquad (3.3)
\end{aligned}$$

By induction, we have $\|x_n - z\| \leq \max\{\|x_1 - z\|, \frac{\|f(z)-z\|}{(1-a)}\}$ $\forall n \in \mathbb{N}$, this implies $\{x_n\}$ bounded. It follows that $\{z_n\}$ and $\{K_nz_n\}$ are also bounded. Next, we will show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Putting $u_n = x_n - r_nAx_n$. Then, we have $z_{n-1} = T_{r_{n-1}}u_{n-1}$ and $z_n = T_{r_n}u_n$. By definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + (1-\alpha_n)K_nz_n - \alpha_{n-1}f(x_{n-1}) - (1-\alpha_{n-1})K_{n-1}z_{n-1}\| \\
&= \|\alpha_n f(x_n) - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) - \alpha_{n-1}f(x_{n-1}) \\
&\quad +(1-\alpha_n)K_nz_n - (1-\alpha_n)K_{n-1}z_{n-1} + (1-\alpha_n)K_{n-1}z_{n-1} \\
&\quad -(1-\alpha_{n-1})K_{n-1}z_{n-1}\| \\
&\leq \alpha_n\|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\quad +(1-\alpha_n)\|K_nz_n - K_{n-1}z_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|K_{n-1}z_{n-1}\| \\
&\leq \alpha_n a\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\quad +(1-\alpha_n)\|K_nz_n - K_{n-1}z_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|K_{n-1}z_{n-1}\| \\
&= \alpha_n a\|x_n - x_{n-1}\| + (1-\alpha_n)\|K_nz_n - K_{n-1}z_{n-1}\| \\
&\quad +|\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|K_{n-1}z_{n-1}\|) \\
&\leq \alpha_n a\|x_n - x_{n-1}\| + (1-\alpha_n)[\|K_nz_n - K_nz_{n-1}\| \\
&\quad +\|K_nz_{n-1} - K_{n-1}z_{n-1}\|] + |\alpha_n - \alpha_{n-1}|L \\
&\leq \alpha_n a\|x_n - x_{n-1}\| + (1-\alpha_n)[\|z_n - z_{n-1}\| \\
&\quad +\|K_nz_{n-1} - K_{n-1}z_{n-1}\|] + |\alpha_n - \alpha_{n-1}|L, \qquad (3.4)
\end{aligned}$$

where $L = \sup\{\|f(x_n)\| + \|K_nz_n\| : n \in \mathbb{N}\}$. Since T_{r_n} and $I - r_nA$ are nonexpansive, we have

$$\begin{aligned}
\|z_n - z_{n-1}\| &= \|T_{r_n}u_n - T_{r_{n-1}}u_{n-1}\| \\
&= \|T_{r_n}u_n - T_{r_n}u_{n-1} + T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}\|
\end{aligned}$$

$$\begin{aligned} &\leq \|T_{r_n}u_n - T_{r_n}u_{n-1}\| + \|T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}\| \\ &\leq \|u_n - u_{n-1}\| + \|T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}\| \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \|u_n - u_{n-1}\| &= \|x_n - r_nAx_n - x_{n-1} + r_{n-1}Ax_{n-1}\| \\ &= \|x_n - r_nAx_n - r_nAx_{n-1} + r_nAx_{n-1} - x_{n-1} + r_{n-1}Ax_{n-1}\| \\ &= \|(I - r_nA)x_n - (I - r_nA)x_{n-1} + (r_{n-1} - r_n)Ax_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + |r_{n-1} - r_n|\|Ax_{n-1}\|. \end{aligned} \tag{3.6}$$

By Lemma 2.5, we have

$$F(T_{r_{n-1}}u_{n-1}, y) + \frac{1}{r_{n-1}}\langle y - T_{r_{n-1}}u_{n-1}, T_{r_{n-1}}u_{n-1} - u_{n-1}\rangle + \varphi(y) - \varphi(T_{r_{n-1}}u_{n-1}) \geq 0,$$

$\forall y \in C$ and

$$F(T_{r_n}u_{n-1}, y) + \frac{1}{r_n}\langle y - T_{r_n}u_{n-1}, T_{r_n}u_{n-1} - u_{n-1}\rangle + \varphi(y) - \varphi(T_{r_n}u_{n-1}) \geq 0, \ \forall y \in C.$$

In particular, we have

$$\begin{aligned} &F(T_{r_{n-1}}u_{n-1}, T_{r_n}u_{n-1}) + \frac{1}{r_{n-1}}\langle T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}, T_{r_{n-1}}u_{n-1} - u_{n-1}\rangle \\ &+\varphi(T_{r_n}u_{n-1}) - \varphi(T_{r_{n-1}}u_{n-1}) \geq 0, \ \forall y \in C \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &F(T_{r_n}u_{n-1}, T_{r_{n-1}}u_{n-1}) + \frac{1}{r_n}\langle T_{r_{n-1}}u_{n-1} - T_{r_n}u_{n-1}, T_{r_n}u_{n-1} - u_{n-1}\rangle \\ &+\varphi(T_{r_{n-1}}u_{n-1}) - \varphi(T_{r_n}u_{n-1}) \geq 0, \ \forall y \in C. \end{aligned} \tag{3.8}$$

Summing up (3.7) and (3.8) and using (A2), we obtain

$$\begin{aligned} &\frac{1}{r_n}\langle T_{r_{n-1}}u_{n-1} - T_{r_n}u_{n-1}, T_{r_n}u_{n-1} - u_{n-1}\rangle \\ &+\frac{1}{r_{n-1}}\langle T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}, T_{r_{n-1}}u_{n-1} - u_{n-1}\rangle \geq 0 \ \forall y \in C. \end{aligned}$$

It then follows that

$$\langle T_{r_{n-1}}u_{n-1} - T_{r_n}u_{n-1}, \frac{T_{r_n}u_{n-1} - u_{n-1}}{r_n} - \frac{T_{r_{n-1}}u_{n-1} - u_{n-1}}{r_{n-1}}\rangle \geq 0, \ \forall y \in C.$$

This implies that

$$\begin{aligned} 0 &\leq \langle T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}, T_{r_{n-1}}u_{n-1} - u_{n-1} - \frac{r_{n-1}}{r_n}(T_{r_n}u_{n-1} - u_{n-1})\rangle \\ &= \langle T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}, T_{r_{n-1}}u_{n-1} - T_{r_n}u_{n-1} + T_{r_n}u_{n-1} - u_{n-1} \\ &\quad -\frac{r_{n-1}}{r_n}(T_{r_n}u_{n-1} - u_{n-1})\rangle \\ &= \langle T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}, T_{r_{n-1}}u_{n-1} - T_{r_n}u_{n-1} + (1 - \frac{r_{n-1}}{r_n})(T_{r_n}u_{n-1} - u_{n-1})\rangle. \end{aligned}$$

It follows that

$$\|T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}\|^2 \leq |1 - \frac{r_{n-1}}{r_n}|\|T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}\|(\|T_{r_n}u_{n-1}\| + \|u_{n-1}\|).$$

Hence, we obtain

$$\|T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}\| \leq |1 - \frac{r_{n-1}}{r_n}|L' = \frac{1}{r_n}|r_n - r_{n-1}|L' \leq \frac{1}{a}|r_n - r_{n-1}|L', \tag{3.9}$$

where $L' = \sup_{n\in\mathbb{N}} \|T_{r_n}u_{n-1}\| + \sup_{n\in\mathbb{N}} \|u_{n-1}\|$. From (3.5), (3.6), and (3.9), we have

$$\begin{aligned}\|z_n - z_{n-1}\| &\leq \|u_n - u_{n-1}\| + \|T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + |r_{n-1} - r_n|\|Ax_{n-1}\| + \frac{1}{a}|r_n - r_{n-1}|L'. \end{aligned} \tag{3.10}$$

By (3.4) and (3.10), we have

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \alpha_n a\|x_n - x_{n-1}\| + (1-\alpha_n)(\|z_n - z_{n-1}\| + \|K_n z_{n-1} - K_{n-1}z_{n-1}\|) \\ &\quad +|\alpha_n - \alpha_{n-1}|L \\ &\leq \alpha_n a\|x_n - x_{n-1}\| + (1-\alpha_n)(\|x_n - x_{n-1}\| + |r_{n-1} - r_n|\|Ax_{n-1}\| \\ &\quad +\frac{1}{a}|r_n - r_{n-1}|L' + \|K_n z_{n-1} - K_{n-1}z_{n-1}\|) + |\alpha_n - \alpha_{n-1}|L \\ &\leq \alpha_n a\|x_n - x_{n-1}\| + (1-\alpha_n)\|x_n - x_{n-1}\| + |r_{n-1} - r_n|\|Ax_{n-1}\| \\ &\quad +\frac{1}{a}|r_n - r_{n-1}|L' + \|K_n z_{n-1} - K_{n-1}z_{n-1}\| + +|\alpha_n - \alpha_{n-1}|L \\ &= (1-\alpha_n(1-a))\|x_n - x_{n-1}\| + |r_{n-1} - r_n|\|Ax_{n-1}\| \\ &\quad +\frac{1}{a}|r_n - r_{n-1}|L' + \|K_n z_{n-1} - K_{n-1}z_{n-1}\| + |\alpha_n - \alpha_{n-1}|L \\ &= (1-\alpha_n(1-a))\|x_n - x_{n-1}\| + |r_{n-1} - r_n|\|Ax_{n-1}\| \\ &\quad +\frac{1}{a}|r_n - r_{n-1}|L' + \sup_{z\in\{z_n\}}\|K_n z - K_{n-1}z\| + |\alpha_n - \alpha_{n-1}|L. \end{aligned} \tag{3.11}$$

From the conditions (ii), (iii), Lemma 2.9 and Lemma 2.2, we can conclude that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.12}$$

By monotonicity of A and nonexpansiveness of T_{r_n}, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n(f(x_n) - z) + (1-\alpha_n)(K_n z_n - z)\|^2 \\ &\leq \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\ &\leq \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)\|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(z - r_nAz)\|^2 \\ &\leq \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)\|x_n - r_nAx_n - z + r_nAz)\|^2 \\ &= \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)\|(x_n - z) - r_n(Ax_n - Az)\|^2 \\ &= \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)(\|x_n - z\|^2 + r_n^2\|Ax_n - Az\|^2 \\ &\quad -2r_n\langle x_n - z, Ax_n - Az\rangle) \\ &\leq \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)(\|x_n - z\|^2 + r_n^2\|Ax_n - Az\|^2 \\ &\quad -2\alpha r_n\|Ax_n - Az\|^2) \\ &= \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)(\|x_n - z\|^2 \\ &\quad +r_n(r_n - 2\alpha)\|Ax_n - Az\|^2) \\ &\leq \alpha_n\|f(x_n) - z\|^2 + \|x_n - z\|^2 \\ &\quad +r_n(1-\alpha_n)(r_n - 2\alpha)\|Ax_n - Az\|^2, \end{aligned} \tag{3.13}$$

which implies that

$$\begin{aligned} r_n(1-\alpha_n)(2\alpha - r_n)\|Ax_n - Az\|^2 &\leq \alpha_n\|f(x_n) - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &= \alpha_n\|f(x_n) - z\|^2 + (\|x_n - z\| - \|x_{n+1} - z\|) \\ &\quad (\|x_n - z\| + \|x_{n+1} - z\|) \end{aligned}$$

$$= \alpha_n\|f(x_n)-z\|^2 + (\|x_n - x_{n+1}\|)$$
$$(\|x_n - z\| + \|x_{n+1} - z\|).$$

This implies by condition (iii) and (3.12) that

$$\lim_{n\to\infty}\|Ax_n - Az\| = 0. \tag{3.14}$$

By nonexpansiveness of T_{r_n} and $I - r_nA$, we have

$$\begin{aligned}
\|z_n - z\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(z - r_nAz)\|^2 \\
&\le \langle (x_n - r_nAx_n) - (z - r_nAz), z_n - z\rangle \\
&= \frac{1}{2}(\|(x_n - r_nAx_n) - (z - r_nAz)\|^2 + \|z_n - z\|^2 \\
&\quad -\|(x_n - r_nAx_n) - (z - r_nAz) - (z_n - z)\|^2) \\
&\le \frac{1}{2}(\|x_n - z\|^2 + \|z_n - z\|^2 - \|(x_n - z_n) - r_n(Ax_n - Az)\|^2) \\
&= \frac{1}{2}(\|x_n - z\|^2 + \|z_n - z\|^2 - \|x_n - z_n\|^2 \\
&\quad +2r_n\langle x_n - z_n, Ax_n - Az\rangle - r_n^2\|Ax_n - Az\|^2).
\end{aligned} \tag{3.15}$$

It follows that

$$\|z_n - z\|^2 \le \|x_n - z\|^2 \tag{3.16}$$

and

$$\|z_n - z\|^2 \le \|x_n - z\|^2 - \|x_n - z_n\|^2 + 2r_n\|x_n - z_n\|\|Ax_n - Az\|. \tag{3.17}$$

By (3.17), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n(f(x_n) - z) + (1-\alpha_n)(K_nz_n - z)\|^2 \\
&\le \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\
&\le \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)(\|x_n - z\|^2 - \|x_n - z_n\|^2 \\
&\quad +2r_n\|x_n - z_n\|\|Ax_n - Az\|) \\
&\le \alpha_n\|f(x_n) - z\|^2 + \|x_n - z\|^2 - (1-\alpha_n)\|x_n - z_n\|^2 \\
&\quad + 2r_n\|x_n - z_n\|\|Ax_n - Az\|.
\end{aligned}$$

This implies

$$\begin{aligned}
(1-\alpha_n)\|x_n - z_n\|^2 &\le \alpha_n\|f(x_n) - z\|^2 + (\|x_n - x_{n+1}\|)(\|x_n - z\| + \|x_{n+1} - z\|) \\
&\quad +2r_n\|x_n - z_n\|\|Ax_n - Az\|.
\end{aligned}$$

Again by the condition (iii), (3.12) and (3.14), we obtain

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.18}$$

Since $x_{n+1} = \alpha_nf(x_n) + (1-\alpha_n)K_nz_n$, we have $x_{n+1} - K_nz_n = \alpha_n(f(x_n) - K_nz_n)$. This implies that

$$\lim_{n\to\infty}\|x_{n+1} - K_nz_n\| = 0. \tag{3.19}$$

By (3.12), (3.18) and (3.19), we have

$$\|K_nz_n - z_n\| \le \|K_nz_n - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_n\| \to 0 \text{ as } n \to \infty. \tag{3.20}$$

Putting $z_0 = P_\Omega f(z_0)$. Then

$$\langle f(z_0) - z_0, z_0 - z\rangle \ge 0 \ \forall z \in \bigcap_{i=1}^{\infty} F(T_i) \cap GMEP(F, A, \varphi). \tag{3.21}$$

We shall show that

$$\limsup_{n\to\infty}\langle f(z_0)-z_0, x_n-z_0\rangle \le 0. \tag{3.22}$$

To show this inequality, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z_0)-z_0, x_n-z_0\rangle = \lim_{k\to\infty}\langle f(z_0)-z_0, x_{n_k}-z_0\rangle. \tag{3.23}$$

Without loss of generality, we may assume that $x_{n_k} \rightharpoonup \omega$ as $k\to\infty$ where $\omega\in C$. We first show $\omega \in GMEP(F, A, \varphi)$. Then we have $z_{n_k} \rightharpoonup \omega$ as $k\to\infty$. From $z_n = T_{r_n}(x_n - r_nAx_n)$, we obtain

$$F(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Ax_n, y-z_n\rangle + \frac{1}{r_n}\langle y-z_n, z_n-x_n\rangle \ge 0,\ \forall y\in C.$$

From $(A2)$, we have
$\varphi(y) - \varphi(z_n) + \langle Ax_n, y-z_n\rangle + \frac{1}{r_n}\langle y-z_n, z_n-x_n\rangle \ge F(y, z_n)$.
Then

$$\varphi(y)-\varphi(z_{n_k})+\langle Ax_{n_k}, y-z_{n_k}\rangle+\frac{1}{r_{n_k}}\langle y-z_{n_k}, z_{n_k}-x_{n_k}\rangle \ge F(y, z_{n_k}),\ \forall y\in C. \tag{3.24}$$

For $t\in[0,1]$ and $y\in C$, put $z_t = ty + (1-t)\omega$. Then, $z_t\in C$. So, from (3.24), we have

$$\begin{aligned}
\langle z_t - z_{n_k}, Az_t\rangle &\ge \langle z_t - z_{n_k}, Az_t\rangle - \langle z_t - z_{n_k}, Ax_{n_k}\rangle + F(z_t, z_{n_k}) - \varphi(z_t) + \varphi(z_{n_k})\\
&\quad -\langle z_t - z_{n_k}, \frac{z_{n_k}-x_{n_k}}{r_{n_k}}\rangle\\
&= \langle z_t - z_{n_k}, Az_t - Az_{n_k}\rangle + \langle z_t - z_{n_k}, Az_{n_k} - Ax_{n_k}\rangle\\
&\quad +F(z_t, z_{n_k}) - \varphi(z_t) + \varphi(z_{n_k})\\
&\quad -\langle z_t - z_{n_k}, \frac{z_{n_k}-x_{n_k}}{r_{n_k}}\rangle\\
&\ge \alpha\|Az_t - Az_{n_k}\|^2 + \langle z_t - z_{n_k}, Az_{n_k} - Ax_{n_k}\rangle\\
&\quad +F(z_t, z_{n_k}) - \varphi(z_t) + \varphi(z_{n_k})\\
&\quad -\langle z_t - z_{n_k}, \frac{z_{n_k}-x_{n_k}}{r_{n_k}}\rangle\\
&\ge \langle z_t - z_{n_k}, Az_{n_k} - Ax_{n_k}\rangle + F(z_t, z_{n_k}) - \varphi(z_t) + \varphi(z_{n_k})\\
&\quad -\langle z_t - z_{n_k}, \frac{z_{n_k}-x_{n_k}}{r_{n_k}}\rangle.
\end{aligned}$$

It follows from (3.18), $(A4)$ and weakly lower semicontinuity of φ that

$$\langle z_t - \omega, Az_t\rangle + \varphi(z_t) - \varphi(\omega) \ge F(z_t, \omega). \tag{3.25}$$

From $(A1)$, $(A4)$ and (3.25), we also have

$$\begin{aligned}
0 &= F(z_t, z_t) \le tF(z_t, y) + (1-t)F(z_t, \omega)\\
&\le tF(z_t, y) + (1-t)(\langle z_t - \omega, Az_t\rangle + \varphi(z_t) - \varphi(\omega))\\
&\le tF(z_t, y) + (1-t)(t\langle y - \omega, Az_t\rangle + t\varphi(y) + (1-t)\varphi(\omega) - \varphi(\omega))\\
&= tF(z_t, y) + (1-t)(t\langle y - \omega, Az_t\rangle + t\varphi(y) - t\varphi(\omega))\\
&= tF(z_t, y) + (1-t)t(\langle y - \omega, Az_t\rangle + \varphi(y) - \varphi(\omega)),
\end{aligned}$$

hence

$$0 \le F(z_t, y) + (1-t)(\langle y - \omega, Az_t\rangle + \varphi(y) - \varphi(\omega)).$$

Letting $t \to 0$, we have

$$0 \le F(\omega, y) + \langle y - \omega, A\omega\rangle + \varphi(y) - \varphi(\omega)$$

for all $y \in C$ and hence $\omega \in GMEP(F, A, \varphi)$.

Next, we show that $\omega \in \bigcap_{i=1}^{\infty} F(T_i)$. Assume that $\omega \neq K\omega$. By using the Opial property, (3.20) and Lemma 2.10 we have

$$\begin{aligned}
\liminf_{k\to\infty} \|z_{n_k} - \omega\| &< \liminf_{k\to\infty} \|z_{n_k} - K\omega\| \\
&\le \liminf_{k\to\infty} (\|z_{n_k} - K_{n_k} z_{n_k}\| + \|K_{n_k} z_{n_k} - K_{n_k}\omega\| + \|K_{n_k}\omega - K\omega\|) \\
&\le \liminf_{k\to\infty} \|z_{n_k} - \omega\|,
\end{aligned}$$

which is a contradiction. Thus $K\omega = \omega$, so $\omega \in F(K)$. By Lemma 2.8, we obtain that $\omega \in \bigcap_{i=1}^{\infty} F(T_i)$. Hence $\omega \in \bigcap_{i=1}^{\infty} F(T_i) \bigcap GMEP(F, A, \varphi)$.
Since $x_{n_k} \rightharpoonup \omega$ and $\omega \in \bigcap_{i=1}^{\infty} F(T_i) \bigcap GMEP(F, A, \varphi)$, by (3.21), we have

$$\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle = \lim_{k\to\infty} \langle f(z_0) - z_0, x_{n_k} - z_0\rangle = \langle f(z_0) - z_0, \omega - z_0\rangle \le 0. \tag{3.26}$$

From Lemma 2.3 and (3.16) we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\alpha_n(f(x_n) - z_0) + (1-\alpha_n)(K_n z_n - z_0)\|^2 \\
&\le (1-\alpha_n)^2 \|K_n z_n - z_0\|^2 + 2\alpha_n \langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\
&\le (1-\alpha_n)^2 \|z_n - z_0\|^2 + 2\alpha_n \langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\
&\le (1-\alpha_n)^2 \|x_n - z_0\|^2 + 2\alpha_n \langle f(x_n) - f(z_0), x_{n+1} - z_0\rangle \\
&\quad + 2\alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\le (1-\alpha_n)^2 \|x_n - z_0\|^2 + 2\alpha_n a \|x_n - z_0\| \|x_{n+1} - z_0\| \\
&\quad + 2\alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\le (1-\alpha_n)^2 \|x_n - z_0\|^2 + \alpha_n a \{\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2\} \\
&\quad + 2\alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0\rangle.
\end{aligned} \tag{3.27}$$

This implies

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\le \frac{(1-\alpha_n)^2 + \alpha_n a}{1 - \alpha_n a} \|x_n - z_0\|^2 + \frac{2\alpha_n}{1-\alpha_n a} \langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&= \frac{1 - 2\alpha_n + \alpha_n a}{1 - \alpha_n a} \|x_n - z_0\|^2 + \frac{\alpha_n^2}{1 - \alpha_n a} \|x_n - z_0\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n a} \langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\le (1 - \frac{2(1-a)\alpha_n}{1-\alpha_n a}) \|x_n - z_0\|^2 \\
&\quad + \frac{2(1-a)\alpha_n}{1-\alpha_n a} \{ \frac{\alpha_n M}{2(1-a)} + \frac{1}{1-a} \langle f(z_0) - z_0, x_{n+1} - z_0\rangle \},
\end{aligned} \tag{3.28}$$

where $M = \sup\{\|x_n - z_0\|^2 : n \in \mathbb{N}\}$. Put $\beta_n = \frac{2(1-a)\alpha_n}{1-a\alpha_n}$ and
$\delta_n = \frac{2(1-a)\alpha_n}{1-\alpha_n a} \{ \frac{\alpha_n M}{2(1-a)} + \frac{1}{1-a} \langle f(z_0) - z_0, x_{n+1} - z_0\rangle \}$. Then

$$\|x_{n+1} - z_0\|^2 \le (1 - \beta_n) \|x_n - z_0\|^2 + \delta_n \ \forall n \in \mathbb{N}. \tag{3.29}$$

It follows from assumption (iii) that $\sum_{n=1}^{\infty} \beta_n = \infty$. By (3.26) and $\lim_{n\to\infty} \alpha_n = 0$, we have $\limsup_{n\to\infty} \frac{\delta_n}{\beta_n} \le 0$. By Lemma 2.2, we obtain that $\|x_n - z_0\| \to 0$ as $n \to \infty$, This completes the proof. $\square$

By setting $A \equiv 0$ in Theorem 3.1, we obtain the following result.

Corollary 3.2. *Let C be a bounded closed convex subset of a real Hilbert space, $F : C \times C \to \mathbb{R}$ be a bifunction sastisfing the condition (A1)-(A4) and $\varphi : C \to \mathbb{R} \bigcup \{+\infty\}$ be a proper lower semi-continuous and convex function and $f : C \to C$ be a contraction map with coefficient $0 < a < 1$. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself with $\Omega = \bigcap_{i=1}^{\infty} F(T_i) \bigcap MEP(F, \varphi) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Let K_n be the K-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$ for each $n \in \mathbb{N}$. Let $x_1 \in C$ and $\{z_n\}$ and $\{x_n\}$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \varphi(y) - \varphi(z_n) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) K_n z_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{3.30}$$

where

$\{\alpha_n\} \subset [0, 1]$ *and* $\{r_n\} \subset [0, 2\alpha]$ *sastisfy the following conditions:*

(i) $0 < a \leq r_n \leq b < 2\alpha,$

(ii) $\displaystyle\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty,$

(iii) $\displaystyle\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\displaystyle\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$

Then $\{x_n\}$ and $\{z_n\}$ converge strongly to $z_0 \in \Omega$, where $z_0 = P_\Omega f(z_0)$.

4. NUMERICAL RESULT

In this section, we give a numerical example of using the iterative method introduced in our main result. Let $H = \mathbb{R}$ and $C = [-5, 5]$. For $n \in \mathbb{N}$, let $T_n : C \longrightarrow C$ be defined by

$$T_n x = \begin{cases} x & \text{, if } x \geq 0. \\ -\frac{n}{n+1} x & \text{, if } x < 0. \end{cases} \tag{4.1}$$

and Let $F : C \times C \to \mathbb{R}$, $\varphi : C \to \mathbb{R}$, $f : C \to C$ and $A : C \to C$ be defined by
$F(y, z) = 2y^2 + 2zy - 4z^2,$
$\varphi(y) = -y^2,$
$Ax = x,$
$f(x) = \frac{x}{2}.$
It can be shown that that $GMEP(F, \varphi, A) = \{0\}$. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated by (3.1). Then we have

$$F(y, z_n) + \varphi(y) - \varphi(z_n) + < Ax_n, y - z_n > + \frac{1}{r_n} < y - z_n, z_n - x_n > \geq 0,$$

$$2y^2 + 2z_n y - 4z_n^2 - y^2 + z^2 + Ax_n y - Ax_n z + \frac{1}{r_n}(yz_n - yx_n - z_n^2 + z_n x_n) \geq 0,$$

$$r_n y^2 + 2r_n z_n y - 3r_n z^2 + r_n x_n y - r_n x_n z_n + yz_n - yx_n - z_n^2 + z_n x_n \geq 0,$$

$$r_n y^2 + 2r_n z_n y + z_n y - r_n x_n y - x_n y + r_n x_n z_n + x_n z_n - 3r_n z_n^2 - z_n^2 \geq 0,$$

$$r_n y^2 + [(2r_n + 1z_n) - (r_n + 1)]\, y + (r_n x_n z_n + x_n z_n - 3r_n z^2 z_n^2 - z_n^2) \geq 0.$$

Let $G(y) = r_n y^2 + [(2r_n + 1z_n) - (r_n + 1)]\, y + (r_n x_n z_n + x_n z_n - 3r_n z^2 z_n^2 - z_n^2)$
$G(y)$ is a quadratic function of y with coeffient $a = r_n$, $b_n = (2r_n + 1z_n) - (r_n + 1)$, $c = r_n x_n z_n + x_n z_n - 3r_n z^2 z_n^2 - z_n^2$
Determine the discriminant Δ of G as follows

$$\begin{aligned}
\Delta &= b^2 - 4ac \\
&= [(2r_n + 1)z_n - (r_n + 1)x_n]^2 - 4r_n(r_n x_n z_n + x_n z_n - 3r_n z_n^2 - z_n^2) \\
&= r_n^2 x_n^2 + 2r_n x_n^2 + x_n^2 - 8r_n^2 x_n z_n - 10r_n x_n z_n + 2x_n z_n + 16r_n^2 z_n^2 + +8r_n z_n^2 + z_n^2 \\
&= \left[(r_n + 1)^2 x_n^2 - 2(r_n + 1)(4r_n + 1)x_n z_n + (4r_n + 1)^2 z_n^2\right] \\
&= [(r_n + 1)x_n - (4r_n + 1)z_n]^2
\end{aligned}$$

We know that $G(y) \geq 0\ \forall y \in C$. If it has most one solution in C, then $\Delta \leq 0$, so $z_n = \frac{(r_n+1)}{4r_n+1} x_n$. Now (3.1) becomes

$$x_{n+1} = \alpha_n \frac{x_n}{2} + (1 - \alpha_n) K_n \left((\frac{r_n + 1}{4r_{n+1}}) x_n \right). \tag{4.2}$$

Now, we set $\alpha_n = \frac{1}{10n}$, $r_n = \frac{n}{n+1}$, $\lambda_n = \frac{1}{2^n}$ and $x_1 = 1$. The following table shows numerical results of $\{x_n\}$ and $\{z_n\}$.

n	x_n	z_n
1	1.000000000	0.500000000
2	0.500000000	0.227272727
3	0.228409091	0.099928977
4	0.100404829	0.043030641
5	0.043209935	0.018281126
6	0.018347603	0.007694156
7	0.007718817	0.003216174
8	0.003225368	0.001337349
⋮	⋮	⋮
18	0.000000441	0.000000179
19	0.000000179	0.000000072
20	0.000000072	0.000000028

Table 1:

We observe that $\{x_n\}$ and $\{z_n\}$ converge to $0 \in \bigcap_{i=1}^{\infty} F(T_i) \cap GMEP(F, A, \varphi)$ and $x_{20} = 0.000000072$ and $z_{20} = 0.000000028$ are approximate solutions with accuracy at 7 decimal places.

Remark 4.1. If we use W_n instead of K_n in (3.1) we also obtain a strong convergence theorem as Theorem 3.1. The next table give comparisons of numerical results among algorithm 1, algorithm 2 and algorithm 3.

When the initial point are $x_1 = 1$ for algorithm 1 and algorithm 2, $x_1 = 0.005$ for algorithm 3. We set $\alpha_n = \frac{1}{10n}$, $r_n = \frac{n}{n+1}$, $\lambda_n = \frac{1}{2^n}$.

	Using K-mapping and $f(x) = \frac{x}{2}$	Using W-mapping and $f(x_n) = \frac{x}{2}$	Using K-mapping and $f(x) = 0.005$
n	x_n	x_n	x_n
1	-1.000000000	-1.000000000	0.005000000
2	-0.162500000	-0.162500000	0.002750000
3	-0.014295691	-0.028914536	0.001437500
4	-0.000927083	-0.004701357	0.000077461
5	-0.000050769	-0.000073748	0.000044867
6	-0.000002513	-0.000011301	0.000028603
7	-0.000000116	-0.000001704	0.000020128
8	-0.000000005	-0.000000254	0.000015410

Table 2:

Algorithm 1:

$$\begin{cases} F(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Ax_n, y - z_n \rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n \rangle \geq 0 \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) K_n z_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{4.3}$$

Algorithm 2:

$$\begin{cases} F(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Ax_n, y - z_n \rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n \rangle \geq 0 \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) W_n z_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{4.4}$$

Algorithm 3:

$$\begin{cases} F(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Ax_n, y - z_n \rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n \rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) K_n z_n, \ \forall n \in \mathbb{N}. \end{cases}$$

Acknowledgments

The authors would like to thank the Center of Excellence in Mathematics for financial support during the preparation of this paper and Dr. Chaiporn Thangthong for his advice about using Mathlab programme in computation. The first author was supported by the graduate school Chiang Mai University.

References

1. A. Moudafi, M. Th´era, Proximal and dynamical approaches to equilibrium problems, in: Lecture Notes in Economics and Mathematical Systems, vol. 477, Springer, (1999), pp. 187—201.
2. K. Goebel, W.A. Kirk, Topics in metric fixed point theory, Cambridge Stud. Adv. Math., Vol. 28, Cambridge University Press, Cambridge, (1990).
3. H. Iiduka, W. Takahashi, Weak convergence theorem by Ces‘aro means for nonexpansive mappings and inverse-strongly monotone mappings, J. Nonlinear Convex Anal. 7 (2006) 105—113.
4. E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Stud. 63 (1994) 123—145.
5. P.L. Combettes, A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (2005) 117—136.
6. J.W. Peng, J.C. Yao, A New hybrid-extragradient method for generalized mixed equilibrium problems and fixed point problems and variational inequality problems, Taiwan. J. Math. 12 (6) (2008) 1401-1432.
7. K. Shimoji, W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, Taiwan. J. Math. 5 (2001) 387-404.
8. P.-E. Maing, Approximation method for common fixed points of nonexpansive mappings in Hilbert spaces, Journal of Mathematical Analysis and Applications 325 (2007) 469-479.
9. F.E. Browder, Convergence of approximate to fixed points of nonexpansive nonlinear mappings in Banach space, Arch. Rational Mech. Anal. 24 (1967), 82-89.
10. L.C. Ceng, J.C. Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems, J. Comput. Appl. Math. 214 (2008) 186-201.
11. H.K. Xu, An iterative approach to quadratic optimization, J. Optim. Theory Appl. 116 (2003) 659-678.
12. J. W. Peng, J. C. Yao, Two extragradient methods for generalized mixed equilibrium problems, nonexpansive mappings and monotone mappings, Computers and Mathematics with Applications. 58 (2009), 1287-1301.
13. A. Kangtunyakarn, Iterative algorithms for finding a common solution of system of the set of variational inclusion problems and the set of fixed point problems, Fixed Point Theory and Application, In press (2011).
14. S. Takahashi, W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Analysis 69 (2008) 1025-1033.
15. S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, Journal of Mathematical Analysis and Applications 331 (2007) 506-515.
16. Z. Opial, Weak convergence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591-567.