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A GENERAL ITERATIVE SCHEME FOR STRICT PSEUDONONSPREADING MAPPING RELATED TO OPTIMIZATION PROBLEM IN HILBERT SPACES

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ABSTRACT. In this paper, we introduce a general iterative scheme for finding fixed points of a strictly pseudononspreading mapping. We show that, under some suitable conditions, the sequence which is generated by the proposed iterative scheme converges strongly to a fixed point of the mapping. Moreover, such a fixed point is a solution of a certain optimization problem that induced by a strongly positive bounded linear operator. Consequently, since the class of strictly pseudononspreading mapping is the largest one, the main results presented in this paper extend various results existing in the current literature.

KEYWORDS : Strictly pseudononspreading mapping; Nonspreading mapping; Strictly pseudo-contractive mapping; Optimization problem; Fixed point problem.

1. INTRODUCTION AND PRELIMINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Recall that a mapping $T : D(T) \subset H \to H$ is said to be nonspreading if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle, \quad \forall x, y \in D(T). \tag{1.1}$$

It is worth to point out that the class of nonspreading mappings has been used for studying the resolvents of maximal monotone operators, which is one important problem (see [7, 8, 9]).

A mapping $T : D(T) \subset H \to H$ is said to be k-strictly pseudo-contraction if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2, \quad \forall x, y \in D(T). \tag{1.2}$$

Note that the class of k-strictly pseudo-contractions includes strictly the class of nonexpansive mappings (i.e., $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in D(T)$) as a subclass. In

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fact, T is a nonexpansive mapping if and only if T is a 0-strictly pseudo-contractive mapping. Moreover, strictly pseudo-contractive mappings is one of the most important class that have powerful applications among nonlinear mappings, as in solving inverse problem. Consequently, many authors have been devoting the studies on the problems of finding fixed points for strictly pseudo-contractions, see, for example, [1, 2, 5] and the references therein.

Recently, Osilike and Isiogugu [12] introduced a new class of mappings, so-called k-strictly pseudononspreading, that is, a mapping $T : D(T) \subset H \to H$ is said to be k-strictly pseudononspreading if there exists $k \in [0, 1)$ such that

$$\|Tx-Ty\|^2 \leq \|x-y\|^2+k\|x-Tx-(y-Ty)\|^2+2\langle x-Tx, y-Ty\rangle, \quad \forall x, y \in D(T). \tag{1.3}$$

They showed that the class of nonspreading mappings is properly contained in the class of strictly pseudononspreading mappings. Moreover, by using an idea of mean convergence, they also introduced the following iterative process for k-strictly pseudononspreading mapping on a closed convex subset K of H. Let $T : K \longrightarrow K$ be a k-strictly pseudononspreading mapping and $\zeta \in [k, 1)$ be chosen. Starting with an arbitrary initial $x_0 \in K$, define the sequences $\{x_n\}$ and $\{y_n\}$ recursively by

$$\begin{aligned} x_{n+1} &= \alpha_n u + (1-\alpha_n) y_n, \\ y_n &= \frac{1}{n}\sum_{i=1}^{n-1} T_\zeta^i x_n, \end{aligned} \tag{1.4}$$

for all $n \geq 1$, where $\zeta \in [k, 1)$, $T_\zeta = \zeta I + (1-\zeta)T$, and $\{\alpha_n\}$ is a sequence in $[0, 1)$. It is proved that, under certain appropriate conditions imposed on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.4) strongly converges to $P_{F(T)}u$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$.

On the other hand, let A be a strongly positive bounded linear operator on H, that is, there exists a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2 \ \text{ for all } x \in H. \tag{1.5}$$

The optimization problems are of very interesting and have been studying by many authors. A kind of optimization problem is the following :

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle, \tag{1.6}$$

where $u \in H$ and $C = \bigcap_{i=1}^{\infty} C_i$, when $C_1, C_2, ...$ are infinitely many closed convex subsets on H such that $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$. For more detailed accounts on optimization problems and related problems, we refer to [3, 4, 6, 14].

Let T be a nonexpansive mapping with a nonempty fixed point set. In 2003, Xu [14] considered the following iterative algorithm:

$$x_{n+1} = \alpha_n u + (I - \alpha_n A)Tx_n, \ n \geq 0, \tag{1.7}$$

where $x_0 \in H$ is chosen arbitrary and $\{\alpha_n\}$ is a sequence of real numbers. He proved that if the $\{\alpha_n\}$ satisfies certain conditions, then the sequence $\{x_n\}$ defined by (1.7) converges strongly to the unique solution of the optimization problem (1.6), when $C = F(T)$. Moreover, by using the viscosity approximation method introduced by Moudafi [11], Marino and Xu [10] considered the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, n \geq 0, \tag{1.8}$$

where $\gamma > 0$ and $f : H \longrightarrow H$ is a contractive mapping. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies certain conditions, then the sequence $\{x_n\}$ generated by (1.8) converges strongly to the unique solution $x^* \in F(T)$ of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T),$$

which is, in fact, the optimality condition for the minimization problem:

$$\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf.

In this paper, motivated by the above-mentioned results, we introduce a general iterative scheme for finding fixed points of a strictly pseudononspreading mapping and then prove that the sequence generated by the proposed iterative scheme converges strongly to a fixed point of such mapping, which is also a solution of the optimization problem (1.6). Additional results of the main result are also obtained. Our results improve and develop the corresponding results of Osilike and Isiogugu [12], Moudafi [11] and Marino and Xu [10].

To do so, we need the following well known results.

Lemma 1.1. [10] *Assume that A is a strongly positive linear bounded self-adjoint operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 1.2. [12] *Let C be a nonempty closed convex subset of a real Hilbert space H, and $T : C \to C$ be a k-strictly pseudononspreading mapping. If $F(T) \neq \emptyset$, then it is closed and convex.*

Lemma 1.3. *Let H be a real Hilbert space. Then for any $x, y \in H$ we have*
- (a) $\|\zeta x + (1-\zeta)y\|^2 = \zeta\|x\|^2 + (1-\zeta)\|y\|^2 - \zeta(1-\zeta)\|x-y\|^2$, *for each* $\zeta \in [0, 1]$.
- (b) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$.

Lemma 1.4. [13] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n\gamma_n,$$

where $\{\gamma_n\}$ is a sequence in (0, 1) and $\{\delta_n\}$ is a sequence such that
- (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
- (ii) $\limsup_{n\longrightarrow\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n\gamma_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

2. MAIN RESULTS

We start with an important useful lemma.

Lemma 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, and $T : C \to C$ be a k-strictly pseudononspreading mapping with a nonempty fixed point set. Let $\zeta \in [k, 1)$ be fixed and define $T_\zeta : C \to C$ by*

$$T_\zeta(x) = \zeta x + (1 - \zeta)Tx, \quad \forall x \in C. \tag{2.1}$$

Then $F(T) = F(T_\zeta)$.

Proof. This follows immediately from the fact that $x - T_\zeta x = (1 - \zeta)(x - Tx)$, for each $x \in C$. □

Now we are in a position to prove our main results.

Algorithm. Let $\gamma > 0$ be a constant and $T : H \to H$ be a k-strictly pseudononspreading mapping, $f : H \to H$ be a contraction and A be a strongly positive bounded linear operator on H. Let $\zeta \in [k, 1)$ and $\{\alpha_n\}$ be a sequence of real numbers in $[0, 1]$. For given $x_0 \in H$, we define a sequence $\{x_n\}$ in H by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n \tag{2.2}$$

where $y_n = \frac{1}{n}\sum_{i=0}^{n-1} T_\zeta^i x_n$ and T_ζ is defined by (2.1).

Theorem 2.2. *Let H be a real Hilbert space, and $T : H \longrightarrow H$ be a k-strictly pseudononspreading mapping with a nonempty fixed point set. Let $f : H \longrightarrow H$ be a contraction with coefficient $\alpha \in (0, 1)$. Let A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$. Let the sequence $\{x_n\}$ be defined by (2.2). If the following control conditions are satisfied:*

(i) $0 < \gamma < \overline{\gamma}/\alpha$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$,
(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$ which solves the following optimization problem:

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{2.3}$$

where h is a potential function for γf.

Proof. We divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is a bounded sequence. Indeed, taking $p \in F(T)$ and $z \in H$, we see that

$$\begin{aligned}
\|T_\zeta z - p\|^2 &= \|[\zeta z + (1-\zeta)Tz] - [\zeta p + (1-\zeta)Tp]\|^2 \\
&= \|\zeta(z-p) + (1-\zeta)(Tz - Tp)\|^2 \\
&= \zeta\|z-p\|^2 + (1-\zeta)\|Tz - Tp\|^2 - \zeta(1-\zeta)\|z - Tz\|^2 \\
&\le \zeta\|z-p\|^2 + (1-\zeta)\left[\|z-p\|^2 + k\|z - Tz\|^2\right] - \zeta(1-\zeta)\|z - Tz\|^2 \\
&= \|z-p\|^2 + (1-\zeta)(k-\zeta)\|z - Tz\|^2 \\
&\le \|z-p\|^2.
\end{aligned}$$

Thus for each $z \in H$ and $p \in F(T)$, we have

$$\|T_\zeta^i z - p\| = \|T_\zeta(T_\zeta^{i-1}z) - p\| \le \|T_\zeta^{i-1}z - p\| \le \|T_\zeta^{i-2}z - p\| \le \cdots \le \|z - p\|, \tag{2.4}$$

for all $i \ge 0$. Consequently,

$$\begin{aligned}
\|y_n - p\| &= \left\|\frac{1}{n}\sum_{i=0}^{n-1} T_\xi^i x_n - p\right\| \\
&\le \frac{1}{n}\sum_{i=0}^{n-1} \left\|T_\xi^i x_n - p\right\| \\
&\le \frac{1}{n}\sum_{i=0}^{n-1} \|x_n - p\| = \|x_n - p\|.
\end{aligned} \tag{2.5}$$

On the other hand, by condition (ii), without loss of generality we may assume that $\alpha_n \le \|A\|^{-1}$. Thus, by using Lemma 1.1 and (2.5), we obtain

$$\|x_{n+1} - p\| = \|\alpha_n(\gamma f(x_n) - Ap) + (I - \alpha_n A)(y_n - p)\|$$

$$\begin{aligned}
&\leq (1-\alpha_n\overline{\gamma})\|y_n - p\| + \alpha_n\|\gamma f(x_n) - Ap\| \\
&\leq (1-\alpha_n\overline{\gamma})\|x_n - p\| + \alpha_n\gamma\|f(x_n) - f(p)\| + \alpha_n\|\gamma f(p) - Ap\| \\
&\leq (1-\alpha_n(\overline{\gamma} - \gamma\alpha)\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\|. \qquad (2.6)
\end{aligned}$$

Using (2.6) and induction, we know that

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \gamma\alpha}\right\}, \quad \forall n \geq 0.$$

Therefore, $\{x_n\}$ is a bounded sequence. Consequently, $\{y_n\}$, $\{Ay_n\}$ and $\{f(x_n)\}$ are also bounded. Moreover, in view of (2.4), we know that $\|T_\zeta^n x_n - p\| \leq \|x_n - p\|$ for all $n \geq 1$, this means $\{T_\zeta^n x_n\}$ is also bounded.

Step 2. We show

$$\limsup_{n\longrightarrow\infty}\langle \gamma f(x^*) - Ax^*, x_n - x^*\rangle \leq 0,$$

where $x^* \in F(T)$ is the unique solution of the optimization problem (2.3).

Observe that, since $\{x_n\}$ is a bounded sequence, we can find a subsequence $\{x_{n_j+1}\}$ of $\{x_{n+1}\}$ and $q \in H$ such that $x_{n_j+1} \rightharpoonup q$ as $j \to \infty$, and

$$\limsup_{n\longrightarrow\infty}\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle = \lim_{j\longrightarrow\infty}\langle \gamma f(x^*) - Ax^*, x_{n_j+1} - x^*\rangle. \qquad (2.7)$$

On the other hand, since $\{Ay_n\}, \{f(x_n)\}$ are bounded sequences and $\lim_{n\to\infty}\alpha_n = 0$, we have

$$\lim_{n\to\infty}\|x_{n+1} - y_n\| = \lim_{n\to\infty}\alpha_n\|\gamma f(x_n) - Ay_n\| = 0. \qquad (2.8)$$

Consequently, for arbitrary bounded linear functional g on H we have

$$\begin{aligned}
|g(y_{n_j}) - g(q)| &\leq |g(y_{n_j}) - g(x_{n_j+1})| + |g(x_{n_j+1}) - g(q)| \\
&\leq \|g\|\|y_{n_j} - x_{n_j+1}\| + |g(x_{n_j+1}) - g(q)|. \qquad (2.9)
\end{aligned}$$

Thus, from (2.8) and (2.9), we conclude that $y_{n_j} \rightharpoonup q$ as $j \to \infty$.

Now for each $j \geq 1$ and $i = 0, 1, 2, ..., n_j - 1$, we put $w_{n_j}^i := T_\zeta^i x_{n_j}$. Then we see that

$$\begin{aligned}
\|T_\zeta^{i+1}x_{n_j} - T_\zeta q\|^2 &= \|T_\zeta w_{n_j}^i - T_\zeta q\|^2 \\
&= \|\zeta(w_{n_j}^i - q) + (1-\zeta)(Tw_{n_j}^i - Tq)\|^2 \\
&= \zeta\|w_{n_j}^i - q\|^2 + (1-\zeta)\|Tw_{n_j}^i - Tq)\|^2 \\
&\quad -\zeta(1-\zeta)\|w_{n_j}^i - Tw_{n_j}^i - (q - Tq)\|^2 \\
&\leq (1-\zeta)\left[\|w_{n_j}^i - q\|^2 + k\|w_{n_j}^i - Tw_{n_j}^i - (q - Tq)\|^2\right] \\
&\quad \zeta\|w_{n_j}^i - q\|^2 + 2(1-\zeta)\langle w_{n_j}^i - Tw_{n_j}^i, q - Tq\rangle \\
&\quad -\zeta(1-\zeta)\|w_{n_j}^i - Tw_{n_j}^i - (q - Tq)\|^2 \\
&= \|w_{n_j}^i - q\|^2 + 2(1-\zeta)\langle w_{n_j}^i - Tw_{n_j}^i, q - Tq\rangle \\
&\quad -(1-\zeta)(\zeta - k)\|w_{n_j}^i - Tw_{n_j}^i - (q - Tq)\|^2 \\
&\leq \|w_{n_j}^i - q\|^2 + 2(1-\zeta)\langle w_{n_j}^i - Tw_{n_j}^i, q - Tq\rangle \\
&= \|w_{n_j}^i - q\|^2 + \frac{2}{(1-\zeta)}\langle w_{n_j}^i - T_\zeta w_{n_j}^i, q - T_\zeta q\rangle \\
&= \|w_{n_j}^i - T_\zeta q + T_\zeta q - q\|^2 + \frac{2}{(1-\zeta)}\langle w_{n_j}^i - T_\zeta w_{n_j}^i, q - T_\zeta q\rangle
\end{aligned}$$

$$\begin{aligned} &= \|w^i_{n_j} - T_\zeta q\|^2 + \|T_\zeta q - q\|^2 + 2\langle w^i_{n_j} - T_\zeta q, q - T_\zeta q\rangle \\ &\quad + \frac{2}{(1-\zeta)}\langle w^i_{n_j} - T_\zeta w^i_{n_j}, q - T_\zeta q\rangle. \end{aligned}$$

Using this one, by summing from $i = 0$ to $n_j - 1$ and dividing by n_j, we know that

$$\begin{aligned} \frac{1}{n_j}\|T^{n_j}_\zeta x_{n_j} - T_\zeta q\|^2 &\leq \frac{1}{n_j}\|x_{n_j} - T_\zeta q\|^2 + \|T_\zeta q - q\|^2 + \langle y_{n_j} - T_\zeta q, T_\zeta q - q\rangle \\ &\quad + \frac{2}{n_j(1-\zeta)}\langle x_{n_j} - T^{n_j}_\zeta x_{n_j}, q - T_\zeta q\rangle. \end{aligned} \tag{2.10}$$

Since $\{x_n\}, \{T^n_\zeta x_n\}$ are bounded sequences and $y_{n_j} \rightharpoonup q$ as $j \to \infty$, we see that (2.10) gives

$$0 \leq \|T_\zeta q - q\|^2 + 2\langle q - T_\zeta q, T_\zeta q - q\rangle = -\|T_\zeta q - q\|^2.$$

This implies that $q \in F(T_\zeta) = F(T)$. Consequently, since x^* is the solution of optimization problem (2.3), from (2.7) we have

$$\begin{aligned} \lim_{n\longrightarrow\infty}\langle(\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle &= \limsup_{j\longrightarrow\infty}\langle(\gamma f(x^*) - Ax^*, x_{n_j+1} - x^*\rangle \\ &= \langle\gamma f(x^*) - Ax^*, q - x^*\rangle \\ &\leq 0. \end{aligned}$$

Step 3. We prove $\{x_n\}$ converges strongly to $x^* \in F(T)$, where x^* is the unique solution of the optimization problem (2.3).

Consider,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_n A)(y_n - x^*) + \alpha_n(\gamma f(x_n) - Ax^*)\|^2 \\ &\leq \|(1 - \alpha_n A)(y_n - x^*)\|^2 + 2\alpha_n\langle\gamma f(x_n) - Ax^*, x_{n+1} - x^*\rangle \\ &\leq (1 - \alpha_n\overline{\gamma})^2\|x_n - x^*\|^2 + \alpha_n\gamma\alpha\{\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\} \\ &\quad + 2\alpha_n\langle\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle, \end{aligned}$$

this implies that,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \frac{1 - 2\alpha_n\overline{\gamma} + \alpha_n^2\overline{\gamma}^2 + \alpha_n\gamma\alpha}{1 - \alpha_n\gamma\alpha}\|x_n - x^*\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \alpha_n\gamma\alpha}\langle u + \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \\ &\leq \left[1 - \frac{2\alpha_n(\overline{\gamma} - \gamma\alpha)}{1 - \alpha_n\gamma\alpha}\right]\|x_n - x^*\|^2 \\ &\quad + \frac{2\alpha_n(\overline{\gamma} - \gamma\alpha)}{1 - \alpha_n\gamma\alpha}\left[\frac{\alpha_n\overline{\gamma}^2 R}{2(\overline{\gamma} - \gamma\alpha)} + \frac{1}{\overline{\gamma} - \gamma\alpha}\langle\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle\right] \\ &= (1 - \kappa_n)\|x_n - x^*\|^2 + \kappa_n\sigma_n, \end{aligned}$$

where

$$R = \sup\{\|x_n - x^*\|^2 : n \geq 1\}, \quad \kappa_n = \frac{2((1+\mu)\overline{\gamma} - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha},$$

$$\sigma_n = \frac{\alpha_n\overline{\gamma}^2 R}{2(\overline{\gamma} - \gamma\alpha)} + \frac{1}{(1+\mu)\overline{\gamma} - \gamma\alpha}\langle\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle.$$

It is easy to see that $\sum_{n=1}^\infty \kappa_n = \infty$ and $\limsup_{n\longrightarrow\infty}\sigma_n \leq 0$. Hence, by Lemma 1.4, we conclude that the sequence $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Setting $A =: I$, the identity mapping, in Theorem 2.2, we obtain the following result.

Corollary 2.3. *Let H be a real Hilbert space and K be a closed convex subset of H. Let $T : K \longrightarrow K$ be a k-strictly pseudononspreading mapping on with a nonempty fixed point set. Let $f : K \longrightarrow K$ be a contraction with coefficient $\alpha \in (0, 1)$. Let $x_0 \in K$ and define a sequence $\{x_n\}$ by*

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)y_n, \\ y_n &= \frac{1}{n}\sum_{i=1}^{n-1} T_\zeta^i x_n, \end{aligned} \tag{2.11}$$

for all $n \geq 1$, where T_ζ is defined by (2.1). If the following control conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$.

Proof. Since the identity mapping is a 1- strongly positive bounded linear operator and $\alpha \in (0, 1)$, we can set $\gamma = 1$ in (2.2). Consequently, (2.2) reduces to (2.11) and the required result is followed from Theorem 2.2 immediately. □

Remark 2.4. Let u be a fixed element in K and setting $f := u$, a constant mapping. Then our Corollary 2.3 recovers the results presented in [12].

In view of Theorem 2.2, we can obtain the following results as special cases.

Theorem 2.5. *Let H be a real Hilbert space, and $T : H \longrightarrow H$ be a nonspreading mapping with a nonempty fixed point set. Let $f : H \longrightarrow H$ be a contraction with coefficient $\alpha \in (0, 1)$. Let A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$. Let $\gamma > 0$ and $\{\alpha_n\} \subset [0, 1]$ be a sequence of real numbers. Let $x_0 \in H$ and define a sequence $\{x_n\}$ by*

$$\begin{aligned} x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \\ y_n &= \frac{1}{n}\sum_{i=1}^{n-1} T^i x_n, \end{aligned} \tag{2.12}$$

for all $n \geq 1$. If the following control conditions are satisfied:

(i) $0 < \gamma < \overline{\gamma}/\alpha$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$,
(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$ which solves the following optimization problem (2.3).

Proof. Since every nonspreading mapping is a 0-strictly pseudononspreading mapping, by choosing $\zeta = 0$ in (2.2), we see that (2.2) is reduced to (2.12). Consequently, by Theorem 2.2, the proof is completed. □

Immediately, form Theorem 2.5, we also have the following results.

Corollary 2.6. *Let H be a real Hilbert space and K be closed convex subset of H. Let $T : K \longrightarrow K$ be a nonspreading mapping with a nonempty fixed point set and $f : K \longrightarrow K$ be a contraction mapping. Let $\{\alpha_n\} \subset [0, 1]$ be a sequence of real numbers and define a sequence $\{x_n\}$ by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n,$$

$$y_n = \frac{1}{n}\sum_{i=1}^{n-1} T^i x_n, \tag{2.13}$$

for all $n \geq 1$. If the following control conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$.

Corollary 2.7. *Let H be a real Hilbert space and K be closed convex subset of H. Let $T : K \longrightarrow K$ be a nonspreading mapping with a nonempty fixed point set. Let $u \in H$ be fixed and $\{\alpha_n\} \subset [0,1]$ be a sequence of real numbers. Let $x_0 \in K$ and define a sequence $\{x_n\}$ by*

$$\begin{aligned} x_{n+1} &= \alpha_n u + (1-\alpha_n) y_n, \\ y_n &= \frac{1}{n}\sum_{i=1}^{n-1} T^i x_n, \end{aligned} \tag{2.14}$$

for all $n \geq 1$. If the following control conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$.

Remark 2.8. It is worth to noting that, since the class of strictly pseudononspreading mappings contains a large number of mappings, our results extend and improve various related results existing in this present time.

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