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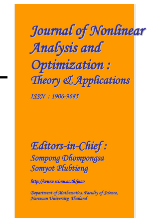

THE GENERALIZED B^r- DIFFERENCE RIESZ χ^2 SEQUENCE SPACES AND UNIFORM OPIAL PROPERTY

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ABSTRACT. We define the new generalized difference Riesz sequence spaces $\Lambda_r^{2q}(p, B^r)$ and $\chi_r^{2q}(p, B^r)$ which consist of all the sequences whose B^r- transforms are in the Riesz sequence spaces $r_\infty^q(p), r_c^q(p)$ and $r_0^q(p)$, respectively, introduced by Altay and Başar (2006). We examine some topological properties and compute the $\alpha-, \beta-,$ and $\gamma-$ duals of the spaces $\Lambda_r^{2q}(p, B^r)$ and $\chi_r^{2q}(p, B^r)$. Fianlly, we determine the necessary and sufficient conditions on the matrix transform from the spaces $\Lambda_r^{2q}(p, B^r)$ and $\chi_r^{2q}(p, B^r)$ to the spaces Λ^2 and χ^2 and prove that sequence space $\chi_r^{2q}(p, B^r)$ have the uniform Opial property for $p_{mn} \geq 1$ for all $m, n \in \mathbb{N}$.

KEYWORDS : Gai Sequence; Analytic Sequence; Double Sequences; Riesz Sequence; Opial Property.
AMS Subject Classification: 40A05, 40C05, 40D05.

1. INTRODUCTION

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial work on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [5], Moricz [9], Moricz and Rhoades [10], Basarir and Solankan [2], Tripathy [17], Turkmenoglu [19], and many others.

Let us define the following sets of double sequences:

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$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : sup_{m,n \in N} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - \Upsilon|^{t_{mn}} = 1 \, for \, some \, \Upsilon \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [21,22] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [27] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also have examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [28] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and have examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [29] have studied the space $\chi^2_M(p, q, u)$ of double sequences and have given some inclusion relations.

Spaces are strongly summable sequences was discussed by Kuttner [31], Maddox [32], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [8] as an extension of the definition of strongly Cesàro summable sequences. Connor [33] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [34] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [35]-[38], and [39] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a^{mn}_{k\ell} x_{mn}$ was studied extensively by Robison and Hamilton. In their work and throughout this paper, the four dimensional matrices and double sequences have real-valued entries unless specified otherwise. In this paper we extend a few results known in the literature for ordinary(single) sequence spaces to multiply sequence spaces.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and

$0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see[1]).

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{all\, finite\, sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.
If X is a sequence space, we give the following definitions:

(i)X' = the continuous dual of X;

(ii)$X^{\alpha} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn} x_{mn}| < \infty,\ for\, each\, x \in X\right\}$;

(iii)$X^{\beta} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn} x_{mn}\, is\, convegent,\ for\, each\, x \in X\right\}$;

(iv)$X^{\gamma} = \left\{a = (a_{mn}) : sup_{mn} \geq 1 \left|\sum_{m,n=1}^{M,N} a_{mn} x_{mn}\right| < \infty, for\, each\, x \in X\right\}$;

(v)$let\, X\, beanFK-space \supset \phi;\, then\, X^{f} = \left\{f(\Im_{mn}) : f \in X'\right\}$;

(vi)$X^{\delta} = \left\{a = (a_{mn}) : sup_{mn} |a_{mn} x_{mn}|^{1/m+n} < \infty,\ for\, each\, x \in X\right\}$;

$X^{\alpha}.X^{\beta}, X^{\gamma}$ are called $\alpha-(or K\ddot{o}the-Toeplitz)$dual of $X, \beta-(or\, generalized-K\ddot{o}the-Toeplitz)\, dual\, of\, X, \gamma-dual\, of\, X, \delta-dual\, of\, X\, respectively. X^{\alpha}$ is defined by Gupta and Kamptan [20]. It is clear that $x^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [30] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.
Here c, c_0 and ℓ_{∞} denote the classes of convergent,null and bounded sclar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by BaŞar and Altay in [42] and in the case $0 < p < 1$ by Altay and BaŞar in [43]. The spaces $c(\Delta), c_0(\Delta), \ell_{\infty}(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$

A linear topological space X over the real field R is said to be a paranormed space if there is a subadditive function $g : X \to R$ such that $g(\theta) = 0, g(x) = g(-x)$ and scalar multiplication is continous; that is $|\alpha_{mn} - \alpha| \to 0$ and $g(x_{mn} - x) \to 0$ imply $g(\alpha_{mn} x_{mn} - \alpha x) \to 0$ for all α's in R and all x's in X, where θ is the zero vector in the linear space X. Assume here and after that $p = (p_{mn})$ is a double analytic sequence of strictly positive real numbers with $supp_{mn} = H$ and $M = max(1, H)$.

Let λ and μ be two sequence spaces and $A = \left(a_{k,\ell}^{mn}\right)$ be an four dimensional infinite matrix of real numbers $\left(a_{k,\ell}^{mn}\right)$, where $m, n, k, \ell \in \mathbb{N}$. Then, we say A defines a matrix mapping from λ into μ and we denoe it by writing $A : \lambda \to \mu$ if for every sequence $x = (x_{mn}) \in \lambda$ the sequence $Ax = \{(Ax)_{k\ell}\}$, the $A-$ transform of x, is in μ, where

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn} \, (k, \ell \in \mathbb{N}) \tag{1.2}$$

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus $A \in (\lambda : \mu)$ if and only if the series on the right side of (1.2) converges for each $k, \ell \in \lambda$. A sequence x is said to be $A-$ summable to α if Ax converges to α which is called as the $A-$ limit of x.

Let (q_{mn}) be a sequence of positive numbers and

$$Q_{k\ell} = \sum_{m=0}^{k} \sum_{n=0}^{\ell} q_{mn} \, (k, \ell \in \mathbb{N}) \tag{1.3}$$

Then, the matrix $R^q = (r_{k\ell}^{mn})^q$ of the Riesz mean is given by

$$(r_{k\ell}^{mn})^q = \begin{cases} \frac{q_{mn}}{Q_{k\ell}} & \text{if } 0 \leq m, n \leq k, \ell \\ 0 & \text{if } (m, n) > k\ell \end{cases} \tag{1.4}$$

The double Riesz sequence spaces are defined as follows:

$$\Lambda_r^{2q}(p) = \left\{x = (x_{mn}) \in w^2 : sup_{k\ell \in N} \left|\frac{1}{Q_{k\ell}} \sum_{m=0}^{k} \sum_{n=0}^{\ell} q_{mn} (x_{mn})^{1/m+n}\right|^{p_{mn}} < \infty\right\},$$

$$\chi_r^{2q}(p) = \left\{x = (x_{mn}) \in w^2 : lim_{k\ell \to \infty} \left|\frac{1}{Q_{k\ell}} \sum_{m=0}^{k} \sum_{n=0}^{\ell} q_{mn} ((m+n)! x_{mn})^{1/m+n}\right|^{p_{mn}} = 0\right\},$$

which are the sequence spaces of the sequences x whose R^q- transforms are in $\Lambda^2(p)$ and $\chi^2(p)$, respectively.

The main purpose of this paper is to introduce the B^r- difference Riesz sequence spaces $\Lambda_r^{2q}(p)$ and $\chi_r^{2q}(p)$ of the sequences whose $R^q B^r-$ transform are in $\Lambda^2(p)$ and $\chi^2(p)$, respectively and to investigate some topological and geometric properties of them. For simplicity, we take the matrix $R^q B^r = T$.

2. B^r- DIFFERENCE RIESZ DOUBLE SEQUENCE SPACES

Let us define the sequence $y = \{y_{k\ell}(q)\}$, which is used, as the $R^q B^r = T-$ transform of a sequence $x = (x_{mn})$, that is,

$$y_{k\ell}(q) = (Tx)_{k\ell} = \frac{1}{Q_{k\ell}} \sum_{m=0}^{k-1} \sum_{n=0}^{\ell-1} \left[\sum_{i=m}^{k} \sum_{j=n}^{\ell} b_{k\ell}^{mn} q_{ij} \left|(m+n)! x_{mn}\right|^{1/m+n}\right] + \frac{1}{Q_{k\ell}} q_{k\ell} \left|(k+\ell)! x_{k\ell}\right|^{1/k+\ell}, (k, \ell \in \mathbb{N}).$$

We define the B^r- difference Riesz sequence spaces
$\Lambda_r^{2q}(p,B^r)=\left\{x=(x_{mn})\in w^2:((Tx)_{k\ell})\in\Lambda^2(p)\right\},$
$\chi_r^{2q}(p,B^r)=\left\{x=(x_{mn})\in w^2:((Tx)_{k\ell})\in\chi^2(p)\right\}$
If $r=1$ then they are reduced the spaces $\Lambda_r^{2q}(p,B)$ and $\chi_r^{2q}(p,B)$. If we take $B=\Delta$ then we have $\Lambda_r^{2q}(p,\Delta^r)$ and $\chi_r^{2q}(p,\Delta^r)$. If we take $B=\Delta$ and $r=1$ then we have $\Lambda_r^{2q}(p,\Delta)$ and $\chi_r^{2q}(p,\Delta)$. If we take $p_{mn}=p=1$ for all m,n then we have $\Lambda_r^{2q}(B^r)$ and $\chi_r^{2q}(B^r)$.
We have the following :

2.1. **DEFINTION.** Let $A=\left(a_{k,\ell}^{mn}\right)$ denotes a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the $k,\ell-$ th term to Ax is as follows:

$$(Ax)_{k\ell}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{k\ell}^{mn}x_{mn}$$

such transformation is said to be nonnegative if $a_{k\ell}^{mn}$ is nonnegative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman [40] and Toeplitz [41]. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which they both added an adiditional assumption of boundedness. This assumption was made because a double sequence which is $P-$ convergent is not necessarily bounded.

2.2. **THEOREM.** $\chi_r^{2q}(p,B^r)$ is a complete metric space paranormed by g_B, defined by

$$g_B(x,y)=sup_{m,n\in\mathbb{N}}\left|((m+n)!\,((Tx)_{mn}-(Ty)_{mn}))^{1/m+n}\right|^{p_{mn}/M},$$

g_B is a paranorm for the spaces $\Lambda_r^{2q}(p,B^r)$ only in the trivial case with $inf p_{mn}>0$ when $\Lambda_r^{2q}(p,B^r)=\Lambda_r^{2q}(B^r)$.

Proof: We prove the theorem for the space $\chi_r^{2q}(p,B^r)$. The linearity of $\chi_r^{2q}(p,B^r)$ with respect to the coordinatewise addition and scalar multiplication that follow from the inequalities which are satisfied for $u,v\in\chi_r^{2q}(p,B^r)$

$$sup_{mn\in\mathbb{N}}\left|\frac{1}{Q_{mn}}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\left[\sum_{i=j}^{m}\sum_{i=j}^{n}b_{mn}^{ij}q_{ij}\left|(i+j)!\,(u_{ij}+v_{ij})\right|^{1/i+j}\right]\right|^{p_{mn}/M}+$$
$$\left|\frac{1}{Q_{mn}}q_{mn}\left|(m+n)!\,(u_{mn}+v_{mn})\right|^{1/m+n}\right|^{p_{mn}/M}\le$$
$$sup_{mn\in\mathbb{N}}\left|\frac{1}{Q_{mn}}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\left[\sum_{i=j}^{m}\sum_{i=j}^{n}b_{mn}^{ij}q_{ij}\left|(i+j)!u_{ij}\right|^{1/i+j}\right]\right|^{p_{mn}/M}+$$
$$\left|\frac{1}{Q_{mn}}q_{mn}\left|(m+n)!u_{mn}\right|^{1/m+n}\right|^{p_{mn}/M}+$$
$$sup_{mn\in\mathbb{N}}\left|\frac{1}{Q_{mn}}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\left[\sum_{i=j}^{m}\sum_{i=j}^{n}b_{mn}^{ij}q_{ij}\left|(i+j)!v_{ij}\right|^{1/i+j}\right]\right|^{p_{mn}/M}+$$
$$\left|\frac{1}{Q_{mn}}q_{mn}\left|(m+n)!v_{mn}\right|^{1/m+n}\right|^{p_{mn}/M}$$

and for any $\alpha\in\mathbb{R}$.

$$|\alpha|^{p_{mn}}\le max\left\{1,|\alpha|^{M}\right\} \tag{2.1}$$

It is clear that $g_B(\theta)=0$ and $g_B(-x)=g_B(x)$ for all $x\in\chi_r^{2q}(p,B^r)$. Again the inequality (2.1) yield the subadditivity of g_B and

$$g_B(\alpha u)\le max\{1,|\alpha|\}\,g_B(u).$$

Let $\left\{x^{k\ell}\right\}$ be any sequence of the elements of the space $\chi_r^{2q}(p,B^r)$ such that

$$g_B\left(x^{k\ell}-x\right)\to 0$$

and $(\lambda_{k\ell})$ also be any sequence of scalars such that $\lambda_{k\ell} \to \lambda$, as $k, \ell \to \infty$. Then, since the inequality

$$g_B\left(x^{k\ell}\right) \leq g_B(x) + g_B\left(x^{k\ell} - x\right)$$

holds by sub additivity of g_B, $\left\{g_B\left(x^{k\ell}\right)\right\}$ is analytic, and thus we have

$$g_B\left(\lambda_{k\ell}x^{k\ell} - \lambda x\right) \leq |\lambda_{k\ell} - \lambda|^{1/M} g_B\left(x^{k\ell}\right) + |\lambda|^{1/M} g_B\left(x^{k\ell} - x\right),$$

which tends to zero as $k\ell \to \infty$. Hence, the scalar multiplication is continuous. Finally, it is clear to say that g_B is a paranorm on the space $\chi_r^{2q}(p, B^r)$. Moreover, we will prove the completeness of the space $\chi_r^{2q}(p, B^r)$. Let x^{ij} be a Cauchy sequence inthe $\chi_r^{2q}(p, B^r)$, where

$$x^{ij} = \left\{x_{mn}^{ij}\right\} = \begin{pmatrix} x_{01}^{ij} & x_{02}^{ij} & x_{03}^{ij}\cdots & x_{0n}^{ij} & 0 \\ x_{11}^{ij} & x_{12}^{ij} & x_{13}^{ij}\cdots & x_{1n}^{ij} & 0 \\ \vdots & & & & \\ x_{m1}^{ij} & x_{m2}^{ij} & x_{m3}^{ij}\cdots & x_{mn}^{ij} & 0 \\ 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix} \in \chi_r^{2q}(p, B^r).$$

Then, for a given $\epsilon > 0$, there exists a positive integer $k_0\ell_0(\epsilon)$ such that

$$g_B\left(x^{ij} - x^{pq}\right) < \epsilon$$

for all $i, j, p, q \geq k_0\ell_0(\epsilon)$. If we use the definition of g_B, we obtain for each fixed $m, n \in \mathbb{N}$ that

$$\left|\left(Tx^{ij}\right)_{mn} - (Tx^{pq})_{mn}\right| \leq sup_{mn\in\mathbb{N}}\left|\left((m+n)!\left(Tx^{ij}\right)_{mn} - (Tx^{pq})_{mn}\right)^{1/m+n}\right|^{p_{mn}/M} < \epsilon \tag{2.2}$$

for $i, j, p, q \geq k_0\ell_0(\epsilon)$ which leads us to the fact that

$$\begin{pmatrix} Tx_{mn}^{01} & Tx_{mn}^{02} & Tx_{mn}^{03}\cdots & Tx_{mn}^{oq} & 0 \\ Tx_{11}^{ij} & Tx_{12}^{ij} & Tx_{13}^{ij}\cdots & Tx_{1n}^{ij} & 0 \\ \vdots & & & & \\ Tx_{mn}^{p1} & Tx_{mn}^{p2} & Tx_{mn}^{p3}\cdots & Tx_{mn}^{pq} & 0 \\ 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix}$$

is a Cauchy sequence of real numbers for every fixed $m, n \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, so we write $\left(Tx^{ij}\right)_{mn} \to (Tx)_{mn}$ as $i, j \to \infty$. Hence, by using these infinitely many limits $(Tx)_{01}, (Tx)_{02} \cdots (Tx)_{mn}, 0, 0, \cdots$ we define the sequence

$$\begin{pmatrix} Tx_{01} & Tx_{02} & Tx_{03}\cdots & Tx_{on} & 0 \\ Tx_{11} & Tx_{12} & Tx_{13}\cdots & Tx_{1n} & 0 \\ \vdots & & & & \\ Tx_{m1} & Tx_{m2} & Tx_{m3}\cdots & Tx_{mn} & 0 \\ 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix}$$

From (2.2) with $p, q \to \infty$, we have

$$\left|\left(Tx^{ij}\right)_{mn} - (Tx)_{mn}\right| \leq \epsilon,\ i, j \geq k_0\ell_0(\epsilon) \tag{2.3}$$

for every fixed $m, n \in \mathbb{N}$. Since $x^{ij} = \left\{x_{mn}^{(ij)}\right\} \in \chi_r^{2q}(p, B^r)$,

$$\left|\left(Tx^{ij}\right)_{mn}\right|^{p_{mn}/M} < \epsilon,$$

for all $m, n \in \mathbb{N}$. Therefore, by (2.3), we obtain that

$$|(Tx)_{mn}|^{p_{mn}/M} \leq \left|(Tx)_{mn} - \left(Tx^{ij}\right)_{mn}\right|^{p_{mn}/M} + \left|\left(Tx^{ij}\right)_{mn}\right|^{p_{mn}/M} < \epsilon \tag{2.4}$$

for all $i,j \geq k_0\ell_0(\epsilon)$. This shows that the sequence Tx belongs to the space $\chi^2(p)$. Since $\left\{x^{(ij)}\right\}$ was an arbitrary Cauchy sequence, the space $\chi_r^{2q}(p,B^r)$ is complete. This completes the proof.

3. THE BASIS FOR THE SPACE $\chi_r^{2q}(p,B^r)$

In this section, we give sequence of the points of the spaces $\chi_r^{2q}(p,B^r)$ which form the basis for those space.

If a sequence space λ paranormed by h contains a sequence $(b_{k\ell})$ with the property that for every $x \in \lambda$, there is a unique sequence of scalars $(\alpha_{k\ell})$ such that

$$lim_{k\ell\to\infty} h\left(x - \sum_{m=0}^{k}\sum_{n=0}^{\ell}\alpha_{mn}\left((m+n)!x_{mn}\right)^{1/m+n}\right) = 0$$

then $(b_{k\ell})$ is a Schauder basis for $\lambda = 0$. The series $\sum\sum \alpha_{mn}\beta_{mn}$ which has the sum x is then called the expansion of x with respect to $(b_{k\ell})$ and written as $x = \sum\sum \alpha_{mn}\beta_{mn}$.

3.1. **THEOREM.** Let $\mu_{mn}(q) = (Tx)_{mn}$ for all $m,n \in \mathbb{N}$ and also $0 < p_{mn} \leq H < \infty$. Define the sequence $b_{mn}(q)$ of the elements of the space $\chi_r^{2q}(p,B^r)$ for every fixed $m,n \in \mathbb{N}$ then the sequence $\{b_{mn}(q)\}_{m,n\in\mathbb{N}}$ is a basis for the space $\chi_r^{2q}(p,B^r)$ and any $x \in \chi_r^{2q}(p,B^r)$ has a unique representation of the form

$$x = \sum_m\sum_n \mu_{mn}(q)\, b_{mn}(q) \tag{3.1}$$

Proof: It is clear that $\{b_{mn}(q)\} \subset \chi_r^{2q}(p,B^r)$, since

$$Tb_{mn}(q) = \Im_{mn} \in \chi^2(p)\,(for\, m,n \in \mathbb{N}) \tag{3.2}$$

for $0 < p_{mn} \leq H < \infty$, where $\Im_{mn}$ denotes the double sequence whose only nonzero term is 1 in the $(mn)^{th}$ place for each $m,n \in \mathbb{N}$. Let $x \in \chi_r^{2q}(p,B^r)$ be given. For every nonnegative integer r,s, we put

$$x^{[rs]} = \sum_{m=0}^{r}\sum_{n=0}^{s}\mu_{mn}(q)\, b_{mn}(q) \tag{3.3}$$

Then, we obtain by applying T to (3.3) with (3.2) that

$$Tx^{[rs]} = \sum_{m=0}^{r}\sum_{n=0}^{s}\mu_{mn}(q)\, Tb_{mn}(q) = \sum_{m=0}^{r}\sum_{n=0}^{s}(T)_{mn}\,\Im_{mn},$$

$\left(R^q\left(x - x^{[rs]}\right)\right)_{ij} = \begin{cases} 0, & \text{if } 0 \leq i,j \leq r,s \\ (Tx)_{ij} & \text{if } (i,j) > (rs) \end{cases}$ Given $\epsilon > 0$, then there exists an integer r_0s_0 such that

$$sup_{i,j\geq r,s}\left|(Tx)_{ij}\right|^{p_{mn}/M} < \frac{\epsilon}{2} \tag{3.4}$$

for all $r,s \geq r_0s_0$. Hence,

$g_B\left(x - x^{[rs]}\right) = sup_{i,j\geq r,s}\left|(Tx)_{ij}\right|^{p_{mn}/M} \leq sup_{i,j\geq r_0,s_0}\left|(Tx)_{ij}\right|^{p_{mn}/M} < \frac{\epsilon}{2} < \epsilon$, for all $r,s \geq r_0s_0$, which proves that $x \in \chi_r^{2q}(p,B^r)$ is represented as in (3.1).

To show the uniqueness of this representation, we suppose that there exists a representation

$$x = \sum_m\sum_n \lambda_{mn}(q)\, b_{mn}(q).$$

Therefore the transformation $\chi_r^{2q}(p,B^r)$ to $\chi^2(p)$ and also continuous we have

$(Tx)_{k\ell} = \sum_m\sum_n \lambda_{mn}(q)\{Tb_{mn}(q)\}_{k\ell} = \sum_m\sum_n \lambda_{mn}(q)\,\Im_{mn} = \lambda_{k\ell}(q); k,\ell \in \mathbb{N}$, which contradicts the fact that $(Tx)_{k\ell} = \mu_{mn}(q)\; for\, all\, m,n \in \mathbb{N}$. Hence the representation (3.1) of $x \in \chi_r^{2q}(p,B^r)$ is unique. This completes the proof.

4. UNIFORM OPIAL PROPERTY OF $\chi_r^{2q}(p, B^r)$ − DIFFERENCE RIESZ SEQUENCE SPACE

In this section, we investigate the uniform Opial property of the sequence spaces $\chi_r^{2q}(p, B^r)$.

The Opial propertyplays an important role in the study of weak convergence of iterates of mapping of Banach spaces and of the asympototic behavior of nonlinear semigroup. The Opial property is important because Banach spaces with this property have the weak fixed point property.

WE give the definition of uniform Opial property in a linear metric space and obtain that $\chi_r^{2q}(p, B^r)$ have uniform Opial property for $p_{mn} \geq 1$.

For a sequence $x = (x_{k\ell}) \in \chi_r^{2q}(p, B^r)$ and for $i, j \in \mathbb{N}$, we use the notation

$$x_{|ij} = \begin{pmatrix} x_{11} & x_{12} & x_{13}\cdots & x_{1j} & 0 \\ x_{21} & x_{22} & x_{13}\cdots & x_{2j} & 0 \\ \vdots & & & & \\ x_{i1} & x_{i2} & x_{i3}\cdots & x_{ij} & 0 \\ 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix}$$

and

$$x_{|\mathbb{N}-ij} = \begin{pmatrix} 0 & 0 & 0\cdots & 0 & 0 \\ x_{i+11} & x_{i+12} & x_{i+13}\cdots & x_{i+1j} & 0 \\ x_{i+21} & x_{i+22} & x_{i+23}\cdots & x_{i+2j} & 0 \\ x_{i+31} & x_{i+32} & x_{i+33}\cdots & x_{i+3j} & 0 \\ \vdots & & & & \\ x_{i+m1} & x_{i+m2} & x_{i+m3}\cdots & x_{i+mn} & 0 \end{pmatrix}$$

We know that every total paranormed space becomes a linear metric space with the metric given by

$$d(x, y) = g\left((m+n)!\,|x_{mn} - y_{mn}|^{1/m+n}\right).$$

It is clear that $\chi_r^{2q}(p, B^r)$ is total paranormed space with

$$d(x, y) = g_B\left((m+n)!\,|x_{mn} - y_{mn}|^{1/m+n}\right).$$

Now, we can give the definition of uniform Opial property in a linear metric space.

A linear metric space (X, d) has the uniform Opial property if for each $\epsilon > 0$ there exists $\tau > 0$ such that for any weakly gai sequence $\{x_{k\ell}\} \in S(0, r)$ and $x \in \chi_r^{2q}(p, B^r)$ with $d(x, 0) \geq \epsilon$ the following inequality holds:

$$r + \tau \leq lim\,inf_{k\ell\to\infty} d(x_{k\ell} + x, 0).$$

4.1. **LEMMA.** If $lim\,inf_{mn\to\infty} p_{mn} > 0$ then for any $L > 0$ and $\epsilon > 0$, there exists $\delta = \delta(\epsilon, L) > 0$ for $u, v \in \chi_r^{2q}(p, B^r)$ such that

$$d^M(u + v, 0) < d^M(u, 0) + \epsilon$$

whenever $d^M(u, 0) \leq L$ and $d^M(v, 0) \leq \delta$.

4.2. **THEOREM.** If $p_{mn} \geq 1$, then $\chi_r^{2q}(p, B^r)$ have uniform Opial property.

Proof: For any $\epsilon > 0$, we can find a positive number $\epsilon_0 \in (0, \epsilon)$ such that

$$r^M + \frac{\epsilon^M}{4} > (r + \epsilon_0)^M.$$

Take any $x \in \chi_r^{2q}(p, B^r)$ with $d^M(x, 0) \geq \epsilon^M$ and $(x_{k\ell})$ to be weakly gai sequence in $S(0, r)$. By this, we write

$$d^M(x_{k\ell}, 0) = r^M.$$

There exists $p_0q_0 \in \mathbb{N}$ such that

$$d^M\left(x_{|\mathbb{N}-p_0q_0}, 0\right) = \sum_{m=p_0+1}^{\infty} \sum_{n=q_0+1}^{\infty} |Tx_{mn}|^{p_{mn}} < \frac{\epsilon_0}{r}^M < \frac{\epsilon^M}{4} \tag{4.1}$$

Furthermore, we have

$$\epsilon^M \leq d^M(x,0) = \sum_{m=0}^{p_0} \sum_{n=0}^{q_0} |Tx_{mn}|^{p_{mn}} + \sum_{m=p_0+1}^{\infty} \sum_{n=q_0+1}^{\infty} |Tx_{mn}|^{p_{mn}},$$

$$\epsilon^M \leq \sum_{m=0}^{p_0} \sum_{n=0}^{q_0} |Tx_{mn}|^{p_{mn}} + \frac{\epsilon^M}{4}, \tag{4.2}$$

$$\tfrac{3\epsilon^M}{4} \leq \sum_{m=0}^{p_0} \sum_{n=0}^{q_0} |Tx_{mn}|^{p_{mn}}.$$

By $x_{k\ell} \to 0$, weakly, this implies that $x_{k\ell} \to 0$, coordinatewise, hence there exists $k_0\ell_0 \in \mathbb{N}$ such that with (4.2)

$$\frac{3\epsilon^M}{4} \leq \sum_{m=0}^{p_0} \sum_{n=0}^{q_0} \left| T_{k\ell}x_{mn} + ((m+n)!x_{mn})^{1/m+n} \right|^{p_{mn}}, \tag{4.3}$$

for all $k\ell \geq n_0\ell_0$. Lemma (4.1) asserts that

$$d^M(y+z,0) \leq d^M(y,0) + \frac{\epsilon^M}{4}, \tag{4.4}$$

whenever $d^M(y,0) \leq r^M$ and $d^M(z,0) \leq \epsilon_0$. Again by $x_{k\ell} \to 0$, weakly, there exists $k_1\ell_1 > k_0\ell_0$ such that $d^M\left(x_{k\ell|p_0q_0}, 0\right) < \epsilon_0$ for all $k\ell > k_1\ell_1$, so by (4.4), we obtain that

$$d^M\left(x_{k\ell|\mathbb{N}-p_0q_0} + x_{k\ell|p_0q_0}, 0\right) < d^M\left(x_{k\ell|\mathbb{N}-p_0q_0}, 0\right) + \frac{\epsilon^M}{4}, \tag{4.5}$$

hence,

$$d^M(x_{k\ell},0) - \frac{\epsilon^M}{4} < d^M\left(x_{k\ell|\mathbb{N}-p_0q_0}, 0\right) = \sum_{m=p_0+1}^{\infty} \sum_{n=q_0+1}^{\infty} |T_{k\ell}x_{mn}|^{p_{mn}}, \tag{4.6}$$

$$r^M - \frac{\epsilon^M}{4} < \sum_{m=p_0+1}^{\infty} \sum_{n=q_0+1}^{\infty} |T_{k\ell}x_{mn}|^{p_{mn}}, \tag{4.7}$$

for all $k\ell > k_1\ell_1$. This, together with (4.1),(4.2), implies that for any $k\ell > k_1\ell_1$,

$$d^M(x_{k\ell}+x,0) = \sum_{m=0}^{p_0} \sum_{n=0}^{q_0} \left| T_{k\ell}x_{mn} + ((m+n)!x_{mn})^{1/m+n} \right|^{p_{mn}} + \sum_{m=p_0+1}^{\infty} \sum_{n=q_0+1}^{\infty} \left| T_{k\ell}x_{mn} + ((m+n)!x_{mn})^{1/m+n} \right|^{p_{mn}} \geq$$

$$\sum_{m=0}^{p_0} \sum_{n=0}^{q_0} \left| T_{k\ell}x_{mn} + ((m+n)!x_{mn})^{1/m+n} \right|^{p_{mn}} + \left| \sum_{m=p_0+1}^{\infty} \sum_{n=q_0+1}^{\infty} |T_{k\ell}x_{mn}|^{p_{mn}} \right| - \left| \sum_{m=p_0+1}^{\infty} \sum_{n=q_0+1}^{\infty} |T_{k\ell}x_{mn}|^{p_{mn}} \right|$$

$$> \tfrac{3\epsilon^M}{4} + \left(r^M + \tfrac{\epsilon^M}{4}\right) - \tfrac{\epsilon^M}{4} = r^M + \tfrac{\epsilon^M}{4} > (r+\epsilon_0)^M.$$

References

[1] T.Apostol, Mathematical Analysis, *Addison-wesley , London*, **1**978.

[2] M.Basarir and O.Solancan, On some double sequence spaces, *J. Indian Acad. Math.*, **21(2)** (1999), 193-200.

[3] C.Bektas and Y.Altin, The sequence space $\ell_M(p,q,s)$ on seminormed spaces, *Indian J. Pure Appl. Math.*, **34(4)** (2003), 529-534.

[4] T.J.I'A.Bromwich, An introduction to the theory of infinite series *Macmillan and Co.Ltd.* ,New York, **(**1965).

[5] G.H.Hardy, On the convergence of certain multiple series, *Proc. Camb. Phil. Soc.*, **19** (1917), 86-95.

[6] M.A.Krasnoselskii and Y.B.Rutickii, Convex functions and Orlicz spaces, *Gorningen, Netherlands*, **1961**.

[7] J.Lindenstrauss and L.Tzafriri, On Orlicz sequence spaces, *Israel J. Math.*, **10** (1971), 379-390.
[8] I.J.Maddox, Sequence spaces defined by a modulus, *Math. Proc. Cambridge Philos. Soc*, **100(1)** (1986), 161-166.
[9] F.Moricz, Extentions of the spaces c and c_0 from single to double sequences, *Acta. Math. Hung.*, **57(1-2)**, (1991), 129-136.
[10] F.Moricz and B.E.Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, *Math. Proc. Camb. Phil. Soc.*, **104**, (1988), 283-294.
[11] M.Mursaleen,M.A.Khan and Qamaruddin, Difference sequence spaces defined by Orlicz functions, *Demonstratio Math.* , **Vol. XXXII** (1999), 145-150.
[12] H.Nakano, Concave modulars, *J. Math. Soc. Japan*, **5**(1953), 29-49.
[13] W.Orlicz, $\ddot{U}$ber Raume $\left(L^M\right)$ *Bull. Int. Acad. Polon. Sci. A*, **(**1936), 93-107.
[14] S.D.Parashar and B.Choudhary, Sequence spaces defined by Orlicz functions, *Indian J. Pure Appl. Math.* , **25(4)**(1994), 419-428.
[15] K.Chandrasekhara Rao and N.Subramanian, The Orlicz space of entire sequences, *Int. J. Math. Math. Sci.*, **68**(2004), 3755-3764.
[16] W.H.Ruckle, FK spaces in which the sequence of coordinate vectors is bounded, *Canad. J. Math.*, **25**(1973), 973-978.
[17] B.C.Tripathy, On statistically convergent double sequences, *Tamkang J. Math.*, **34(3)**, (2003), 231-237.
[18] B.C.Tripathy,M.Et and Y.Altin, Generalized difference sequence spaces defined by Orlicz function in a locally convex space, *J. Anal. Appl.*, **1(3)**(2003), 175-192.
[19] A.Turkmenoglu, Matrix transformation between some classes of double sequences, *J. Inst. Math. Comp. Sci. Math. Ser.*, **12(1)**, (1999), 23-31.
[20] P.K.Kamthan and M.Gupta, Sequence spaces and series, Lecture notes, Pure and Applied Mathematics, *65 Marcel Dekker, In c., New York* , **1**981.
[21] A.Gökhan and R.Çolak, The double sequence spaces $c_2^P(p)$ and $c_2^{PB}(p)$, *Appl. Math. Comput.*, **157(2)**, (2004), 491-501.
[22] A.Gökhan and R.Çolak, Double sequence spaces ℓ_2^∞, *ibid.*, **160(1)**, (2005), 147-153.
[23] M.Zeltser, Investigation of Double Sequence Spaces by Soft and Hard Analitical Methods, Dissertationes Mathematicae Universitatis Tartuensis 25, *Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu*, **2001**.
[24] M.Mursaleen and O.H.H. Edely, Statistical convergence of double sequences, *J. Math. Anal. Appl.*, **288(1)**, (2003), 223-231.
[25] M.Mursaleen, Almost strongly regular matrices and a core theorem for double sequences, *J. Math. Anal. Appl.*, **293(2)**, (2004), 523-531.
[26] M.Mursaleen and O.H.H. Edely,Almost convergence and a core theorem for double sequences, *J. Math. Anal. Appl.*, **293(2)**, (2004), 532-540.
[27] B.Altay and F.Basar, Some new spaces of double sequences, *J. Math. Anal. Appl.*, **309(1)**, (2005), 70-90.
[28] F.Basar and Y.Sever, The space $\mathcal{L}_p$ of double sequences, *Math. J. Okayama Univ*, **51**, (2009), 149-157.
[29] N.Subramanian and U.K.Misra, The semi normed space defined by a double gai sequence of modulus function, *Fasciculi Math.*, **46**, (2010).
[30] H.Kizmaz, On certain sequence spaces, *Cand. Math. Bull.*, **24(2)**, (1981), 169-176.
[31] B.Kuttner, Note on strong summability, *J. London Math. Soc.*, **21**(1946), 118-122.
[32] I.J.Maddox, On strong almost convergence, *Math. Proc. Cambridge Philos. Soc.*, **85(2)**, (1979), 345-350.
[33] J.Cannor, On strong matrix summability with respect to a modulus and statistical convergence, *Canad. Math. Bull.*, **32(2)**, (1989), 194-198.
[34] A.Pringsheim, Zurtheorie derzweifach unendlichen zahlenfolgen, *Math. Ann.*, **53**, (1900), 289-321.
[35] H.J.Hamilton, Transformations of multiple sequences, *Duke Math. J.*, **2**, (1936), 29-60.
[36] ———-, A Generalization of multiple sequences transformation, *Duke Math. J.*, **4**, (1938), 343-358.
[37] ———-, Change of Dimension in sequence transformation , *Duke Math. J.*, **4**, (1938), 341-342.
[38] ———-, Preservation of partial Limits in Multiple sequence transformations, *Duke Math. J.*, **4**, (1939), 293-297.
[39] , G.M.Robison, Divergent double sequences and series, *Amer. Math. Soc. Trans.*, **28**, (1926), 50-73.
[40] L.L.Silverman, On the definition of the sum of a divergent series, *un published thesis*, **University of Missouri studies, Mathematics series**.
[41] O.Toeplitz, $\ddot{U}$ber allgenmeine linear mittel bridungen, *Prace Matemalyczno Fizyczne (warsaw)*, **22**, (1911).

[42] F.Basar and B.Atlay, On the space of sequences of $p-$ bounded variation and related matrix mappings, *Ukrainian Math. J.*, **55(1)**, (2003), 136-147.

[43] B.Altay and F.Basar, The fine spectrum and the matrix domain of the difference operator Δ on the sequence space $\ell_p, (0 < p < 1)$, *Commun. Math. Anal.*, **2(2)**, (2007), 1-11.

[44] R.Çolak,M.Et and E.Malkowsky, Some Topics of Sequence Spaces, Lecture Notes in Mathematics, *Firat Univ. Elazig, Turkey*, **2004**, pp. 1-63, Firat Univ. Press, (2004), ISBN: 975-394-0386-6.