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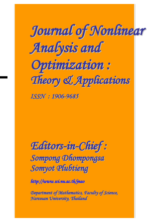

OPTIMALITY CONDITIONS FOR (G, α)-INVEX MULTIOBJECTIVE PROGRAMMING

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ABSTRACT. In this paper, a generalization of convexity, namely (G, α)-invexity, is considered in the case of nonlinear multiobjective programming problems where the functions constituting vector optimization problems are differentiable. Two auxiliary programming problems are constructed to present the modified Kuhn-Tucker necessary optimality conditions for (CVP). With the help of auxiliary programming problems (G-CVP), the relation between (CVP) and (G-CVP) is discussed; while with the help of (φ_G P), a new Kuhn-Tucker necessary condition for (CVP) is presented. Furthermore, the sufficiency of the introduced G-Karush-Kuhn-Tucker(G-Kuhn-Tucker) necessary optimality conditions, for nonconvex multiobjective programming probleminvolving (G, α)-invex functions, is proved.

KEYWORDS : (G, α)-invexity; Kuhn-Tucker constraint qualification; (weakly) efficient solution; G-Kuhn-Tucker necessary optimality conditions.
AMS Subject Classification: 90C29, 90C46.

1. INTRODUCTION

Convexity plays a central role in many aspects of mathematical programming including analysis of stability, sufficient optimality conditions and duality. Based on convexity assumptions, nonlinear programming problems can be solved efficiently. There have been many attempts to weaken the convexity assumptions in order to treat many practical problems. Therefore, many concepts of generalized convex functions have been introduced and applied to mathematical programming problems in the literature [1, 2, 10]. One of these concepts, invexity, was introduced by Hanson in [7]. Hanson has shown that invexity has a common property in mathematical programming with convexity that Karush-Kuhn-Tucker conditions are sufficient for global optimality of nonlinear programming under the invexity assumptions. Ben-Israel and Mond [6] introduced the concept of pre-invex functions which is a special case of invexity.

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Recently, Antczak extended further invexity to G-invexity [3] for scalar differentiable functions and introduced new necessary optimality conditions for differentiable mathematical programming problem. Antczak also applied the introduced G-invexity notion to develop sufficient optimality conditions and new duality results for differentiable mathematical programming problems. Furthermore, in the natural way, Antczak's definition of G-invexity was also extended to the case of differentiable vector-valued functions. In [4], Antczak defined vector G-invex (G-incave) functions with respect to η, and applied this vector G-invexity to develop optimality conditions for differentiable multiobjective programming problems with both inequality and equality constraints. He also established the so-called G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable vector optimization problems under the Kuhn-Tucker constraint qualification [4]. With this vector G-invexity concept, Antczak proved new duality results for nonlinear differentiable multiobjective programming problems[5]. A number of new vector duality problems such as G-Mond–Weir, G-Wolfe and G-mixed dual vector problems to the primal one were also defined in [5].

Motivated by [4, 5, 9], we present new classes of generalized convexity, namely vector (G, α)-invexity, in this paper. Basing on this new vector (G, α)-invexity, we have managed to deal with nonlinear programming problems under some assumptions. The rest of the paper is organized as follows: In section 2, we present concepts regarding vector (G, α)-invexity, and discuss In section 3, we firstly present G-Karush-Kuhn-Tucker necessary optimality conditions for mathematical programming problems with a different method from Antczak's one in [4]. Moreover, with the vector (G, α)-invexity assumption, we prove G-Karush-Kuhn-Tucker sufficient optimality conditions for mathematical programming problems.

2. VECTOR (G, α)-INVEX FUNCTIONS

In this section, we provide some definitions and some results that we shall use in the sequel. The following convention for equalities and inequalities will be used throughout the paper. For any $x = (x_1, x_2, \cdots, x_n)^T$, $y = (y_1, y_2, \cdots, y_n)^T$, we define:

$$x > y \text{ if and only if } x_i > y_i, \text{ for } i = 1, 2, \cdots, n;$$
$$x \geqq y \text{ if and only if } x_i \geq y_i, \text{ for } i = 1, 2, \cdots, n;$$
$$x \geqslant y \text{ if and only if } x_i \geq y_i, \text{ for } i = 1, 2, \cdots, n, \text{ but } x \neq y;$$
$$x \not> y \text{ is the negation of } x > y.$$

We say that a vector $z \in \mathbb{R}^n$ is negative if $z \leqq 0$ and strictly negative if $z < 0$.

Let $f = (f_1, \cdots, f_k) : X \to \mathbb{R}^k$ be a vector-valued differentiable function defined on a nonempty set $X \subset \mathbb{R}^n$, $I_{f_i}(x)$, $i = 1, \cdots, k$, be the range of f_i, that is, the image of X under f_i. Let $G_f = (G_{f_1}, \cdots, G_{f_k}) : R \to \mathbb{R}^k$ be a vector-valued function such that any of its component $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$ is a strictly increasing function on its domain.

Definition 2.1. Let $f = (f_1, \cdots, f_k) : X \to \mathbb{R}^k$ be a vector-valued differentiable function defined on a nonempty open set $X \subset \mathbb{R}^n$, $I_{f_i}(x)$, $i = 1, \cdots, k$, be the range of f_i. If there exist a differentiable vector-valued function $G_f = (G_{f_1}, \cdots, G_{f_k}) : R \to \mathbb{R}^k$ such that any of its component $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$ is a strictly increasing function on its domain, a vector-valued function $\eta : X \times X \to \mathbb{R}^n$ and real function $\alpha_i : X \times X \to \mathbb{R}_+ (i \in K)$ such that, for all $x \in X (x \neq u)$,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) \geq \alpha_i(x, u) G'_{f_i}(f_i(u)) \triangledown f_i(u) \eta(x, u), i = 1, \cdots, k \qquad (2.1)$$

then f is said to be a (strictly) vector (G_f,α)-invex function at u on X (with respect to η) (or shortly, (G_f,α)-invex function at u on X), where $\alpha := (\alpha_1,\alpha_2,\cdots,\alpha_n)^T$. If (2.1) is satisfied for each $u\in X$, then f is vector (G_f,α)-invex on X with respect to η.

Remark 2.2. In order to define an analogous class of (strictly) vector (G_f,α)-incave functions with respect to η, the direction of the inequality in the definition of these functions should be changed to the opposite one.

Remark 2.3. (1) If f is (G_f,α)-invex function, then, by Definition 2.1 and definition of α-invex in [9], $G_f(f)(x) = (G_{f_1}(f_1(x)), G_{f_2}(f_2(x)),\cdots,G_{f_k}(f_k(x))$, $G_f(f)$ is α-invex.

(2) If $G_{f_i}(a)=a, \forall a\in\mathbb{R}$, then (G_f,α)-invex function is α-invex defined in [9].

(3) If $\alpha_i(x,u)=1, \forall\, x,u\in X,\ i\in K$, then (G_f,α)-invex function is vector G_f-invex defined in [4]. Further, if $|K|=1$, then (G_f,α)-invex function is G_f-invex defined in [3].

Hence, the (G_f,α)-invex function is a generalization of α-invex and G_f-invex function.

In general, a multiobjective programming problem is formulated as the following vector minimization problem:

$$
\begin{aligned}
(CVP)\quad & \min f(x) := (f_1(x), f_2(x),\cdots,f_k(x)),\\
& s.t.\ g(x) := (g_1(x),g_2(x),\cdots,g_m(x)) \leqq 0\\
& \quad h(x) := (h_1(x),h_2(x),\cdots,h_p(x)) = 0\\
& \quad x\in X
\end{aligned}
$$

where X is a nonempty set of $\mathbb{R}^n$, and f_i denotes a real-valued differentiable function on X. We denote by $K := \{1,2,\cdots,k\}$, $M := \{1,2,\cdots,m\}$ and $P := \{1,2,\cdots,p\}$.

Let $E_{CVP} = \{x\in X : g_j(x)\le 0, j\in M, h_t(x)=0, t\in P\}$ be the set of all feasible solutions for problem (CVP). Further, we denote by $J(\bar{x}) := \{j\in M : g_j(\bar{x})=0\}$ the set of constraint indices active at $\bar{x}\in E_{CVP}$.

For the convenience, we need the following vector minimization problem:

$$
\begin{aligned}
(G-CVP)\quad & \min G_f f(x) := (G_{f_1}(f_1(x), G_{f_2}(f_2(x)),\cdots,G_{f_k}(f_k(x)))\,,\\
& s.t.\ G_g g(x) := (G_{g_1}(g_1(x)), G_{g_2}(g_2(x)),\cdots,G_{g_m}(g_m(x))) \leqq G_g(0)\\
& \quad G_h h(x) := \big(G_{h_1}(h_1(x)), G_{h_2}(h_2(x)),\cdots,G_{h_p}(h_p(x))\big) = G_h(0)\\
& \quad x\in X
\end{aligned}
$$

where

$$
G_g(0) := (G_{g_1}(0), G_{g_2}(0),\cdots,G_{g_m}(0))\,, G_h(0) := \big(G_{h_1}(0), G_{h_2}(0),\cdots,G_{h_p}(0)\big)\,.
$$

We denote by $E_{G-CVP} = \{x\in X : G_g g(x)\le G_g(0), G_h h(x) = G_h(0)\}$, $J'(\bar{x}) := \{j\in M : G_{g_j} g_j(\bar{x}) = G_{g_j}(0)\}$. Then, it is easy to see that $E_{CVP} = E_{G-CVP}$ and $J(\bar{x}) = J'(\bar{x})$. So, we represent the set of all feasible solutions and the set of constraint active indices for either (CVP) or (G-CVP) by the notations E and $J(\bar{x})$, respectively.

Before studying optimality in multiobjective programming, one has to define clearly the concepts of optimality and solutions in multiobjective programming problem. Note that, in vector optimization problems there is a multitude of competing definitions and approaches. The dominated ones are now various scalarizations and (weak) Pareto optimality. The (weak) Pareto optimality in multiobjective programming associates the concept of a solution with some property that seems intuitively natural.

Definition 2.4. A feasible point $\bar{x}$ is said to be an efficient solution for a multiobjective programming problem (CVP) if and only if there exists no $x \in E$ such that

$$f(x) \leq f(\bar{x}).$$

Definition 2.5. A feasible point $\bar{x}$ is said to be a weakly efficient solution for a multiobjective programming problem (CVP) if and only if there exists no $x \in E$ such that

$$f(x) < f(\bar{x}).$$

Theorem 2.6. *Let $G_{f_i}(i \in K)$ be strictly increasing function defined on $I_{f_i}(X)$, $G_{g_j}(j \in M)$ be strictly increasing function defined on $I_{g_j}(X)$ and $G_{h_t}(t \in P)$ be strictly increasing function defined on $I_{h_t}(X)$. Further, let $0 \in I_{g_j}(X)$, $j \in M$, and $0 \in I_{h_t}(X)$, $t \in M$. Then $\bar{x}$ is an efficient solution (a weakly efficient solution) for (CVP) if and only if $\bar{x}$ is an efficient solution (a weakly efficient solution) for (G-CVP).*

Proof Here, we only prove the case that $\bar{x}$ is an efficient solution, weakly efficient solution case can be proved in similar way.

"if" part, we prove that if $\bar{x}$ is an efficient solution for (G-CVP), then $\bar{x}$ is an efficient solution for (G-CVP). On the contrary, Let $\bar{x}$ be an efficient solution for (G-CVP) but not an efficient solution for (CVP). Then there exists $x_0 \in E$, such that

$$f(x_0) \leq f(\bar{x}).$$

That is

$$f_i(x_0) \leqslant f_i(\bar{x}), i = 1, \cdots, k$$

and there exists $i_0 \in K$ such that

$$f_{i_0}(x_0) < f_{i_0}(\bar{x}).$$

Note that the strictly monotonicity of G_{f_i}, $i = 1, \cdots, k$, we have

$$G_{f_i}(f_i(x_0)) \leqslant G_{f_i}(f_i(\bar{x}))$$

and

$$G_{f_i}(f_i(x_0)) < G_{f_i}(f_i(\bar{x})).$$

This contradict to the assumption that $\bar{x}$ be an efficient solution for (G-CVP).

The proof of "only if" part is similar to "if " part, we omitted it.

Example 2.7. We now consider the following multiobjective programming problem

$$\min f(x) := (f_1(x), f_2(x)) = \left(e^{x^2-4x}, \arctan x\right)$$
$$g(x) = \ln(x^2 - x + 1) \leq 0, x \in \mathbb{R}.$$

Let $G_{f_1}(f_1(t)) = \frac{1}{2}\ln t, G_{f_2}(f_2(t)) = \tan t, G_g(g(t)) = e^t$, and consider the following multiobjective programming problem:

$$\min G_f(f(x)) := (G_{f_1}(f_1(x)), G_{f_2}(f_2(x))) = \left(\frac{1}{2}\ln(e^{x^2-4x}), \tan(\arctan x)\right)$$
$$G_g(g(x)) = e^{\ln(x^2-x+1)} \leq e^0, x \in \mathbb{R}.$$

That is the following multiobjective programming problem:

$$\min G_f(f(x)) := (G_{f_1}(f_1(x)), G_{f_2}(f_2(x))) = \left(\frac{1}{2}(x^2 - 4x), x\right)$$

$$G_g(g(x)) = x^2 - x + 1 \leq 1, x \in [0, 1].$$

Note that $G_{f_1}(f_1(x)) = \frac{1}{2}(x^2 - 4x)$ is decreasing and $G_{f_2}(f_2(x)) = x$ is increasing on the interval $[0, 1]$, we can say $x = 0$ and $x = 1$ are efficient solution for G-CVP. Hence, by Theorem 2.6, $x = 0$ and $x = 1$ are efficient solution for CVP.

Definition 2.8. Let E be a set of all feasible solutions in the multiobjective programming problem (CVP) and $\bar{x} \in E$. The multiobjective programming problem (CVP) is said to satisfy the Kuhn-Tucker constraint qualification at $\bar{x}$ if,

$$C(E, \bar{x}) = \{d \in \mathbb{R}^n : \nabla g_j(\bar{x})d \leqq 0, j \in J(\bar{x}), \nabla h_t(\bar{x})d = 0, t \in P\}$$

where $C(E, \bar{x})$, the Bouligand tangent cone to E at $\bar{x}$, is defined as follows:

$$C(E, \bar{x}) = \left\{d \in \mathbb{R}^n : \exists x_k \in E, \lambda_k \in \mathbb{R}_+, \lim_{k\to\infty} x_k = \bar{x}, \lim_{k\to\infty} \lambda_k = 0, d = \lim_{k\to\infty} \frac{x_k - \bar{x}}{\lambda_k}\right\}$$

Proposition 2.9. *Let ϕ be a real strictly increasing and differentiable function defined on interval $(a, b) \subset \mathbb{R}$. Then*

$$\phi'(x) \geqslant 0, \forall\, x \in (a, b).$$

Denote $F_{CVP} =: \{d \in \mathbb{R}^n : \nabla g_j(\bar{x})d \leqq 0, j \in J(\bar{x}), \nabla h(\bar{x})d = 0\}$ and $F_{G-CVP} =: \{d \in \mathbb{R}^n : G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})d \leqq 0, j \in J(\bar{x}), G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})d = 0, t \in P\}$.

Theorem 2.10. *Let $G_{g_j}(j \in M)$ be strictly increasing function defined on $I_{g_j}(X)$ and $G_{h_t}(t \in P)$ be strictly increasing function defined on $I_{h_t}(X)$. Then $F_{CVP} \subset F_{G-CVP}$. Further, we assume that*

$$G'_{g_j}(g_j(\bar{x})) > 0, j \in J,$$
$$G'_{h_t}(h_t(\bar{x})) > 0, t \in P.$$

Then $F_{CVP} \supset F_{G-CVP}$.

Proof Since $G_{g_i}(j \in J)$, $G_{h_t}(t \in T)$ are strictly increasing functions, by Proposition 2.9, we have

$$G'_{g_j}(u) \geqslant 0, u \in I_{g_j}(X), j \in M$$
$$G'_{h_t}(u) \geqslant 0, u \in I_{h_t}(X), t \in P$$

Therefore, for $d \in F_{CVP}$, we have

$$G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})d \leqq 0, j \in M$$
$$G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})d = 0, t \in P$$

That is $d \in F_{G-CVP}$. On the other hand, if $d \in F_{G-CVP}$, then $d \in F_{CVP}$ by assumption and the proof is complete.

Theorem 2.11. *Let $G_{g_j}(j \in M)$ be strictly increasing function defined on $I_{g_j}(X)$ and $G_{h_t}(t \in P)$ be strictly increasing function defined on $I_{h_t}(X)$. Further, let $\bar{x} \in E$, $G'_{g_j}(g_j(\bar{x})) > 0, j \in J(\bar{x})$, and $G'_{h_t}(h_t(\bar{x})) > 0, t \in P$. Then the multiobjective programming problem (CVP) satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$ if only if the multiobjective programming problem (G-CVP) satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$.*

Proof From Definition 2.8 and Theorem 2.10, we get the desired result.

3. OPTIMALITY CONDITIONS IN MULTIOBJECTIVE PROGRAMMING

In [3], Antczak introduced the so-called G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable mathematical programming problem. In a natural way, he extended the so-called G-Karush-Kuhn-Tucker necessary optimality conditions to the vectorial case for differentiable multiobjective programming problems. In this section, we firstly prove G-Karush-Kuhn-Tucker necessary optimality which is Theorem 18 in [4] with a different technique. Moreover, we present a different G-Kuhn-Tucker necessary optimality conditions for differentiable multiobjective programming problems through a scalar assisted programming problem.

Theorem 3.1 (G-Karush-Kuhn-Tucker necessary optimality conditions)**.** *Let $G_{f_i}(i \in K)$ be strictly increasing function defined on $I_{f_i}(X)$, $G_{g_j}(j \in M)$ be strictly increasing function defined on $I_{g_j}(X)$ and $G_{h_t}(t \in P)$ be strictly increasing function defined on $I_{h_t}(X)$. Let $\bar{x}$ be a weakly efficient solution (an efficient solution) for (CVP). Moreover, we assume that the multiobjective programming problem (CVP) satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$. Then, there exist $\bar{\lambda} \in \mathbb{R}^n$, $\bar{\xi} \in \mathbb{R}^m$ and $\bar{\mu} \in \mathbb{R}^p$ such that*

$$\sum_{i=1}^{k} \bar{\lambda}_i G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\xi}_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x}) + \sum_{t=1}^{p} \bar{\mu}_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x}) = 0 \tag{3.1}$$

$$\bar{\xi}_j \left(G_{g_j}(g_j(x)) - G_{g_j}(g_j(\bar{x}))\right) \leqslant 0,\ j \in M, \forall x \in E \tag{3.2}$$

$$\bar{\lambda} \geq 0, \bar{\xi} \geqq 0 \tag{3.3}$$

Proof From Theorem 2.10, we have $F_{CVP} \subset F_{G-CVP}$. That is

$$\begin{cases} G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})d \leqq 0, & j \in J(\bar{x}) \\ G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})d = 0, & t \in P \end{cases}$$

for all $d \in C(E, \bar{x})$. Since $\bar{x}$ is a weakly efficient solution (an efficient solution) for (CVP), then, by Theorem 2.6, $\bar{x}$ is a weakly efficient solution (an efficient solution) for (G-CVP). Note that X be a nonempty open set, therefore

$$\nabla(G_{f_i}(f_i))(\bar{x})d = G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x})d \geq 0, \forall d \in \mathbb{R}^n, i \in K,$$

Therefore, the following system

$$\begin{cases} G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) < 0, & i \in K \\ G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})d \leqq 0, & j \in J(\bar{x}) \\ G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})d = 0, & t \in P \end{cases} \tag{3.4}$$

is inconsistent on $\mathbb{R}^n$.

Since the system (3.4) has no solution, then, from Motzkin's theorem [8], there exist $\lambda \in \mathbb{R}^k$, $\xi \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^p$ such that

$$\sum_{i \in K} \lambda_i G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \xi_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x}) + \sum_{t \in P} \mu_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x}) = 0$$

$$\xi_j \left(G_{g_j}(g_j(x)) - G_{g_j}(g_j(\bar{x}))\right) \leqslant 0,\ j \in J(\bar{x}), \forall x \in E$$

$$\lambda \geq 0, \xi \geqq 0.$$

Denote by $\bar{\lambda}_i = \lambda_i, i \in K$, $\bar{\xi}_j = \xi_j, j \in J(\bar{x})$, $\bar{\xi}_j = 0, j \in M/J(\bar{x})$, $\bar{\mu}_t = \mu_t, t \in P$, and we get the desired result.

Remark 3.2. In proof of Theorem 3.1, the method we used here is different from the one used by Antczak in [3].

Theorem 3.3 (G-Karush-Kuhn-Tucker necessary optimality conditions). *Let $\bar{x}$ be a weakly efficient solution (an efficient solution) for (CVP), $G_{f_i}(i \in K)$ be strictly increasing function defined on $I_{f_i}(X)$, $G_{g_j}(j \in M)$ be strictly increasing function defined on $I_{g_j}(X)$ and $G_{h_t}(t \in P)$ be strictly increasing function defined on $I_{h_t}(X)$. Moreover, we assume that the multiobjective programming problem (G-CVP) satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$. Then, there exist $\bar{\lambda} \in \mathbb{R}^n$, $\bar{\xi} \in \mathbb{R}^m$ and $\bar{\mu} \in \mathbb{R}^p$ such that*

$$\sum_{i=1}^{k} \bar{\lambda}_i G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\xi}_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x}) + \sum_{t=1}^{p} \bar{\mu}_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x}) = 0 \tag{3.5}$$

$$\bar{\xi}_j \left(G_{g_j}(g_j(x)) - G_{g_j}(g_j(\bar{x}))\right) \leqslant 0, \ j \in M \tag{3.6}$$

$$\bar{\lambda} \geq 0, \bar{\xi} \geqq 0 \tag{3.7}$$

Proof Since $\bar{x}$ is a weakly efficient solution (an efficient solution) for (CVP), then, by Theorem 2.6, $\bar{x}$ is a weakly efficient solution (an efficient solution) for (G-CVP). Again, the multiobjective programming problem (G-CVP) satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$. Therefore, the system

$$\begin{cases} G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x})d < 0, & i \in K \\ G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})d \leqq 0, & j \in J, \\ G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})d = 0, & t \in P \end{cases} \tag{3.8}$$

is inconsistent. The following part is similar to Theorem 3.1

In order to present a strongly G-Karush-Kuhn-Tucker necessary optimality conditions, namely G-Kuhn-Tucker necessary optimality conditions, we need the following scalar programming problem:

$$(\varphi_G P) \quad \min \quad \varphi_G(x) = \sum_{i=1}^{k} G_{f_i}(f_i(x)) \tag{3.9}$$

$$s.t. G_f(f(x)) \leqq G_f(f(\bar{x})) \tag{3.10}$$

$$G_g(g(x)) \leqq G_g(0) \tag{3.11}$$

$$G_h(h(x)) = G_h(0) \tag{3.12}$$

$$x \in E \tag{3.13}$$

Let $\bar{E} = \{x \in E : G_f(f(x)) \leqq G_f(f(\bar{x})), G_g(g(x)) \leqq G_g(0), G_h(h(x)) = G_h(0)\}$. we say that ($\varphi_G$P)satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$ if

$$C(\bar{E}, \bar{x}) = \{d \in \mathbb{R}^n : \nabla(G_f(f))(\bar{x})d \leqq 0, \nabla(G_g(g))(\bar{x})d \leqq 0, \nabla(G_h(h))(\bar{x})d = 0\}.$$

Theorem 3.4 (G-Kuhn-Tucker necessary optimality conditions). *Let $\bar{x}$ be a weakly efficient solution (an efficient solution) for (CVP), $G_{f_i}(i \in K)$ be strictly increasing function defined on $I_{f_i}(X)$, $G_{g_j}(j \in M)$ be strictly increasing function defined on $I_{g_j}(X)$ and $G_{h_t}(t \in P)$ be strictly increasing function defined on $I_{h_t}(X)$. Moreover, we assume that the scalar programming problem (φ_GP) satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$. Then, there exist $\bar{\lambda} \in \mathbb{R}^n$, $\bar{\xi} \in \mathbb{R}^m$ and $\bar{\mu} \in \mathbb{R}^p$ such that*

$$\sum_{i=1}^{k} \bar{\lambda}_i G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\xi}_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x}) + \sum_{t=1}^{p} \bar{\mu}_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x}) = 0 \tag{3.14}$$

$$\bar{\xi}_j \left(G_{g_j}(g_j(\bar{x})) - G_{g_j}(0)\right) \leqslant 0, \ j \in M \tag{3.15}$$

$$\bar{\lambda} > 0, \bar{\xi} \geqq 0 \tag{3.16}$$

Proof Since $\bar{x}$ is a weakly efficient solution (an efficient solution) for (CVP), then, by Theorem 2.6, $\bar{x}$ is a weakly efficient solution (an efficient solution) for (G-CVP). Further, we can prove that $\bar{x}$ is a optimal solution for (φ_G P). Therefore, from the Theorem 4.14 in [11],

$$\nabla(\varphi_G)(\bar{x})d \geq 0, \forall d \in C(\bar{E}, \bar{x}).$$

Again, the scalar programming problem (φ_GP) satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$. Therefore, the following system:

$$\begin{cases} \nabla(\varphi_G)(\bar{x})d < 0, \\ \nabla(G_f(f))(\bar{x})d \leqq 0, \\ \nabla(G_g(g))(\bar{x})d \leqq 0, \\ \nabla(G_h(h))(\bar{x})d = 0 \end{cases} \tag{3.17}$$

is inconsistent. Note that

$$\nabla(\varphi_G)(\bar{x}) = \sum_{i=1}^{k} G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}),$$

then the following system

$$\begin{cases} \left(\sum_{i=1}^{k} G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x})\right) d < 0 & \\ G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x})d \leqq 0, & i \in K \\ G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})d \leqq 0, & j \in J, \\ G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})d = 0, & t \in P \end{cases} \tag{3.18}$$

is inconsistent. Hence, from Farkas's theorem, there exist $\lambda \in \mathbb{R}^k$, $\xi \in \mathbb{R}^{|J|}$ and $\mu \in \mathbb{R}^p$ such that

$$\sum_{i=1}^{k} G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) + \sum_{i \in K} \lambda_i G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \xi_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x}) + \sum_{t \in P} \mu_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x}) = 0$$

$$\sum_{i=1}^{k} \lambda_i \left(G_{f_i}(f_i(\bar{x})) - G_{f_i}(f_i(\bar{x}))\right) + \sum_{j \in J(\bar{x})} \xi_j \left(G_{g_j}(g_j(\bar{x})) - (G_{g_j}(0))\right) \leqslant 0,$$

$$\lambda \geqq 0, \xi \geqq 0.$$

That is

$$\sum_{i=1}^{k}(1+\lambda_i) G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \xi_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x}) \tag{3.19}$$
$$+ \sum_{t \in P} \mu_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x}) = 0$$
$$\xi_j \left(G_{g_j}(g_j(\bar{x})) - (G_{g_j}(0))\right) \leqslant 0, j \in J(\bar{x}), \lambda \geqq 0, \xi \geqq 0.$$

Denote by $\bar{\lambda}_i = 1 + \lambda_i$, $\bar{\xi}_j = \xi_j (j \in J(\bar{x}))$, $\bar{\xi}_j = 0 (j \in M/J(\bar{x}))$ and $\bar{\mu}_t = \mu_t$, we get the desired result.

Now, we establish the sufficient optimality conditions for multiobjective programming problems of such a type. In the following theorem, we assume that the functions constituting the considered vector optimization problem (CVP) belong to the introduced class of nonconvex functions. Then we prove that a feasible point $\bar{x}$, at which the G-Karush-Kuhn-Tucker necessary optimality conditions are fulfilled, is a weakly efficient solution.

Theorem 3.5 (G-Karush-Kuhn-Tucker sufficent optimality conditions)**.** *Let $\bar{x}$ be a feasible point for (CVP), $G_{f_i}(i \in K)$ be a differentiable real-valued strictly increasing function defined on $I_{f_i}(X)$, $G_{g_j}(j \in M)$ be a differentiable real-valued strictly increasing function defined on $I_{g_j}(X)$, and $G_{h_t}(t \in P)$ be a differentiable real-valued strictly increasing function defined on $I_{h_t}(X)$, such that G-Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) are satisfied at $\bar{x}$. Further, assume that f is vector (G_f, α)-invex with respect to η at $\bar{x}$ on X, g is vector (G_g, β)-invex with respect to the same function η at $\bar{x}$ on X, h_t $(t \in P^+(\bar{x}))$ is (G_{h_t}, γ_t)-invex with respect to the same function η at $\bar{x}$ on X, and h_t $(t \in P^-(\bar{x}))$ is (G_{h_t}, γ_t)-incave with respect to the same function η at $\bar{x}$ on X, where $P^+(\bar{x}) = \{t \in P : \bar{\mu}_t > 0\}$ and $P^-(\bar{x}) = \{t \in P : \bar{\mu}_t < 0\}$. Then $\bar{x}$ is a weakly efficient solution for (CVP).*

Proof Suppose, contrary to the result, that $\bar{x}$ is not a weakly efficient solution for (CVP). By Theorem 2.6, $\bar{x}$ is not a weakly efficient solution for (G-CVP). Hence, there exists $x_0 \in X$ such that

$$G_{f_i}(f_i(x_0)) < G_{f_i}(f_i(\bar{x})), i \in K \tag{3.20}$$

By the generalized invexity assumption of f, g and h, we have

$$G_{f_i}(f_i(x_0)) - G_{f_i}(f_i(\bar{x})) \geq \alpha_i(x_0, \bar{x})G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x})\eta(x_0, \bar{x}), i \in K \tag{3.21}$$

$$G_{g_j}(g_j(x_0)) - G_{g_j}(g_j(\bar{x})) \geq \beta_j(x_0, \bar{x})G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})\eta(x_0, \bar{x}), j \in M \tag{3.22}$$

$$G_{h_t}(h_t(x_0)) - G_{h_t}(h_t(\bar{x})) \geq \gamma_t(x_0, \bar{x})G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})\eta(x_0, \bar{x}), t \in P^+(\bar{x}) \tag{3.23}$$

$$G_{h_t}(h_t(x_0)) - G_{h_t}(h_t(\bar{x})) \leq \gamma_t(x_0, \bar{x})G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})\eta(x_0, \bar{x}), t \in P^-(\bar{x}) \tag{3.24}$$

Multiplying (3.22),(3.23) and (3.24) by $\xi_j(j \in M)$, $\mu_t(t \in P^+)$ and $\mu_t(t \in P^-)$, we get

$$\bar{\xi}_j(G_{g_j}(g_j(x_0)) - G_{g_j}(g_j(\bar{x}))) \geq \bar{\xi}_j\beta_j(x_0, \bar{x})G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})\eta(x_0, \bar{x}), j \in M \tag{3.25}$$

$$\bar{\mu}_t(G_{h_t}(h_t(x_0)) - G_{h_t}(h_t(\bar{x}))) \geq \bar{\mu}_t\gamma_t(x_0, \bar{x})G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})\eta(x_0, \bar{x}), t \in P^+ \tag{3.26}$$

$$\bar{\mu}_t(G_{h_t}(h_t(x_0)) - G_{h_t}(h_t(\bar{x}))) \geq \bar{\mu}_t\gamma_t(x_0, \bar{x})G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})\eta(x_0, \bar{x}), t \in P^- \tag{3.27}$$

From (3.2), (3.20), (3.21), (3.25), (3.26) and (3.27), we have

$$\begin{aligned}
&G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x})\eta(x_0, \bar{x}) < 0, i \in K \\
&\bar{\xi}_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})\eta(x_0, \bar{x}) \leq 0, j \in M \\
&\bar{\mu}_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})\eta(x_0, \bar{x} \leq 0, t \in P^+ \\
&\bar{\mu}_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})\eta(x_0, \bar{x}) \leq 0, t \in P^-
\end{aligned}$$

Note that $\lambda \geq 0$, then

$$\left(\sum_{i=1}^{k} \bar{\lambda}_i G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\xi}_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x}) + \sum_{t=1}^{p} \bar{\mu}_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})\right) \times \eta(x_0, \bar{x}) < 0$$

which contradicts the G-Karush-Kuhn-Tucker necessary optimality condition (3.1). Hence, $\bar{x}$ is a weakly efficient solution for (CVP), and the proof is complete.

Theorem 3.6 (G-Karush-Kuhn-Tucker sufficient optimality conditions). *Let $\bar{x}$ be a feasible point for (CVP), $G_{f_i}(i \in K)$ be a differentiable real-valued strictly increasing function defined on $I_{f_i}(X)$, $G_{g_j}(j \in M)$ be a differentiable real-valued strictly increasing function defined on $I_{g_j}(X)$, and $G_{h_t}(t \in P)$ be a differentiable real-valued strictly increasing function defined on $I_{h_t}(X)$, such that G-Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) are satisfied at $\bar{x}$. Further, assume that f is vector strictly (G_f, α)-invex with respect to η at $\bar{x}$ on X, g is vector (G_g, β)-invex with respect to the same function η at $\bar{x}$ on X, h_t $(t \in P^+(\bar{x}))$ is (G_{h_t}, γ_t)-invex with respect to the same function η at $\bar{x}$ on X, and h_t $(t \in P^-(\bar{x}))$ is (G_{h_t}, γ_t)-incave with respect to the same function η at $\bar{x}$ on X, where $P^+(\bar{x}) = \{t \in P : \bar{\mu}_t > 0\}$ and $P^-(\bar{x}) = \{t \in P : \bar{\mu}_t < 0\}$. Then $\bar{x}$ is an efficient solution for (CVP).*

Proof Proof for efficient optimality is similar to the proof of Theorem 3.5.

4. CONCLUSION

This paper represents a new type of generalized invexity, namely (G_f, α)-invexity. This new invexity unified the G-invexity and α-invexity presented in [4] and [9], respectively. We have constructed two auxiliary mathematical programmings (G-CVP) and (φ_G P), and have discussed the relations between programming (G-CVP) and (CVP). We have illustrated the relation result proved by a suitable example of the multiobjective programming problem (CVP) involving (G, α)-invex functions. With assisted mathematical programming (G-CVP), we have proved G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable multiobjective programming problems with both inequality and equality constraints, through a easier method than presented in [4]. Furthermore, we have proved G-Kuhn-Tucker necessary optimality conditions for (CVP) with the help of auxiliary mathematical programming (φ_G P). As mentioned in [4], our statement of the so-called G-Kuhn-Tucker necessary optimality conditions established in this paper is more general than the classical Kuhn- Tucker necessary optimality conditions found in the literature. Also, we have proved the sufficiency of the introduced G-Karush-Kuhn-Tucker (G-Kuhn-Tucker) necessary optimality conditions for multiobjective programming problems involving (G, α)-invexity. More exactly, this result has been proved for such multiobjective programming problems in which the objective functions, the inequality constraints and the equality constraints (for which associated Lagrange multipliers are positive) are (G, α)-invex with respect to the same function η and the equality constraints (for which associated Lagrange multipliers are negative) are (G, α)-incave with respect to the same function η, but not necessarily with respect to the same function G. Hence, we can establish dual result, which is similar to ones presented in [5], for the programming problem (VCP).

REFERENCES

1. T. Antczak, Lipschitz r-invex functions and nonsmooth programming, Numer. Funct. Anal. Optim., **23**, 265-283 (2002).
2. T. Antczak, Generalized (p, r)-invexity in mathematical programming, Numer. Funct. Anal. Optim., **24**, 437-453(2003).
3. T.Antczak,New optimality conditions and duality results of G-type in differentiable mathematical programming, Nonlinear Anal., 66, 1617-1632 (2007).
4. T.Antczak, On G-invex multiobjective programming. Part I. Optimality, J. Global Optim., 43(1), 97-109 (2009).
5. T.Antczak, On G-invex multiobjective programming. Part II. Duality, J. Global Optim., 43(1), 111-140 (2009).
6. A.Ben-Israel and B.Mond,What is invexity? J. Aust. Math. Soc. Series B, **28**, 1-9 (1986).

7. M.A.Hanson, On sufficiency of the Kuhn-Tucker conditions,J. Math. Anal. Appl., **80**, 545-550(1981).
8. O.L. Mangasarian, Nonlinear programming, McGraw-Hill, New York (1969).
9. M.Noor,On generalized preinvex functions and monotonicities, JIPAM. J.Inequal. Pure Appl. Math., 5(4), 1-9(2004).
10. D.H. Yuan, X.L. Liu, A.Chinchuluun and P.M.Pardalos, Nondifferentiable minimax fractional programming problems with (C, α, ρ, d)-convexity, J. Optim. Theory Appl., **129(1)**, 185-199(2006).
11. Johannes Jahn, Introduction to the theory of nonlinear optimization, Springer, Heidelberg (2007).