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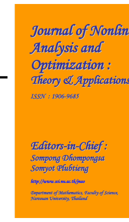

AN APPROACH FOR SIMULTANEOUSLY DETERMINING THE OPTIMAL TRAJECTORY AND CONTROL OF A HEATING SYSTEM◇

HAMID REZA SAHEBI

Department of Mathematics, Islamic Azad University,Ashtian Branch,Ashtian, Iran.

ABSTRACT. In the recent decade, a considerable number of optimal control problems have been solved successfully based on the properties of the measures. Even the method, has many useful benefits, in general, it is not able to determine the optimal trajectory and control at the same time; moreover, it rarely uses the advantages of the classical solutions of the involved systems. In this article, for a heating control system, we are going to present a new solution path. First, by considering all necessary conditions, the problem is represented in a variational format in which the trajectory is shown by a trigonometric series with the unknown coefficients. Then the problem is converted into a new one that the unknowns are the mentioned coefficients and a positive Radon measure. It is proved that the optimal solution is exited and it is also explained how the optimal pair would be identified from the results deduced by a finite linear programming problem. A numerical examples is also given.

KEYWORDS : Simultaneously Determining; Optimal Trajectory; Control of a Heating System.
AMS Subject Classification: 49QJ20, 49J45,49M25, 76D33.

1. INTRODUCTION

According to an idea of L. C. Young, by transferring the problem in to a theoretical measure optimization, in 1986 Rubio introduced a powerful method for solving optimal control problems ([11]). The important properties of the method (globality, automatic existence theorem a linear treatment even for extremely nonlinear problems, ...) caused it to be applied for the wide variety of problems. Even the method has been used frequency for solving several kinds of problems, like [3], [1], [4] and [7], but at least two important points were not considered in applying the method yet. Generally the method can not be able to produce the acceptable optimal trajectory and control directly at the same time; and moreover, the classical format of the system solution, usually is not taken into account. Therefore, there is no any possibility to use this important fact and their related literatures in the analysis of the system.

Email address : sahebi@mail.aiau.ac.ir (H.R.Sahebi).
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In this article, we try to bring attention these two facts; for these perposes, an optimal control problem governed by a one-dimensional wave equation system (a heating system) with initial and boundary conditions and an integral criterion is considered as a sample. Regarding a general format of the classical solution, the problem is presented in a variational format and then by a doing deformation it is converted into a measure theoretical one with some positive coefficient. Next, extending the underlying space, using the density properties and applying some discretization scheme cause to approximate the optimal pair as a result of a finite linear programming. The approach would be improved if the number of constraints and nods are exceeded . In this manner, the optimal trajectory and control is determined at the same time.

2. THE CONTROL SYSTEM

For all $t \in [0, T] \subset R$ the deflection of the shell at an arbitrary point x in time t, is denoted by $u(t, x)$ which satisfies in the following equation (see [7] and [13]):

$$u_t = cu_{xx} \ (1)$$

where c is a constant that it dependents on physical structure of the heating. Since the heating at its boundary there is no any heating at these points and hence we have the following boundary conditions:

$$u(t, 0) = u(t, L), \ \forall \ 0 \leq x \leq L. \ (2)$$

If the initial deflection are denoted by $f(x)$, then the initial conditions of the system are defined as:

$$f(x) = u(0, x); \ (3)$$

According to [8], $u(t, x)$ belongs to the class of homogeneous Cauchy problems. Thus it can have a unique bounded classic solution on $D = [0, T] \times [0, L]$, if $f(x)$, and the different orders of their partial derivatives are continuous. Moreover, as mentioned in [7], the one-dimensional equation problem have the following Fourier series as the solution:

$$u(t, x) = \sum_{n=1}^{\infty} A_n e^{-k(\frac{n^2\pi^2}{L^2})t} sin\frac{n\pi}{L}x \ (4)$$

Where

$A_n = \frac{2}{L}\int_0^L f(x)sin\frac{n\pi}{L}x \ dx$ for $n = 1, 2, \ldots$

Convergence of the above series (to a bounded solution of the problem) indicates that one can approximate the solution by the finite number terms of the series.
Controlling the one-dimensional of heat system, needs to inter some power to the system somehow. This fact can be done by inserting a shock on a specified place of the shell. Without lose of the generality, suppose that the place for inserting the shock be $\frac{L}{2}$. Now, let $V \subseteq R$ be a bounded set, and $\vartheta = \vartheta(t) : [0, t] \rightarrow V$ be a Lebesgue-measurable control function. Moreover suppose
$f_0 = f_0(t, x, \vartheta(t)) : D \times V \rightarrow R$ be a continues function.

The aim is to find the optimal pair of the trajectory and control functions $u(t, x)$ and $\vartheta = \vartheta(t)$ simultaneously, as an optimal solution for the following control problem:

$Min: \ I(P) \equiv \int_D f_0(t,x,\vartheta)dA$

$S.to: \ u_t = cu_{xx}; \ (5-1)$

$u(t,0) = u(t,L) = 0; \ (5-2)$

$f(x) = u(0,x); \ (5-3) \ (5)$

$u_t|_{\frac{L}{2}} = \vartheta(t). \ (5-4)$

We remind that the objective functional $\int_D f_0(t,x,\vartheta)dA$ can explain error of the System or so on.

Definition: As a classical form, a pair $P \equiv (u,\vartheta)$ is called admissible if conditions (5-2) to (5-4) are satisfied, and u be a bounded solution of (5-1). The set of all admissible pairs is denoted by P .

Therefore, we wish to find the admissible minimizer pair for the functional $I(P)$ over P. It is necessary to indicate that the controllability and the observability of the above system were discussed in many references such as [3]. Thus, we can suppose that P is nonempty. In the next, we will try to find the solution of (5) according to the trigonometrical series and use of the embedding method, as mentioned in section one. For reaching to our purposes, we need to present the problem in a new formulation.

3. NEW REPRESENTATION OF THE PROBLEM

For a fixed N, the optimal trajectory of (5) can be approximated by the first N terms of a trigonometric series; i.e.:

$u(t,x) = \sum_{n=1}^{N} A_n e^{-k(\frac{n^2\pi^2}{L^2})t} sin\frac{n\pi}{L}x \ (6)$

where A_n for $n = 1, 2, ..., N$ are unknown real coefficients that must be determined under the conditions (5-2) to (5-4). Since this coefficients are unknown, the amount of the eliminated part of the solution in (4) (the tail of the series), can be considered in the calculated amount for unknowns. We define:

$$\overline{u}^n(t,x) = A_n e^{-k(\frac{n^2\pi^2}{L^2})t} sin\frac{n\pi}{L}x \ (7)$$

one can easily show that $\overline{u}^n_{xxxt}(t,x) = -\frac{n^2\pi^2}{L^2}\overline{u}^n_{xt}(t,x)$ and then we have:

$$u_{xxxt}(t,x,y) = \sum_{n=1}^{N} \overline{u}^n_{xxxt}(t,x) = \sum_{n=1}^{N} -\frac{n^2\pi^2}{L^2}\overline{u}^n_{xt}(t,x);$$

then, integrating over $[0,T] \times [0,\frac{L}{2}]$ gives

$$\int_0^T \int_0^{\frac{L}{2}} u_{xxxt}dxdt = \sum_{n=1}^{N} -\frac{n^2\pi^2}{L^2} \int_0^T \int_0^{\frac{L}{2}} \overline{u}^n_{xt}(t,x)dxdt;$$

By regarding the continuity of $\overline{u}^n(t,x)$and its partial derivatives, one can change the order of the integration. Then, by some simple calculations, the constraint (5-4) can be appeared as the following new format:

$$\int_0^T \vartheta(t)dt = \sum_{n=1}^{N} A_n sin\frac{n\pi}{2}(e^{-k(\frac{n^2\pi^2}{L^2})T} - 1). \text{ (8)}$$

Now, by considering the equations (6) and (8), the problem (5) can be represented as the new following exhibition:

$Min: \ I(P) = \int_D f_0(t, x, \vartheta(t))dA$

$S.to: \ f(x) = \sum_{n=1}^{N} A_n sin\frac{n\pi x}{L}$ (9)

$\int_0^T \vartheta(t)dt = \sum_{n=1}^{N} A_n sin\frac{n\pi}{2}(e^{-k(\frac{n^2\pi^2}{L^2})T} - 1).$

Let $x_0 = 0, x_1, x_2, ..., x_m = L$ be belong to a dense subsequence of $[0, L]$. If $m \to \infty$ then obviously the solution of the following problem converges to the solution of (10). Thus, for a suitable numbers m, the solution of the problem (9) can be approximated by the solution of the following one:

$Min: \ I(P) = \int_0^T [\sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} f_0(\vartheta, x_i, t)dx]dt = \int_0^T F_0(t, \vartheta)dt$

$S.to: \ f(x) = \sum_{n=1}^{N} A_n sin\frac{n\pi x}{L}$ (10)

$\int_0^T \vartheta(t)dt = \sum_{n=1}^{N} A_n sin\frac{n\pi}{2}(e^{-k(\frac{n^2\pi^2}{L^2})T} - 1).$

4. METAMORPHOSIS

To show the existence of the optimal solution of (10) and introducing a simple linear treatment for obtaining it, we follow [2] and [10] by applying some new ideas. Hence, we do the metamorphosis step in this section, to deform the problem and define it in a new space in which it has many advantages.

Let $\Omega = [0, T] \times V$, then for each $(u, \vartheta) \in P$, $\Lambda_\vartheta : C(\Omega) \to R$ that $\Lambda_\vartheta(h) = \int_0^T h(t, \vartheta)dt$ be a positive continuous linear functional. Based on the Riesz Representation Theorem ([13]), there exists a positive Radon measure $\mu_\vartheta \in M^+(\Omega)$ (the space of all positive Radon measures on Ω) so that for all $h \in C(\Omega)$, $\mu_\vartheta(h) = \int_\Omega h \ d\mu = \Lambda_\vartheta(h)$. Therefore, problem (9) is changed into a new one in which its unknowns are the coefficients A_n ($n = 1, 2, ..., N$) and a positive Radon measure, say μ , produced by the Riesz Representation Theorem. To be sure that we are able to elastraite the global solution, like [10], we enlarge the underlying space and seek on a subset of $M^+(\Omega)$ which is defined by the last equations of (9). This means that instead of searching for the optimal measure, say μ^* , between the introduced measures from the Risez Representation Theorem, we seek in the set of all positive Radon measures in which they just satisfy in the conditions of (10); hence, the induced measures from the Risez theorem are belonged in this set and therefore our minimization is global. Thus we try to solve the following problem:

$Min: \ \mu(F_0)$

$S.to: \ f(x_i) = \sum_{n=1}^{N} A_n sin\frac{n\pi x_i}{L};$

$\mu(\vartheta) = \sum_{n=1}^{N} A_n sin\frac{n\pi}{2}(e^{-k(\frac{n^2\pi^2}{L^2})T} - 1); \ (11)$

$i = 1, 2, \ldots, m;$

$\mu(\xi) = a_\xi, \ \forall\xi \in C^1(\Omega).$

where the unknown measure μ belongs to $M^+(\Omega)$, $C^1(\Omega)$ is a subset of functions in $C(\Omega)$ that depends only on variable t and a_ξ is the Lebesgue integral of ξ over $[0, T]$; indeed the last set of equations is added to guarantee this property of an admissible measure that its projection on the real line is the Lebesgue measure (see for instance [11] and [2]).
Now, suppose that A_n's are obtained by solving the following linear equations:

$$f(x_i) = \sum_{n=1}^{N} A_n sin\frac{n\pi x_i}{L}; \ i = 1, 2, \ldots, m$$

Then by substituting the obtained coefficients in the third equation of (11), the problem is converted into one in which the unknown is just the measure $\mu \in M^+(\Omega)$ which satisfied in the last two conditions of (11). Thus, if Q be the space of all measure in $M^+(\Omega)$ which satisfied the conditions of (11), as Rubio shown in [11], Q is compact in the sense of weak* topology; moreover $\mu \to \mu(F_0)$ is a continuous function. Since each continuous function has an infimum on a compact space there exists an optimal measure, say μ^*, which minimizes the objective function of (11) and together with the obtained unknown are satisfied in the conditions of (11).Thus , we have the following proposition.

Proposition 1: Problem (11) has the optimal solution.

Proof. see [9]. □

By regarding the result of Rosenblooms works in [10], the optimal measure has the form

$$\mu^* = \sum_{j=1}^{M} \alpha_j \delta(z_j) \ (12)$$

where $\delta(z_j)$ is an atomic measure with the support of the singleton set $\{z_j\}$, α_j is a nonnegative real coefficient, and z_j is a point belongs to Ω. Using (12) in (11), changes the problem into a nonlinear one in which its unknowns are the coefficients A_n, α_j, and the supporting points z_j for $n = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$. We know that, by doing a discretization on Ω with nodes $z_j = (t_j, \vartheta_j), \ j = 1, 2, \ldots, M$ in a dense subset of $\omega \subseteq \Omega$, the supporting points can be determined; hence the problem can be converted into a linear one. But, regarding the last set of equations in (11), the number of constraints are still infinite. It would be more convenient if somehow we could change the problem into a finite linear programming one. Then in the next step of approximation, by choosing a dense countable subset of $C^1(\Omega)$ and then selecting a finite number of its elements as ξ_h for $h = 1, 2, ..., K$, the total number of the constraints of the problem would be finite. Therefore, the solution of (11) can be approximated by the following linear programming problem with variables $\alpha_j, \ j = 1, 2, \ldots, M$, and A_n^+, A_n^- that $A_n = A_n^+ - A_n^-$.

$Min: \sum_{j=1}^{M} \alpha_j F_0(t_j, \vartheta_j)$

$S.to: f(x_i) = \sum_{n=1}^{N} (A_n^+ - A_n^-) sin\frac{n\pi x_i}{L};$ (11)

$\sum_{j=1}^{M} \alpha_j \vartheta_j = \sum_{n=1}^{M} (A_n^+ - A_n^- sin\frac{n\pi}{2}(e^{-k(\frac{n^2\pi^2}{L^2})T} - 1);$

$\sum_{j=1}^{M} \alpha_j \xi_h(t_j, \vartheta_j) = a_h, \ h = 1, 2, ..., K.$

$\alpha_j \geq 0, \ j = 1, 2, \ldots, M;, A_n^+, A_n^- \geq 0, \ n = 1, 2, ..., N;$

The density properties of the applied sets, indicate that if N, m, n, h chosen bigger and bigger, the optimal solution of (13) convergence into the solution of (10), or more precisely (5)(see [10]). Therefore, the solution of (5) can be approximated from the results of the finite linear programming problem (13).
To set up (13), as mentioned in [10] and some other literatures (like [4]), for $h = 1, 2, \ldots, K$ we choose $0 = t_0 \leq t_1 \leq \ldots t_K = T$ and $D_h = [t_{h-1}, t_h)$ for $h = 1, 2, ..., K$ and $D_h = [t_{K-1}, t_K]$; hence $\bigcup_{h=1}^{K} D_h = [0, T]$. Now we define the function ξ_h as follow:

$$\xi_k(t_j, \vartheta_j) = \begin{cases} 1 \ t_j \in D_h \\ 0 \ otherwise \end{cases}$$

although these class of functions are not continuous but, when $K \to \infty$ every functions in $C^1(\Omega)$ can be approximated by a finite linear combination of these functions (see [5]). In this manner, for an arbitrary function ξ_h , we have $a_h = \int_0^T \xi_h \ dt = t_h - t_{h-1}$. Now by solving the linear programming problem (13), one can obtain the optimal coefficients $\alpha_j^*, \ A_n^*$ at the same time. Then, according to (6) and the explained method in [10], simultaneously the optimal trajectory and control functions can be determined, which is one of the main aim of this paper.

5. A NUMERICAL EXAMPLE

Based on the explained new approach, we incline to find the optimal pair of the trajectory and control for vibrating system in (5) defined by:

$$u_t = u_{xx}$$
$$u(t, 0) = u(t, L) = 0;$$
$$u(0, x) = x^2;$$

with the performance criterion defined by $F_0(t, \vartheta) = (\vartheta - t^2)^2$; indeed, here was supposed that $c = 1, \ t \in [0, 0.01], \ D = [0, 1] \times [0, 0.01], U = [-0.4, 0.902], \ f(x) = x^2$. Also we choose $N = 20, \ m = 14$. Then for discretization on Ω we chose 30 value for ϑ_l in U as $\vartheta_l = -0.4 + \frac{1.302l}{30}$, 30 value for t_j in $[0, 1]$ as $t_j = \frac{j}{30}$ and 14 value for x_i in $[0, 1]$ as $x_i = \frac{i}{m+1}$, for $l, j = 1, 2, ..., 30$ as also $i = 1, 2, ..., 14$. Therefore, for solving the problem, a similar linear programming problem like (13) with 980 variables was established as follow:

$Min: \sum_{j=1}^{900} \alpha_j (\vartheta - t^2)^2$

$s.to: \sum_{n=1}^{20} (A_n^+ - A_n^-) sin\frac{n\pi x_i}{2} = x_n^2 \ n = 1, 2, ..., 14;$

$\sum_{j=1}^{900} \alpha_j \vartheta_j - \sum_{n=1}^{20} (A_n^+ - A_n^- sin\frac{n\pi}{2}(e^{-k(\frac{n^2\pi^2}{L^2})T} - 1) = 0;$

$$\alpha_1+\alpha_2+\ldots+\alpha_{60}=0.1;\ \alpha_{61}+\alpha_{62}+\ldots+\alpha_{150}=0.1;\ \alpha_{631}+\alpha_{632}+\ldots+\alpha_{690}=0.1;$$
$$\alpha_{151}+\alpha_{152}+\ldots+\alpha_{270}=0.1;\ \alpha_{271}+\alpha_{272}+\ldots+\alpha_{330}=0.1;\ \alpha_{691}+\alpha_{692}+\ldots+\alpha_{780}=0.1;$$
$$\alpha_{331}+\alpha_{332}+\ldots+\alpha_{420}=0.1;\ \alpha_{421}+\alpha_{422}+\ldots+\alpha_{540}=0.1;\ \alpha_{781}+\alpha_{782}+\ldots+\alpha_{900}=0.1;$$
$$\alpha_{541}+\alpha_{542}+\ldots+\alpha_{630}=0.1;$$
$$A_n^+, A_n^-,\ \alpha_r\geq 0,\ r=1,\ldots,900,\ n=1,2,\ldots,20;$$

Then we applied the subroutine **DLPRS** from **IMSL** library of **Compaq Visual Fortran** to solve the above linear programming problem by Revised Simplex Method. The optimal value of the objective function was obtained as 0.0000022463. The optimal value of the variables were as follows:

$A_1^*=0.37628;\ A_2^*=-0.31364$
$A_3^*=0.19562;\ A_4^*=-0.14974$
$A_5^*=0.11342;\ A_6^*=-0.091759$
$A_7^*=0.073306;\ A_8^*=0.060027$
$A_8^*=0.0481107;\ A_{10}^*=0$
$A_{11}^*=0.28193;\ A_{12}^*=0$
$A_{13}^*=0;\ A_{14}^*=0$
$A_{15}^*=0;\ A_{16}^*=0.0070069$
$A_{17}^*=0;\ A_{18}^*=0.021661$
$A_{19}^*=0.25242\ A_{20}^*=0.03849;$

$\alpha_1^*=\alpha_2^*=...=\alpha_{10}^*=0.1$

From the obtained optimal values, the nearly optimal piecewise-constant control was calculated as the explained manner in [11]. Also, by regarding (6) and the above obtained optimal coefficient, the trajectory function $u^*(t,x)$ was determined by :

$$u^*(t,x)=\sum_{n=1}^{20}A_n^*e^{-k(\frac{n^2\pi^2}{L^2})t}sin\frac{n\pi}{L}x$$

The obtained nearly optimal control and trajectory functions are plotted in figures 1 and 2 respectively (since the optimal trajectory is a variable function, it was plotted for some especifeid times).

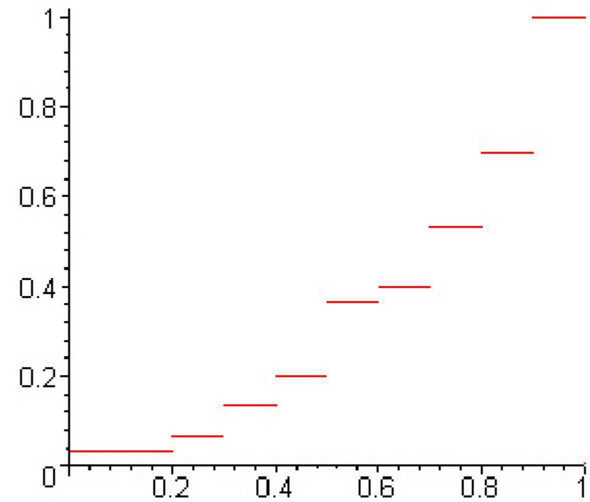

Figure 1:The Optimal Control

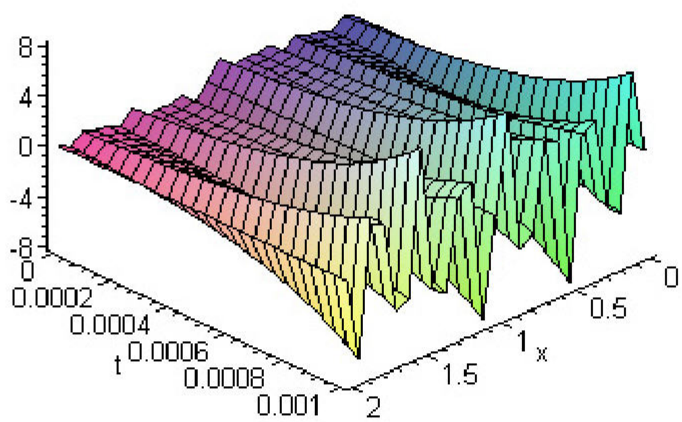

Figure 2: The Optimal Trajectory

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