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STRONG CONVERGENCE OF THREE-STEP ITERATIVE PROCESS WITH ERRORS FOR THREE MULTIVALUED MAPPINGS

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Abstract. In this paper, we introduced a three-step iterative process with errors for three multivalued mappings satisfying the condition (C) in uniformly convex Banach spaces and establish strong convergence theorems for the proposed process under some basic boundary conditions. Our results generalized recent known results in the literature.

Keywords: Fixed point; Condition (C); Three-step iteration process; Strong convergence.

AMS Subject Classification: 47H10, 47H09.

1. INTRODUCTION

The study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [9] and Nadler [10]. Since then the metric fixed point theory of multivalued mappings has been rapidly developed. The theory of multivalued mappings has applications in control theory, convex optimization, differential equations and economics. Theory of multivalued nonexpansive mappings is harder than the corresponding theory of singlevalued nonexpansive mappings. Different iterative processes have been used to approximate fixed points of multivalued nonexpansive mappings. In particular in 2005, Sastry and Babu [14] proved that the Mann and Ishikawa iteration process for multivalued maping T with a fixed point p converge to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from p. Panyanak [12] extended result of Sastry and Babu [14] to uniformly convex Banach spaces. Recently, Song and Wang [17] noted that there was a gap in the proof of the main result in [12]. They further revised the gap and also gave the affirmative answer to Panyanak's open question. Shahzad and Zegeye [16] extended and improved results already appeared in the papers [12, 14, 17]. Very recently, motivated by [16], Cholamjiak and Suantai [2, 3] introduced some new two-step

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iterative process for two multivalued mappings in Banach spaces and prove strong convergence of the proposed iterations.

Glowinski and Le Tallec [5] used three-step iterative process to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [5] that the three-step iterative process gives better numerical results than the two-step and one-step approximate iterations. In 1998, Haubruge et al. [6] studied the convergence analysis of three-step process of Glowinski and Le Tallec [5] and applied these process to obtain new splitting-type algorithms for solving variation inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. Thus we conclude that three-step process plays an important and significant part in solving various problems, which arise in pure and applied sciences.

Now the aim of this paper is to introduce a three-step iterative process with errors for multivalued mappings satisfying condition (C) and then prove some strong convergence theorems for such process in a uniformly convex Banach space. Both Mann and Ishikawa iterative processes for multivalued mappings can be obtained from this process as special cases by suitably choosing the parameters. Our results generalized recent known result in literature.

2. PRELIMINARIES

Recall that a Banach space X is said to be uniformly convex if for each $t \in [0, 2]$, the modulus of convexity of X given by:

$$\delta(t) = inf\{1 - \frac{1}{2}\|x + y\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq t\}$$

satisfies the inequality $\delta(t) > 0$ for all $t > 0$.

A subset $E \subset X$ is called proximal if for each $x \in X$, there exists an element $y \in E$ such that

$$\| x - y \| = dist(x, E) = inf\{\| x - z \| : z \in E\}.$$

It is known that every closed convex subset of a uniformly convex Banach space is proximal.

We denote by $CB(E)$ and $P(E)$ the collection of all nonempty closed bounded subsets and nonempty proximal bounded subsets of E respectively. The Hausdorff metric H on $CB(X)$ is defined by

$$H(A, B) := \max\{\sup_{x \in A} dist(x, B), \sup_{y \in B} dist(y, A)\},$$

for all $A, B \in CB(X)$.

Let $T : X \longrightarrow 2^X$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$. The set of fixed points of T will be denote by $F(T)$.

Definition 2.1. A multivalued mapping $T : X \longrightarrow CB(X)$ is called

(i) nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in X.$$

(ii) quasi nonexpansive if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \| x - p \|$ for all $x \in X$ and all $p \in F(T)$.

In 2008, Suzuki [19] introduced a condition on mappings, called (C) which is weaker than nonexpansiveness and stronger than quasi nonexpansiveness. Very recently, Abkar and Eslamian [1] used a modified Suzuki condition for multivalued mappings as follows:

Definition 2.2. A multivalued mapping $T : X \longrightarrow CB(X)$ is said to satisfy condition (C) provided that

$$\frac{1}{2}dist(x,Tx) \leq \|x-y\| \implies H(Tx,Ty) \leq \|x-y\|, \quad x,y \in X.$$

Lemma 2.3. *([1]) Let $T : X \longrightarrow CB(X)$ be a multivalued nonexpansive mapping, then T satisfies the condition (C).*

Lemma 2.4. *([4]) Let $T : X \longrightarrow CB(X)$ be a multivalued mapping which satisfies the condition (C) and has a fixed point. Then T is a quasi nonexpansive mapping.*

Lemma 2.5. *([4]) Let E be a nonempty subset of a Banach space X. Suppose $T : E \longrightarrow P(E)$ satisfies condition (C) then*

$$H(Tx,Ty) \leq 2dist(x,Tx) + \|x-y\|,$$

holds for all $x, y \in E$.

Lemma 2.6. *([20], Lemma1) Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequence of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\longrightarrow\infty} a_n$ exists. In particular , if $\{a_n\}$ has a subsequence converging to 0, then $\lim_{n\longrightarrow\infty} a_n = 0$.

The following Lemma can be found in ([11], Lemma 1.4)

Lemma 2.7. *Let X ba a uniformly convex Banach space and let $B_r(0) = \{x \in X :\| x \|\leq r\}$, $r > 0$. Then there exist a continuous, strictly increasing, and convex function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ with $\varphi(0) = 0$ such that*

$$\| \alpha x + \beta y + \gamma z + \eta w \|^2 \leq \alpha \| x \|^2 + \beta \| y \|^2 + \gamma \| z \|^2 + \eta \|w\|^2 - \alpha\beta\varphi(\| x - y \|),$$

for all $x, y, z, w \in B_r(0)$, and $\alpha, \beta, \gamma, \eta \in [0, 1]$ with $\alpha + \beta + \gamma + \eta = 1$.

3. MAIN RESULTS

In this section we use the following iteration process.
(A) Let X be a Banach space, E be a nonempty convex subset of X and $T_1, T_2, T_3 : E \longrightarrow CB(E)$ be three given mappings. Then, for $x_1 \in E$, we consider the following iterative process:

$$w_n = (1-a_n-b_n)x_n + a_n z_n + b_n s_n, \qquad n \geq 1,$$
$$y_n = (1-c_n-d_n-e_n)x_n + c_n u_n + d_n u'_n + e_n s'_n, \qquad n \geq 1,$$
$$x_{n+1} = (1-\alpha_n-\beta_n-\gamma_n)x_n + \alpha_n v_n + \beta_n v'_n + \gamma_n s''_n, \qquad n \geq 1,$$

where $z_n, u'_n \in T_1(x_n)$, $u_n, v'_n \in T_2(w_n)$ and $v_n \in T_3(y_n)$ and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in [0, 1]$ and $\{s_n\}, \{s'_n\}$ and $\{s''_n\}$ are bounded sequences in E.

Definition 3.1. A mapping $T : E \longrightarrow CB(E)$ is said to satisfy condition (I) if there is a non decreasing function $g : [0,\infty) \longrightarrow [0,\infty)$ with $g(0) = 0$, $g(r) > 0$ for $r \in (0,\infty)$ such that

$$dist(x,Tx) \geq g(dist(x,F(T)).$$

Let $T_i : E \longrightarrow CB(E), (i = 1, 2, 3)$ be three given mappings. The mappings T_1, T_2, T_3 are said to satisfy condition (II) if there exist a non decreasing function $g : [0,\infty) \longrightarrow [0,\infty)$ with $g(0) = 0$, $g(r) > 0$ for $r \in (0,\infty)$, such that

$$\frac{1}{3}sum_{i=1}^{3} dist(x,T_i x) \geq g(dist(x,\mathcal{F})),$$

where $\mathcal{F} = \bigcap_{i=1}^{3} F(T_i)$.

Theorem 3.1. *Let E be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T_i : E \longrightarrow CB(E), (i = 1, 2, 3)$ be three multivalued mappings satisfying the condition (C) . Assume that $\mathcal{F} = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, 3)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_n + b_n, c_n+d_n+e_n, \alpha_n+\beta_n+\gamma_n \in [a, b] \subset (0, 1)$ and also $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} e_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Assume that T_1, T_2 and T_3 satisfying the condition (II). Then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.*

Proof. Let $p \in \mathcal{F}$. Then, by the boundedness of $\{s_n\}, \{s'_n\}$ and $\{s''_n\}$, we let

$$M = max\{sup_{n\geq 1}\|s_n - p\|, sup_{n\geq 1}\|s'_n - p\|, sup_{n\geq 1}\|s''_n - p\|\}.$$

Using (A) and quasi nonexpansiveness of T_i (i=1,2,3) we have

$$\begin{aligned}
\| w_n - p \| &= \| (1 - a_n - b_n)x_n + a_n z_n + b_n s_n - p \| \\
&\leq (1 - a_n - b_n) \| x_n - p \| + a_n \| z_n - p \| + b_n\|s_n - p\| \\
&= (1 - a_n - b_n \| x_n - p \| + a_n dist(z_n, T_1(p)) + b_n\|s_n - p\| \\
&\leq (1 - a_n - b_n) \| x_n - p \| + a_n H(T_1(x_n), T_1(p)) + b_n\|s_n - p\| \\
&\leq (1 - a_n - b_n) \| x_n - p \| + a_n \| x_n - p \| + b_n\|s_n - p\| \\
&\leq (1 - b_n)\|x_n - p\| + b_n M \\
&\leq \|x_n - p\| + b_n M
\end{aligned}$$

and

$$\begin{aligned}
\| y_n - p \| &= \| (1 - c_n - d_n - e_n)x_n + c_n u_n + d_n u'_n + e_n s'_n - p \| \\
&\leq (1 - c_n - d_n - e_n) \| x_n - p \| + c_n \| u_n - p \| + d_n \| u'_n - p\| + e_n\|s'_n - p\| \\
&\leq (1 - c_n - d_n - e_n) \| x_n - p \| + c_n dist(u_n, T_2(p)) + d_n dist(u'_n, T_1(p)) + e_n\|s'_n - p\| \\
&\leq (1-c_n-d_n-e_n) \| x_n-p \| + c_n H(T_2(w_n), T_2(p)) + d_n H(T_1(x_n), T_1(p)) + e_n\|s'_n-p\| \\
&\leq (1 - c_n - d_n - e_n) \| x_n - p \| + c_n \| w_n - p \| + d_n \| x_n - p \| + e_n\|s'_n - p\| \\
&\leq (1 - c_n - d_n - e_n) \| x_n - p \| + c_n \| x_n - p \| + d_n \| x_n - p \| + c_n b_n M + e_n M \\
&\leq \|x_n - p\| + b_n M + e_n M.
\end{aligned}$$

We also have

$$\begin{aligned}
\| x_{n+1} - p \| &= \| (1 - \alpha_n - \beta_n - \gamma_n)x_n + \alpha_n v_n + \beta_n v'_n + \gamma_n s''_n - p \| \\
&\leq (1 - \alpha_n - \beta_n - \gamma_n) \| x_n - p \| + \alpha_n \| v_n - p \| + \beta_n \| v'_n - p\| + \gamma_n\|s''_n - p\| \\
&\leq (1 - \alpha_n - \beta_n - \gamma_n) \| x_n - p \| + \alpha_n dist(v_n, T_3(p)) + \beta_n dist(v'_n, T_2(p)) + \gamma_n\|s''_n - p\| \\
&\leq (1-\alpha_n-\beta_n-\gamma_n) \| x_n-p \| + \alpha_n H(T_3(y_n), T_3(p)) + \beta_n H(T_2(w_n), T_2(p)) + \gamma_n\|s''_n-p\| \\
&\leq (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p \| + \alpha_n \| y_n - p \| + \beta_n \| w_n - p \| + \gamma_n\|s''_n - p\| \\
&\leq (1-\alpha_n-\beta_n-\gamma_n) \| x_n-p \| + \alpha_n\|x_n-p\| + \alpha_n b_n M + \alpha_n e_n M + \beta_n\|x_n-p\| + \beta_n b_n M + \gamma_n M \\
&\leq (1 - \gamma_n)\|x_n - p\| + M(b_n + e_n + \gamma_n) \\
&\leq \|x_n - p\| + \theta_n. \qquad (3.1)
\end{aligned}$$

where $\theta_n = M(b_n + e_n + \gamma_n)$. By assumption we have $\sum_{n=1}^{\infty} \theta_n < \infty$. Hence by Lemma 2.6 it follows that $\lim \| x_n - p \|$ exist for any $p \in \mathcal{F}$. Since the sequences $\{x_n\}, \{y_n\}$ and $\{w_n\}$ are bounded, we can find $r > 0$ depending on p such that

$x_n - p, y_n - p, w_n - p \in B_r(0)$ for all $n \geq 0$. Denote by

$$N = max\{sup_{n\geq 1}\|s_n - p\|^2, sup_{n\geq 1}\|s'_n - p\|^2, sup_{n\geq 1}\|s''_n - p\|^2\}.$$

From Lemma 2.7, we get

$$\begin{aligned}
&\| w_n - p \|^2 = \| (1 - a_n - b_n)x_n + a_n z_n + b_n s_n - p \|^2 \\
&\leq (1-a_n-b_n) \| x_n-p \|^2 + a_n \| z_n-p \|^2 + b_n\|s_n-p\|^2 - a_n(1-a_n-b_n)\varphi(\|x_n-z_n\|) \\
&\leq (1-a_n-b_n \| x_n-p \|^2 + a_n dist(z_n, T_1(p))^2 + b_n\|s_n-p\|^2 - a_n(1-a_n-b_n)\varphi(\|x_n-z_n\|) \\
&\leq (1-a_n-b_n) \| x_n-p \|^2 + a_n H(T_1(x_n), T_1(p))^2 + b_n\|s_n-p\|^2 - a_n(1-a_n-b_n)\varphi(\|x_n-z_n\|) \\
&\leq (1-a_n-b_n) \| x_n-p \|^2 + a_n \| x_n-p \|^2 + b_n\|s_n-p\|^2 - a_n(1-a_n-b_n)\varphi(\|x_n-z_n\|) \\
&\leq (1-b_n)\|x_n-p\|^2 + b_n N - a_n(1-a_n-b_n)\varphi(\|x_n-z_n\|) \\
&\leq \|x_n-p\|^2 + b_n N - a_n(1-a_n-b_n)\varphi(\|x_n-z_n\|)
\end{aligned}$$

It follows from Lemma 2.7 that

$$\begin{aligned}
&\| y_n - p \|^2 = \| (1 - c_n - d_n - e_n)x_n + c_n u_n + d_n u'_n + e_n s'_n - p \|^2 \\
&\leq (1 - c_n - d_n - e_n) \| x_n - p \|^2 + c_n \| u_n - p \|^2 + d_n \| u'_n - p\|^2 + e_n\|s'_n - p\|^2 \\
&\quad - \frac{1}{2}(1 - c_n - d_n - e_n)d_n\varphi(\|x_n - u'_n\|) - \frac{1}{2}(1 - c_n - d_n - e_n)c_n\varphi(\|x_n - u_n\|) \\
&\leq (1-c_n-d_n-e_n) \| x_n-p \|^2 + c_n dist(u_n, T_2(p))^2 + d_n dist(u'_n, T_1(p))^2 + e_n\|s'_n-p\|^2 \\
&\quad - \frac{1}{2}(1 - c_n - d_n - e_n)d_n\varphi(\|x_n - u'_n\|) - \frac{1}{2}(1 - c_n - d_n - e_n)c_n\varphi(\|x_n - u_n\|) \\
&\leq (1-c_n-d_n-e_n) \| x_n-p \|^2 + c_n H(T_2(w_n), T_2(p))^2 + d_n H(T_1(x_n), T_1(p))^2 + e_n\|s'_n-p\|^2 \\
&\quad - \frac{1}{2}(1 - c_n - d_n - e_n)d_n\varphi(\|x_n - u'_n\|) - \frac{1}{2}(1 - c_n - d_n - e_n)c_n\varphi(\|x_n - u_n\|) \\
&\leq (1 - c_n - d_n - e_n) \| x_n - p \|^2 + c_n \| w_n - p \|^2 + d_n \| x_n - p \|^2 + e_n\|s'_n - p\|^2 \\
&\quad - \frac{1}{2}(1 - c_n - d_n - e_n)d_n\varphi(\|x_n - u'_n\|) - \frac{1}{2}(1 - c_n - d_n - e_n)c_n\varphi(\|x_n - u_n\|) \\
&\leq (1 - c_n - d_n - e_n) \| x_n - p \|^2 + c_n \| x_n - p \|^2 + d_n \| x_n - p \|^2 + c_n b_n N + e_n N \\
&\quad - \frac{1}{2}(1 - c_n - d_n - e_n)d_n\varphi(\|x_n - u'_n\|) - \frac{1}{2}(1 - c_n - d_n - e_n)c_n\varphi(\|x_n - u_n\|) \\
&\leq \|x_n - p\|^2 + b_n N + e_n N \\
&\quad - \frac{1}{2}(1 - c_n - d_n - e_n)d_n\varphi(\|x_n - u'_n\|) - \frac{1}{2}(1 - c_n - d_n - e_n)c_n\varphi(\|x_n - u_n\|).
\end{aligned}$$

By another application of Lemma 2.7 we obtain that

$$\begin{aligned}
&\| x_{n+1} - p \|^2 = \| (1 - \alpha_n - \beta_n - \gamma_n)x_n + \alpha_n v_n + \beta_n v'_n + \gamma_n s''_n - p \|^2 \\
&\leq (1 - \alpha_n - \beta_n - \gamma_n) \| x_n - p \|^2 + \alpha_n \| v_n - p \|^2 + \beta_n \| v'_n - p\|^2 + \gamma_n\|s''_n - p\|^2 \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \gamma_n)\varphi(\|x_n - v_n\|) \\
&\leq (1-\alpha_n-\beta_n-\gamma_n) \| x_n-p \|^2 + \alpha_n dist(v_n, T_3(p))^2 + \beta_n dist(v'_n, T_2(p))^2 + \gamma_n\|s''_n-p\|^2 \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \gamma_n)\varphi(\|x_n - v_n\|) \\
&\leq (1-\alpha_n-\beta_n-\gamma_n) \| x_n-p \|^2 + \alpha_n H(T_3(y_n), T_3(p))^2 + \beta_n H(T_2(w_n), T_2(p)^2 + \gamma_n\|s''_n-p\| \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \gamma_n)\varphi(\|x_n - v_n\|) \\
&\leq (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p \|^2 + \alpha_n \| y_n - p \|^2 + \beta_n \| w_n - p \|^2 + \gamma_n\|s''_n - p\|^2
\end{aligned}$$

$$- \alpha_n(1-\alpha_n-\beta_n-\gamma_n)\varphi(\|x_n-v_n\|)$$
$$\leq (1-\alpha_n-\beta_n-\gamma_n) \parallel x_n-p \parallel^2 +\alpha_n\|x_n-p\|^2+\alpha_n b_n N+\alpha_n e_n N+\beta_n\|x_n-p\|^2+\beta_n b_n N+\gamma_n N$$
$$- \alpha_n(1-\alpha_n-\beta_n-\gamma_n)\varphi(\|x_n-v_n\|) - \frac{1}{2}\alpha_n(1-c_n-d_n-e_n)d_n\varphi(\|x_n-u'_n\|)$$
$$- \frac{1}{2}\alpha_n(1-c_n-d_n-e_n)c_n\varphi(\|x_n-u_n\|) - a_n\beta_n(1-a_n-b_n)\varphi(\|x_n-z_n\|)$$
$$\leq \|x_n-p\|^2+N(b_n+e_n+\gamma_n)-\alpha_n(1-\alpha_n-\beta_n-\gamma_n)\varphi(\|x_n-v_n\|)-\frac{1}{2}\alpha_n(1-c_n-d_n-e_n)d_n\varphi(\|x_n-u'_n\|)$$
$$- \frac{1}{2}\alpha_n(1-c_n-d_n-e_n)c_n\varphi(\|x_n-u_n\|) - a_n\beta_n(1-a_n-b_n)\varphi(\|x_n-z_n\|).$$

So, we have

$$\frac{1}{2}a^2(1-b)\varphi(\|x_n-u'_n\|)$$
$$\leq \frac{1}{2}\alpha_n(1-c_n-d_n-e_n)d_n\varphi(\|x_n-u'_n\|)$$
$$\leq \parallel x_n-p \parallel^2 - \parallel x_{n+1}-p \parallel^2 +N(b_n+e_n+\gamma_n).$$

This implies that

$$\sum_{n=1}^{\infty} a^2(1-b)\varphi(\|x_n-u'_n\|) \leq \|x_1-p\|^2 + \sum_{n=1}^{\infty} N(b_n+e_n+\gamma_n) < \infty$$

from which it follows that $\lim_{n\longrightarrow\infty}\varphi(\|x_n-u'_n\|)=0$. Since φ is continuous at 0 and is strictly increasing, we have

$$\lim_{n\longrightarrow\infty}\|x_n-u'_n\|=0.$$

Similarly we obtain that

$$\lim_{n\longrightarrow\infty}\|x_n-z_n\| = \lim_{n\longrightarrow\infty}\|x_n-u_n\| = \lim_{n\longrightarrow\infty}\|x_n-v_n\|=0.$$

Hence we obtain $dist(x_n,T_1x_n) \leq \|x_n-u'_n\| \longrightarrow 0$ as $n\longrightarrow\infty$. Also we have

$$\lim_{n\longrightarrow\infty}\|x_n-w_n\| = \lim_{n\longrightarrow\infty}(a_n\|z_n-x_n\|+b_n\|s_n-x_n\|)=0.$$

and

$$\lim_{n\longrightarrow\infty}\|x_n-y_n\| = \lim_{n\longrightarrow\infty}(c_n\|u_n-x_n\|+d_n\|u'_n-x_n\|+e_n\|s'_n-x_n\|)=0.$$

Therefore by Lemma 2.5 we have

$$\begin{aligned} dist(x_n,T_2(x_n)) &\leq dist(x_n,T_2(w_n)) + H(T_2(w_n),T_2(x_n)) \\ &\leq dist(x_n,T_2(w_n)) + 2\,dist(w_n,T_2(w_n)) + \|x_n-w_n\| \\ &\leq 3\,\|x_n-w_n\| + 3\,dist(x_n,T_2(w_n)) \\ &\leq 3\,\|x_n-w_n\| + 3\,\|x_n-u_n\| \longrightarrow 0 \qquad as \quad n\longrightarrow\infty. \end{aligned}$$

and

$$\begin{aligned} dist(x_n,T_3x_n) &\leq dist(x_n,T_3(y_n)) + H(T_3(y_n),T_3(x_n)) \\ &\leq dist(x_n,T_3(y_n)) + 2\,dist(y_n,T_3(w_n)) + \|x_n-y_n\| \\ &\leq 3\,\|x_n-y_n\| + 3\,dist(x_n,T_3(y_n)) \\ &\leq 3\,\|x_n-y_n\| + 3\,\|x_n-v_n\| \longrightarrow 0 \qquad as \quad n\longrightarrow\infty. \end{aligned}$$

Note that by our assumption $\lim_{n\longrightarrow\infty} dist(x_n,\mathcal{F})=0$. Hence there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_k\}$ in $\mathcal{F}$ such that $\|x_{n_k}-p_k\|<\frac{1}{2^k}$ for all k. Therefore by inequality 3.1 we get

$$\begin{aligned}\|x_{n_{k+1}}-p\| &\leq \|x_{n_{k+1}-1}-p\|+\theta_{n_{k+1}-1}\\ &\leq \|x_{n_{k+1}-2}-p\|+\theta_{n_{k+1}-2}+\theta_{n_{k+1}-1}\\ &\leq \dots\\ &\leq \|x_{n_k}-p\|+\sum_{i=1}^{n_{k+1}-n_k-1}\theta_{n_k+i}\end{aligned}$$

for all $p\in\mathcal{F}$. This implies that

$$\begin{aligned}\|x_{n_{k+1}}-p\| &\leq \|x_{n_k}-p_k\|+\sum_{i=1}^{n_{k+1}-n_k-1}\theta_{n_k+i}\\ &\leq \frac{1}{2^k}+\sum_{i=1}^{n_{k+1}-n_k-1}\theta_{n_k+i}.\end{aligned}$$

Now, we show that $\{p_k\}$ is a Cauchy sequence in E. Note that

$$\begin{aligned}\|p_{k+1}-p_k\| &\leq \|p_{k+1}-x_{n_{k+1}}\|+\|x_{n_{k+1}}-p_k\|\\ &< \frac{1}{2^{k+1}}+\frac{1}{2^k}+\sum_{i=1}^{n_{k+1}-n_k-1}\theta_{n_k+i}\\ &< \frac{1}{2^{k-1}}+\sum_{i=1}^{n_{k+1}-n_k-1}\theta_{n_k+i}.\end{aligned}$$

This implies that $\{p_k\}$ is a Cauchy sequence in E and hence converges to $q\in E$. Since for $i=1,2,3$

$$dist(p_k,T_i(q))\leq H(T_i(p_k),T_i(q))\leq\|p_k-q\|$$

and $p_k\longrightarrow q$ as $n\longrightarrow\infty$, it follows that $dist(q,T_i(q))=0$ and thus $q\in\mathcal{F}$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\longrightarrow\infty}\|x_n-q\|$ exists, we conclude that $\{x_n\}$ converges strongly to q. □

Theorem 3.2. *Let E be a nonempty compact convex subset of uniformly convex Banach space X. Let $T_i:E\longrightarrow CB(E),(i=1,2,3)$ be three multivalued mappings satisfying the condition (C) . Assume that $\mathcal{F}=\bigcap_{i=1}^3 F(T_i)\neq\emptyset$ and $T_i(p)=\{p\},(i=1,2,3)$ for each $p\in\mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_n+b_n, c_n+d_n+e_n, \alpha_n+\beta_n+\gamma_n\in[a,b]\subset(0,1)$ and also $\sum_{n=1}^\infty b_n<\infty$, $\sum_{n=1}^\infty e_n<\infty$ and $\sum_{n=1}^\infty\gamma_n<\infty$. Then $\{x_n\}$ converges strongly to a common fixed point of T_1,T_2 and T_3.*

Proof. As in the proof of Theorem 3.1, we have $\lim_{n\longrightarrow\infty} dist(T_i(x_n),x_n)=0,(i=1,2,3)$. Since E is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim x_{n_k}=w$ for some $w\in E$. By lemma 2.6, for $i=1,2,3$ we have

$$\begin{aligned}dist(w,T_i(w)) &\leq \|w-x_{n_k}\|+dist(x_{n_k},T_i(w))\\ &\leq \|w-x_{n_k}\|+dist(x_{n_k},T_i(x_{n_k}))+H(T_i(x_{n_k}),T_i(w))\\ &\leq 3dist(x_{n_k},T_i(x_{n_k}))+2\|w-x_{n_k}\|\longrightarrow 0 \quad as \quad k\longrightarrow\infty,\end{aligned}$$

this implies that $w \in \mathcal{F}$. Since $\{x_{n_k}\}$ converges strongly to w and $\lim_{n\longrightarrow\infty} \|x_n - w\|$ exists (as in the proof of Theorem 3.1), this implies that $\{x_n\}$ converges strongly to w. □

Remark: If we put $T_1 = T_2 = T_3 = T$, then Theorem 3.1 also hold even if T is quasi-nonexpansive.

We now intend to remove the restriction that $T_i(p) = p$ for each $p \in \mathcal{F}$. We define the following iteration process.

(B): Let X be a Banach space, E be a nonempty convex subset of X and $T_i : E \longrightarrow P(E), (i = 1, 2, 3)$ be given mappings and

$$P_{T_i}(x) = \{y \in T_i(x) :\| x - y \|= dist(x, T_i(x))\}.$$

Then, for $x_1 \in E$, we consider the following iterative process:

$$w_n = (1 - a_n - b_n)x_n + a_n z_n + b_n s_n, \qquad n \geq 1,$$

$$y_n = (1 - c_n - d_n - e_n)x_n + c_n u_n + d_n u'_n + e_n s'_n, \qquad n \geq 1,$$

$$x_{n+1} = (1 - \alpha_n - \beta_n - \gamma_n)x_n + \alpha_n v_n + \beta_n v'_n + \gamma_n s''_n, \qquad n \geq 1,$$

where $z_n, u'_n \in P_{T_1}(x_n)$,$u_n, v'_n \in P_{T_2}(w_n)$ and $v_n \in P_{T_3}(y_n)$ and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in [0, 1]$ and $\{s_n\}, \{s'_n\}$ and $\{s''_n\}$ are bounded sequences in E.

Theorem 3.3. *Let E be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T_i : E \longrightarrow P(E), (i = 1, 2, 3)$ be multivalued mappings such that P_{T_i} satisfing the condition (C). Let $\{x_n\}$ be the iterative process defined by (B), and $a_n + b_n, c_n + d_n + e_n, \alpha_n + \beta_n + \gamma_n \in [a, b] \subset (0, 1)$ and also $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} e_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Assume that T_1, T_2 and T_3 satisfying the condition (II) and $\mathcal{F} \neq \emptyset$. Then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.*

Proof. Let $p \in \mathcal{F}$. Then, for $i = 1, 2, 3$ we have $p \in P_{T_i}(p) = \{p\}$. Also, we have

$$\|z_n - p\| \leq dist(z_n, P_{T_1}(p)) \leq H(P_{T_1}(x_n), P_{T_1}(p)) \leq \|x_n - p\|$$

and

$$\|u_n - p\| \leq dist(u_n, P_{T_2}(p)) \leq H(P_{T_2}(w_n), P_{T_2}(p)) \leq \|w_n - p\|,$$

and

$$\|v_n - p\| \leq dist(v_n, P_{T_3}(p)) \leq H(P_{T_3}(y_n), P_{T_3}(p)) \leq \|y_n - p\|.$$

Now, by similar argument as in the proof of Theorem 3.1, $\lim_{n\longrightarrow\infty} \|x_n - q\|$ exists. Also we get a sequence $\{p_k\} \in \mathcal{F}$ which converges to some $q \in E$. Since for each $i = 1, 2, 3$

$$dist(p_k, T_i(q)) \leq dist(p_k, P_{T_i}(q)) \leq H(P_{T_i}(p_k), P_{T_i}(q)) \leq\| q - p_k \|,$$

and $p_k \longrightarrow q$ as $k \longrightarrow \infty$, it follows that $dist(q, T_i(q)) = 0$ for $i = 1, 2, 3$. Hence $q \in \mathcal{F}$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\longrightarrow\infty} \|x_n - q\|$ exists, we conclude that $\{x_n\}$ converges strongly to q. □

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