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FIXED POINT THEOREMS FOR GENERALIZED WEAKLY CONTRACTIVE CONDITION IN ORDERED PARTIAL METRIC SPACES

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Abstract. In this paper, we establish some fixed point theorems for generalized weakly contractive condition in ordered partial metric spaces. The results extend the main theorems of Nashine and Altun [17] on the class of ordered partial metric ones. Also, some applications are given to illustrate our results.

Keywords: Partial metric, ordered set, fixed point, common fixed point.

AMS Subject Classification: 54H25, 47H10, 54E50.

1. INTRODUCTION AND PRELIMINARIES

The concept of partial metric space was introduced by Matthews [16] in 1994. In such spaces, the distance of a point to its self may not be zero. Specially, from the point of sequences, a convergent sequence need not have unique limit. Matthews [16] extended the well known Banach contraction principle to complete partial metric spaces. After that, many interesting fixed point results were established in such spaces. In this direction, we refer the reader to Valero [25], Oltra and Valero [23], Altun et al. [4], Romaguera [24], Altun and Erduran [2] and Aydi [6, 7, 8].

First, we recall some definitions and properties of partial metric spaces (see [2, 4, 16, 22, 23, 24, 25] for more details).

Definition 1.1. A partial metric on a non-empty set X is a function $p : X\times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$:
(p1) $x = y \Longleftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p2) $p(x,x) \leq p(x,y)$,
(p3) $p(x,y) = p(y,x)$,
(p4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.
A partial metric space is a pair (X,p) such that X is a non-empty set and p is a partial metric on X.

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Remark 1.2. It is clear that, if $p(x,y)=0$, then from (p1) and (p2), $x=y$. But if $x=y$, $p(x,y)$ may not be 0.

A basic example of a partial metric space is the pair $(\mathbb{R}_+,p)$, where $p(x,y)=\max\{x,y\}$ for all $x,y\in\mathbb{R}_+$.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon),\ x\in X,\varepsilon>0\}$, where $B_p(x,\varepsilon)=\{y\in X : p(x,y)<p(x,x)+\varepsilon\}$ for all $x\in X$ and $\varepsilon>0$.

If p is a partial metric on X, then the function $p^s : X\times X\to\mathbb{R}_+$ given by

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y), \tag{1.1}$$

is a metric on X.

Definition 1.3. Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then
(i) $\{x_n\}$ converges to a point $x\in X$ if and only if $p(x,x)=\lim_{n\to+\infty}p(x,x_n)$,
(ii) $\{x_n\}$ is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty}p(x_n,x_m)$.

Definition 1.4. A partial metric space (X,p) is said to be complete if every Cauchy sequence (x_n) in X converges, with respect to τ_p, to a point $x\in X$, such that $p(x,x)=\lim_{n,m\to+\infty}p(x_n,x_m)$.

Lemma 1.5. *Let (X,p) be a partial metric space. Then*
(a) $\{x_n\}$ *is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s),*
(b) (X,p) *is complete if and only if the metric space (X,p^s) is complete. Furthermore,* $\lim_{n\to+\infty}p^s(x_n,x)=0$ *if and only if*

$$p(x,x)=\lim_{n\to+\infty}p(x_n,x)=\lim_{n,m\to+\infty}p(x_n,x_m).$$

Definition 1.6. ([2]) Suppose that (X,p) is a partial metric space. A mapping $F:(X,p)\to(X,p)$ is said to be *continuous* at $x\in X$, if for every $\varepsilon>0$, there exists $\delta>0$ such that $F(B_p(x,\delta))\subseteq B_p(Fx,\varepsilon)$.

The following result is easy to check.

Lemma 1.7. *Let (X,p) be a partial metric space. $F:X\to X$ is continuous if and only if given a sequence $\{x_n\}\in\mathbb{N}$ and $x\in X$ such that $p(x,x)=\lim_{n\to+\infty}p(x,x_n)$, then $p(Fx,Fx)=\lim_{n\to+\infty}p(Fx,Fx_n)$.*

Remark 1.8. ([22]) Let (X,p) be a partial metric space and $F:(X,p)\to(X,p)$. If F is continuous on (X,p), then $F:(X,p^s)\to(X,p^s)$ is continuous.

On the other hand, fixed point problems of contractive mappings in metric spaces endowed with a partially order have been studied by many authors (see [1, 3, 5, 9, 10, 11, 12, 14, 15, 17, 18, 19, 20, 21]). In particular, Nashine and Altun [17] proved the following:

Theorem 1.9. *Let $(X,\le)$ be a partially ordered set and (X,d) be a complete metric space. Suppose that $T:X\to X$ is a nondecreasing mapping such that for every two comparable elements $x,y\in X$*

$$\psi(d(Tx,Ty))\le\psi(M(x,y))-\varphi(M(x,y)), \tag{1.2}$$

where

$$M(x,y) = ad(x,y) + bd(x,Tx) + cd(y,Ty) + e[d(y,Tx) + d(x,Ty)], \tag{1.3}$$

with $a > 0$; b, c, $e \geq 0$, $a + b + c + 2e \leq 1$, *and* $\psi, \varphi : [0, +\infty) \to [0, +\infty)$, ψ *is a continuous, nondecreasing,* φ *is a lower semi-continuous functions and* $\psi(t) = 0 = \varphi(t)$ *if and only if* $t = 0$. *Also suppose, there exists* $x_0 \in X$ *with* $x_0 \leq Tx_0$. *Assume that :*
(i) T *is continuous, or*
(ii) if a nondecreasing sequence $\{x_n\}$ *converges to* x, *then* $x_n \leq x$ *for all* n.
Then, T *has a fixed point.*

The purpose of this paper is to extend Theorem 1.9 on the class of ordered partial metric spaces. Also, a common fixed point result is given.

2. MAIN RESULTS

Our first result is the following.

Theorem 2.1. *Let* $(X, \leq)$ *be a partially ordered set and* (X, p) *be a complete partial metric space. Suppose that* $T : X \to X$ *is a nondecreasing mapping such that for every two comparable elements* $x, y \in X$

$$\psi(p(Tx,Ty)) \leq \psi(\theta(x,y)) - \varphi(\theta(x,y)), \tag{2.1}$$

where

$$\theta(x,y) = ap(x,y) + bp(x,Tx) + cp(y,Ty) + e[p(y,Tx) + p(x,Ty)], \tag{2.2}$$

with $a, e > 0$; b, $c \geq 0$, $a + b + c + 2e \leq 1$, *and* $\psi, \varphi : [0, +\infty) \to [0, +\infty)$, ψ *is a continuous, nondecreasing,* φ *is lower semi-continuous functions and* $\psi(t) = 0 = \varphi(t)$ *if and only if* $t = 0$. *Also suppose, there exists* $x_0 \in X$ *with* $x_0 \leq Tx_0$. *Assume that :*
(i) T *is continuous, or*
(ii) if a nondecreasing sequence $\{x_n\}$ *converges to* x *in* (X, p), *then* $x_n \leq x$ *for all* n.
Then T *has a fixed point, say* z. *Moreover,* $p(z, z) = 0$.

Proof. If $Tx_0 = x_0$, then the proof is completed. Suppose $Tx_0 \neq x_0$. Now since $x_0 < Tx_0$ and T is nondecreasing we have

$$x_0 < Tx_0 \leq T^2x_0 \leq ... \leq T^nx_0 \leq T^{n+1}x_0 \leq ...$$

Put $x_n = T^nx_0$, hence $x_{n+1} = Tx_n$. If there exists $n_0 \in \{1, 2, ...\}$ such that $\theta(x_{n_0}, x_{n_0-1}) = 0$ then by definition (2.2), it is clear that $p(x_{n_0-1}, x_{n_0}) = p(x_{n_0}, Tx_{n_0-1}) = 0$, so $x_{n_0-1} = x_{n_0} = Tx_{n_0-1}$ and so we are finished. Now we can suppose

$$\theta(x_n, x_{n-1}) > 0, \tag{2.3}$$

for all $n \geq 1$. Let us check that

$$\lim_{n\to+\infty} p(x_{n+1}, x_n) = 0. \tag{2.4}$$

By (2.2), we have using condition (p4)

$$\begin{aligned}\theta(x_n, x_{n-1}) =& ap(x_n, x_{n-1}) + bp(x_n, Tx_n) + cp(x_{n-1}, Tx_{n-1}) \\ &+ e[p(x_{n-1}, Tx_n) + p(x_n, Tx_{n-1})] \\ =& (a + c)p(x_n, x_{n-1}) + bp(x_n, x_{n+1}) + e[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)] \\ \leq& (a + c + e)p(x_n, x_{n-1}) + (b + e)p(x_n, x_{n+1}) \quad \text{[by (p4)]}.\end{aligned}$$

Now we claim that

$$p(x_{n+1}, x_n) \leq p(x_n, x_{n-1}), \tag{2.5}$$

for all $n \geq 1$. Suppose this is not true, that is, there exists $n_0 \geq 1$ such that $p(x_{n_0+1}, x_{n_0}) > p(x_{n_0}, x_{n_0-1})$. Now since $x_{n_0} \leq x_{n_0+1}$, we can use the inequality (2.1), then we have

$$\begin{aligned}
\psi(p(x_{n_0+1}, x_{n_0})) =& \psi(p(Tx_{n_0}, Tx_{n_0-1})) \\
\leq& \psi(\theta(x_{n_0}, x_{n_0-1})) - \varphi(\theta(x_{n_0}, x_{n_0-1})) \\
\leq& \psi((a+c+e)p(x_{n_0}, x_{n_0-1}) + (b+e)p(x_{n_0}, x_{n_0+1})) \\
&- \varphi(\theta(x_{n_0}, x_{n_0-1})) \\
\leq& \psi((a+b+c+2e)p(x_{n_0}, x_{n_0+1})) - \varphi(\theta(x_{n_0}, x_{n_0-1})) \\
\leq& \psi(p(x_{n_0}, x_{n_0+1})) - \varphi(\theta(x_{n_0}, x_{n_0-1})),
\end{aligned}$$

which implies that $\varphi(\theta(x_{n_0}, x_{n_0-1})) \leq 0$, and by property of φ, giving that $\theta(x_{n_0}, x_{n_0-1}) = 0$, this contradicts (2.3). Hence (2.5) holds, and so the sequence $\{p(x_{n+1}, x_n)\}$ is nonincreasing and bounded below. Thus there exists $\rho \geq 0$ such that

$\lim_{n\to+\infty} p(x_{n+1}, x_n) = \rho$. Assume that $\rho > 0$. By (2.2), we have

$$\begin{aligned}
a\rho = \lim_{n\to+\infty} ap(x_n, x_{n-1}) \leq& \limsup_{n\to+\infty} \theta(x_n, x_{n-1}) \\
=& \limsup_{n\to+\infty} [(a+c)p(x_n, x_{n-1}) + bp(x_n, x_{n+1}) \\
&+ e(p(x_{n-1}, x_{n+1}) + p(x_n, x_n))] \\
\leq& \limsup_{n\to+\infty} [(a+c+e)p(x_n, x_{n-1}) + (b+e)p(x_n, x_{n+1})].
\end{aligned}$$

This implies

$$0 < a\rho \leq \limsup_{n\to+\infty} \theta(x_n, x_{n-1}) \leq (a+b+c+2e)\rho \leq \rho,$$

and so there exist $\rho_1 > 0$ and a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\lim_{k\to+\infty} \theta(x_{n(k)}, x_{n(k)-1}) = \rho_1 \leq \rho.$$

By the lower semi-continuity of φ we have

$$\varphi(\rho_1) \leq \liminf_{k\to+\infty} \varphi(\theta(x_{n(k)}, x_{n(k)+1})).$$

From (2.1), we have

$$\begin{aligned}
\psi(p(x_{n(k)+1}, x_{n(k)})) &= \psi(p(Tx_{n(k)}, Tx_{n(k)-1})) \\
&\leq \psi(\theta(x_{n(k)}, x_{n(k)-1})) - \varphi(\theta(x_{n(k)}, x_{n(k)-1})),
\end{aligned}$$

and taking upper limit as $k \to +\infty$, we have using the properties of ψ and φ

$$\begin{aligned}
\psi(\rho) \leq& \psi(\rho_1) - \liminf_{k\to+\infty} \varphi(\theta(x_{n(k)}, x_{n(k)+1})) \\
\leq& \psi(\rho_1) - \varphi(\rho_1) \\
\leq& \psi(\rho) - \varphi(\rho_1),
\end{aligned}$$

that is, $\varphi(\rho_1) = 0$. Thus, by the property of φ, we have $\rho_1 = 0$, which is a contradiction. Therefore we have $\rho = 0$, that is (2.4) holds.

Now, we show that $\{x_n\}$ is a Cauchy sequence in the partial metric space (X, p). From Lemma 1.5, it is sufficient to prove that $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s). Suppose to the contrary. Then there is a $\varepsilon > 0$ such that for an integer k there exist integers $m(k) > n(k) > k$ such that

$$p^s(x_{n(k)}, x_{m(k)}) > \varepsilon. \tag{2.6}$$

For every integer k, let $m(k)$ be the least positive integer exceeding $n(k)$ satisfying (2.6) and such that

$$p^s(x_{n(k)}, x_{m(k)-1}) \le \varepsilon. \tag{2.7}$$

Now, using (2.4)

$$\begin{aligned}\varepsilon < p^s(x_{n(k)}, x_{m(k)}) \le& p^s(x_{n(k)}, x_{m(k)-1}) + p^s(x_{m(k)-1}, x_{m(k)})\\ \le& \varepsilon + p^s(x_{m(k)-1}, x_{m(k)}).\end{aligned}$$

Then by (2.4) it follows that

$$\lim_{k\to+\infty} p^s(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{2.8}$$

Also, by the triangle inequality, we have

$$|p^s(x_{n(k)}, x_{m(k)-1}) - p^s(x_{n(k)}, x_{m(k)})| \le p^s(x_{m(k)-1}, x_{m(k)}).$$

By using (2.4), (2.8) we get

$$\lim_{k\to+\infty} p^s(x_{n(k)}, x_{m(k)-1}) = \varepsilon. \tag{2.9}$$

On the other hand, by definition of p^s,

$$p^s(x_{n(k)}, x_{m(k)}) = 2p(x_{n(k)}, x_{m(k)}) - p(x_{n(k)}, x_{n(k)}) - p(x_{m(k)}, x_{m(k)}),$$

$$p^s(x_{n(k)}, x_{m(k)-1}) = 2p(x_{n(k)}, x_{m(k)-1}) - p(x_{n(k)}, x_{n(k)}) - p(x_{m(k)-1}, x_{m(k)-1}),$$

hence letting $k \to +\infty$, we find thanks to (2.8), (2.9) and the condition (p3) in (2.4)

$$\lim_{k\to+\infty} p(x_{n(k)}, x_{m(k)}) = \frac{\varepsilon}{2}, \tag{2.10}$$

$$\lim_{k\to+\infty} p(x_{n(k)}, x_{m(k)-1}) = \frac{\varepsilon}{2}. \tag{2.11}$$

In view of (2.2), we get

$$\begin{aligned}ap(x_{n(k)}, x_{m(k)-1}) \le& \theta(x_{n(k)}, x_{m(k)-1})\\ =& ap(x_{n(k)}, x_{m(k)-1}) + bp(x_{n(k)}, Tx_{n(k)}) + cp(x_{m(k)-1}, Tx_{m(k)-1})\\ &+ e[p(x_{m(k)-1}, Tx_{n(k)}) + p(x_{n(k)}, Tx_{m(k)-1})]\\ =& ap(x_{n(k)}, x_{m(k)-1}) + bp(x_{n(k)}, x_{n(k)+1}) + cp(x_{m(k)-1}, x_{m(k)})\\ &+ e[p(x_{m(k)-1}, x_{n(k)+1}) + p(x_{n(k)}, x_{m(k)})]\\ \le& ap(x_{n(k)}, x_{m(k)-1}) + bp(x_{n(k)}, x_{n(k)+1}) + cp(x_{m(k)-1}, x_{m(k)})\\ &+ e[p(x_{m(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{n(k)+1}) + p(x_{n(k)}, x_{m(k)})].\end{aligned}$$

Taking upper limit as $k \to +\infty$ and using (2.4), (2.10) and (2.11), we have

$$0 < a\frac{\varepsilon}{2} \le \limsup_{k\to+\infty} \theta(x_{n(k)}, x_{m(k)-1}) \le (a + 2e)\frac{\varepsilon}{2} \le \frac{\varepsilon}{2}.$$

This implies that there exist $\varepsilon_1 > 0$ and a subsequence $\{x_{n(k(p))}\}$ of $\{x_{n(k)}\}$ such that

$$\lim_{p\to+\infty} \theta(x_{n(k(p))}, x_{m(k(p))-1}) = \varepsilon_1 \le \frac{\varepsilon}{2}.$$

By the lower semi-continuity of φ we have

$$\varphi(\varepsilon_1) \le \liminf_{k\to+\infty} \varphi(\theta(x_{n(k)}, x_{m(k)-1})).$$

Now by (2.1) we get

$$\begin{aligned}\psi(\frac{\varepsilon}{2}) =&\limsup_{p\to+\infty}\psi(p(x_{n(k(p))},x_{m(k(p))}))\\ \leq&\limsup_{p\to+\infty}\psi(p(x_{n(k(p))},x_{n(k(p))+1})+p(Tx_{n(k(p))},Tx_{m(k(p))-1}))\\ =&\limsup_{p\to+\infty}\psi(p(Tx_{n(k(p))},Tx_{m(k(p))-1}))\\ \leq&\limsup_{p\to+\infty}[\psi(\theta(x_{n(k(p))},x_{m(k(p))-1}))-\varphi(\theta(x_{n(k(p))},x_{m(k(p))-1}))]\\ =&\psi(\varepsilon_1)-\liminf_{p\to+\infty}\varphi(\theta(x_{n(k(p))},x_{m(k(p))-1}))\\ \leq&\psi(\varepsilon_1)-\varphi(\varepsilon_1)\\ \leq&\psi(\frac{\varepsilon}{2})-\varphi(\varepsilon_1),\end{aligned}$$

which is a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence in the metric space (X,p^s). From Lemma 1.5, (X,p^s) is a complete metric space. Then there is $z\in X$ such that

$$\lim_{n\to+\infty}p^s(x_n,z)=0.$$

Again, from lemma 1.5, we have thanks to (2.4) and the condition (p2)

$$p(z,z)=\lim_{n\to+\infty}p(x_n,z)=\lim_{n\to+\infty}p(x_n,x_n)=0. \tag{2.12}$$

We will prove that $Tz=z$.
1. Assume that (i) holds, that is, T is continuous. By (2.12), the sequence $\{x_n\}$ converges in (X,p) to z, and since T is continuous, hence the sequence $\{Tx_n\}$ converges to Tz, that is

$$p(Tz,Tz)=\lim_{n\to+\infty}p(Tx_n,Tz) \tag{2.13}$$

Again, thanks to (2.12),

$$p(z,Tz)=\lim_{n\to+\infty}p(x_n,Tz)=\lim_{n\to+\infty}p(Tx_{n-1},Tz)=p(Tz,Tz). \tag{2.14}$$

On the other hand, by (2.1), (2.14)

$$\psi(p(z,Tz))=\psi(p(Tz,Tz))\leq\psi(\theta(z,z))-\varphi(\theta(z,z)),$$

where from (2.12) and the condition (p2)

$$\theta(z,z)=ap(z,z)+(b+c+2e)p(z,Tz)=(b+c+2e)p(z,Tz)\leq p(z,Tz).$$

Thus,

$$\begin{aligned}\psi(p(z,Tz))\leq&\psi(\theta(z,z))-\varphi(\theta(z,z))\\ \leq&\psi(p(z,Tz))-\varphi(\theta(z,z)).\end{aligned}$$

It follows that $\varphi(\theta(z,z))=0$, so $\theta(z,z)=(b+c+2e)p(z,Tz)=0$, that is $p(z,Tz)=0$ because $e>0$. Hence $z=Tz$, that is, z is a fixed point of T.
2. Assume that (ii) holds. Then, we have $x_n\leq z$ for all n. Therefore, for all n, we can use the inequality (2.1) for x_n and z. Since

$$\begin{aligned}\theta(z,x_n)=&ap(z,x_n)+bp(z,Tz)+cp(x_n,Tx_n)+e[p(x_n,Tz)+p(z,Tx_n)]\\ =&ap(z,x_n)+bp(z,Tz)+cp(x_n,x_{n+1})+e[p(x_n,Tz)+p(z,x_{n+1})],\end{aligned}$$

hence, from (2.4), (2.12), $\lim_{n\to+\infty}\theta(z,x_n)=(b+e)p(z,Tz)$. We have

$$\begin{aligned}\psi(p(Tz,z)) &= \limsup_{n\to+\infty}\psi(p(Tz,x_{n+1}))\\ &= \limsup_{n\to+\infty}\psi(p(Tz,Tx_n))\\ &\le \limsup_{n\to+\infty}\psi[(\psi(z,x_n))-\varphi(\theta(z,x_n))]\\ &\le \psi((b+e)p(Tz,z))-\varphi((b+e)p(Tz,z))\\ &\le \psi(p(Tz,z))-\varphi((b+e)p(Tz,z)).\end{aligned}$$

Then, $\varphi((b+e)p(Tz,z))=0$, and since $e>0$, hence by the property of φ we have $p(Tz,z)=0$, so $Tz=z$. This completes the proof of Theorem 2.1. □

Remark 2.2. Theorem 2.1 holds for ordered partial metric spaces, so it is an extension of the result of Nashine and Altun [17] given in Theorem 1.9 which is verified just for ordered metric ones.

Corollary 2.3. *Let $(X,\le)$ be a partially ordered set and (X,p) be a complete partial metric space. Suppose that $T:X\to X$ be a nondecreasing mapping such that for every two comparable elements $x,y\in X$*

$$p(Tx,Ty))\le\theta(x,y)-\varphi(\theta(x,y)), \tag{2.15}$$

where

$$\theta(x,y)=ap(x,y)+bp(x,Tx)+cp(y,Ty)+e[p(y,Tx)+p(x,Ty)], \tag{2.16}$$

with $a,e>0$; b, $c\ge 0$, $a+b+c+2e\le 1$, and $\varphi:[0,+\infty)\to[0,+\infty)$, φ is a lower semi-continuous functions and $\varphi(t)=0$ if and only if $t=0$. Also suppose, there exists $x_0\in X$ with $x_0\le Tx_0$. Assume that:
(i) T is continuous, or
(ii) if a nondecreasing sequence $\{x_n\}$ converges to x in (X,p), then $x_n\le x$ for all n.
Then T has a fixed point, say z. Moreover, $p(z,z)=0$.

Proof. It suffices to take $\psi(t)=t$ in Theorem. □

Corollary 2.4. *Let $(X,\le)$ be a partially ordered set and (X,p) be a complete partial metric space. Suppose that $T:X\to X$ be a nondecreasing mapping such that for every two comparable elements $x,y\in X$*

$$p(Tx,Ty))\le k\theta(x,y), \tag{2.17}$$

where

$$\theta(x,y)=ap(x,y)+bp(x,Tx)+cp(y,Ty)+e[p(y,Tx)+p(x,Ty)], \tag{2.18}$$

with $k\in[0,1)$, $a,e>0$; b, $c\ge 0$ and $a+b+c+2e\le 1$. Also suppose, there exists $x_0\in X$ with $x_0\le Tx_0$. Assume that:
(i) T is continuous, or
(ii) if a nondecreasing sequence $\{x_n\}$ converges to x in (X,p), then $x_n\le x$ for all n.
Then T has a fixed point, say z. Moreover, $p(z,z)=0$.

Proof. It suffices to take $\varphi(t)=(1-k)t$ in Corollary 2.3. □

We give in the following a sufficient condition for the uniqueness of the fixed point of the mapping T.

Theorem 2.5. *Let all the conditions of Theorem 2.1 be fulfilled and let the following condition hold: for arbitrary two points $x, y \in X$ there exists $z \in X$ which is comparable with both x and y. If $(a+2b+2e) \leq 1$ or $(a+2c+2e) \leq 1$, then the fixed point of T is unique.*

Proof. Let u and v be two fixed points of T, i.e., $Tu = u$ and $Tv = v$. We have in mind, $p(u, u) = p(v, v) = 0$. Consider the following two cases:
1. u and v are comparable. Then we can apply condition (2.1) and obtain that

$$\psi(p(u,v)) = \psi(p(Tu,Tv)) \leq \psi(\theta(u,v)) - \varphi(\theta(u,v)),$$

where

$$\begin{aligned}\theta(u,v) =& ap(u,v) + bp(u,Tu) + cp(v,Tv) + e[p(u,Tv) + p(v,Tu)]\\ =&(a+2e)p(u,v) + bp(u,u) + cp(v,v)\\ \leq&(a+b+c+2e)p(u,v) \leq p(u,v).\end{aligned}$$

We deduce $\psi(p(u,v)) \leq \psi(p(u,v)) - \varphi(\theta(u,v))$, i,e, $\theta(u,v) = 0$, so $p(u,v) = 0$, meaning that $u = v$, that is the uniqueness of the fixed point of T.
2. Suppose now that u and v are not comparable. Choose an element $w \in X$ comparable with both of them. Then also $u = T^n u$ is comparable with $T^n w$ for each n (since T is nondecreasing). Applying (2.1), one obtains that

$$\begin{aligned}\psi(p(u,T^n w)) =& \psi(p(TT^{n-1}u, TT^{n-1}w))\\ \leq& \psi(\theta(T^{n-1}u, T^{n-1}w)) - \varphi(\theta(T^{n-1}u, T^{n-1}w))\\ =& \psi(\theta(u, T^{n-1}w)) - \varphi(\theta(u, T^{n-1}w))\end{aligned}$$

where

$$\begin{aligned}\theta(u,T^{n-1}w) =& ap(u,T^{n-1}w) + bp(u,TT^{n-1}u) + cp(T^{n-1}w, TT^{n-1}w)\\ &+e[p(u,TT^{n-1}w) + p(T^{n-1}w, Tu)]\\ =& ap(u,T^{n-1}w) + bp(u,u) + cp(T^{n-1}w, T^n w)\\ &+e[p(u,T^n w) + p(T^{n-1}w, u)]\\ =&(a+e)p(u,T^{n-1}w) + cp(T^{n-1}w, T^n w) + ep(u, T^n w)\\ \leq&(a+c+e)p(u,T^{n-1}w) + (c+e)p(u,T^n w).\end{aligned}$$

Similarly as in the proof of Theorem 2.1, it can be shown that, under the condition $(a+2c+2e) \leq 1$

$$p(u,T^n w) \leq p(u, T^{n-1}w).$$

Note that when we consider

$$\psi(p(T^n w, u)) \leq \psi(\theta(T^{n-1}w, u)) - \varphi(\theta(T^{n-1}w, u))$$

where

$$\begin{aligned}\theta(T^{n-1}w, u) =&(a+e)p(u,T^{n-1}w) + bp(T^{n-1}w, T^n w) + ep(u, T^n w)\\ \leq&(a+b+e)p(u,T^{n-1}w) + (b+e)p(u,T^n w),\end{aligned}$$

hence, one finds under $(a+2b+2e) \leq 1$ that

$$p(T^n w, u) \leq p(T^{n-1}w, u).$$

In each case, it follows that the sequence $\{p(u, f^n w)\}$ is nonincreasing and it has a limit $l \geq 0$. Adjusting again as in the proof of Theorem 2.1, one can finds that $l = 0$. In the same way it can be deduced that $p(v, T^n w) \to 0$ as $n \to +\infty$. Now, passing to the limit in $p(u,v) \leq p(u, T^n w) + p(T^n w, v)$, it follows that $p(u,v) = 0$, so $u = v$, and the uniqueness of the fixed point is proved. □

Example 2.6. Let $X = [0, +\infty)$ endowed with the usual partial order (which is a total order). Let $p(x, y) = \max(x, y)$. For any $x, y \in X$, we have $p^s(x, y) = |x - y|$. Then, (X, p^s) is a complete metric space, and so for (X, p). Take $T : X \to X$ be defined as

$$Tx = \frac{1}{5}x.$$

Letting $x_0 = 0$, we have $x_0 = 0 \leq 0 = Tx_0$. The mapping T is nondecreasing. Also, take $a = \frac{1}{2}$, $e = \frac{1}{8}$ and $b = c = 0$, so

$$\theta(x, y) = \frac{1}{2}p(x, y) + \frac{1}{8}[p(x, Ty) + p(y, Tx)] = \frac{1}{2}\max\{x, y\} + \frac{1}{8}[\max\{x, Ty\} + \max\{y, Tx\}].$$

Moreover, define

$$\psi(t) = t, \quad \varphi(t) = \frac{t}{2}.$$

Note that

$$\begin{aligned}\psi(p(Tx, Ty)) =& \frac{1}{5}\max\{x, y\} \leq \frac{1}{4}\max\{x, y\} \\ \leq& \frac{1}{4}\max\{x, y\} + \frac{1}{16}[\max\{x, Ty\} + \max\{y, Tx\}] = \frac{1}{2}\theta(x, y) \\ =& \psi(\theta(x, y)) - \varphi(\theta(x, y)).\end{aligned}$$

Thus, the inequality (2.1) is verified for each comparable x and y. All the hypotheses of Theorem 2.5 are verified. Here, T has a unique fixed point, which is $z = 0$.

Now we will give a common fixed point theorem for two maps. For this, we need the following definition, which is given in [13].

Definition 2.7. Let $(X, \leq)$ be a partially ordered set. Two mappings $S, T : X \to X$ are said to be weakly increasing if $Sx \leq TSx$ and $Tx \leq STx$ for all $x \in X$.

Note that, two weakly increasing mappings need not be nondecreasing. There exist some examples to illustrate this fact in [3]

Theorem 2.8. *Let $(X, \leq)$ be a partially ordered set and (X, p) be a complete partial metric space. Suppose that $T, S : X \to X$ are two weakly increasing mappings such that for every two comparable elements $x, y \in X$*

$$\psi(p(Tx, Sy)) \leq \psi(u(x, y)) - \varphi(u(x, y)), \tag{2.19}$$

where

$$u(x, y) = ap(x, y) + bp(x, Tx) + cp(y, Sy) + e[p(y, Tx) + p(x, Sy)], \tag{2.20}$$

with $a, e > 0$; b, $c \geq 0$, $a + b + c + 2e \leq 1$, and $\psi, \varphi : [0, +\infty) \to [0, +\infty)$, ψ is a continuous, nondecreasing, φ is a lower semi-continuous functions and $\psi(t) = 0 = \varphi(t)$ if and only if $t = 0$. Also suppose, there exists $x_0 \in X$ with $x_0 \leq Tx_0$. Assume that :
(i) T is continuous, or
(ii) S is continuous, or
(iii) if a nondecreasing sequence $\{x_n\}$ converges to x in (X, p), then $x_n \leq x$ for all n.
Then, S and T have a common fixed point.

Proof. Let x_0 be an arbitrary point of X. We can define a sequence $\{x_n\}$ in X as follows:

$$x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n+2} = Tx_{2n+1} \quad \text{for} \quad n \in \mathbb{N}.$$

Since S and T are weakly increasing, we have $x_1 = Sx_0 \leq TSx_0 = Tx_1 = x_2$ and $x_2 = Tx_1 \leq STx_1 = Sx_2 = x_3$. Continuing this process we have

$$x_1 \leq x_2 \leq ... \leq x_n \leq x_{n+1} \leq ...$$

The terms x_{2n-1} and x_{2n} are comparable then we can use the inequality (2.19) and we have

$$\psi(p(Tx_{2n-1}, Sx_{2n})) \leq \psi(u(x_{2n-1}, x_{2n})) - \varphi(u(x_{2n-1}, x_{2n})), \tag{2.21}$$

where

$$\begin{aligned} u(x_{2n-1}, x_{2n}) &= ap(x_{2n-1}, x_{2n}) + bp(x_{2n-1}, Tx_{2n-1}) + cp(x_{2n}, Sx_{2n}) \\ &\quad + e[p(x_{2n}, Tx_{2n-1}) + p(x_{2n-1}, Sx_{2n})] \\ &= (a+b)p(x_{2n-1}, x_{2n}) + cp(x_{2n}, x_{2n+1}) + e[p(x_{2n}, x_{2n}) + p(x_{2n-1}, x_{2n+1})] \\ &\leq (a+b+e)p(x_{2n-1}, x_{2n}) + (c+e)p(x_{2n}, x_{2n+1}), \quad \text{using (p4)}. \end{aligned} \tag{2.22}$$

Now, we claim that

$$p(x_{n+1}, x_n) \leq p(x_n, x_{n-1}), \quad \forall\, n \in \mathbb{N}^*. \tag{2.23}$$

If $p(x_{2n+1}, x_{2n}) > p(x_{2n}, x_{2n-1})$ for some $n \in \{1, 2, ...\}$, then

$$u(x_{2n-1}, x_{2n}) \leq (a+b+c+2e)p(x_{2n+1}, x_{2n}) \leq p(x_{2n+1}, x_{2n}),$$

and so by (2.21) we have

$$\psi(p(x_{2n}, x_{2n+1})) \leq \psi(p(x_{2n+1}, x_{2n})) - \varphi(u(x_{2n-1}, x_{2n})),$$

so $u(x_{2n-1}, x_{2n}) = 0$, then from (2.22), we get

$$(a+b)p(x_{2n-1}, x_{2n}) + cp(x_{2n}, x_{2n+1}) + e[p(x_{2n}, x_{2n}) + p(x_{2n-1}, x_{2n+1})] = 0.$$

Having in mind that $e > 0$ and $a > 0$, hence

$$p(x_{2n-1}, x_{2n}) = p(x_{2n}, x_{2n}) + p(x_{2n-1}, x_{2n+1}) = 0. \tag{2.24}$$

Since $p(x_{2n}, x_{2n}) \leq p(x_{2n-1}, x_{2n})$, then $p(x_{2n}, x_{2n}) = 0$, so by (2.24),

$$p(x_{2n-1}, x_{2n+1}) = 0. \tag{2.25}$$

By assumption, we have $p(x_{2n+1}, x_{2n}) > 0 = p(x_{2n}, x_{2n-1})$. On the other hand, by property (p4)

$$\begin{aligned} 0 &< p(x_{2n+1}, x_{2n}) \leq p(x_{2n+1}, x_{2n-1}) + p(x_{2n-1}, x_{2n}) \\ &= p(x_{2n+1}, x_{2n-1}) + 0 = p(x_{2n+1}, x_{2n-1}), \end{aligned}$$

hence $p(x_{2n+1}, x_{2n-1}) > 0$, which is a contradiction with respect to (2.25). So we have $p(x_{2n+1}, x_{2n}) \leq p(x_{2n}, x_{2n-1})$ for all $n \in \mathbb{N}^*$. Similarly, we have

$$p(x_{2n+1}, x_{2n+2}) \leq p(x_{2n}, x_{2n+1}).$$

Therefore, (2.23) holds for any $n \in \mathbb{N}^*$. Hence, the sequence $\{p(x_{n+1}, x_n)\}$ is nonincreasing and bounded below. Thus there exists $\rho \geq 0$ such that $\lim_{n\to+\infty} p(x_{n+1}, x_n) = \rho$. In particular, we give

$$\lim_{n\to+\infty} p(x_{2n}, x_{2n+1}) = \lim_{n\to+_i i} p(x_{2n-1}, x_{2n}) = \rho.$$

Suppose that $\rho > 0$. Therefore, from (2.22)

$$\begin{aligned} \limsup_{n\to+\infty} ap(x_{2n-1}, x_{2n}) &\leq \limsup_{n\to+\infty} u(x_{2n-1}, x_{2n}) \\ &\leq \limsup_{n\to+\infty} \{(a+b+e)p(x_{2n-1}, x_{2n}) + (c+e)p(x_{2n}, x_{2n+1})\}. \end{aligned}$$

This implies $0 < a\rho \leq \limsup_{n\to+\infty} u(x_{2n-1}, x_{2n}) \leq (a+b+c+2e)\rho \leq \rho$ and so there exist $\rho_1 > 0$ and a subsequence $\{u(x_{2n(k)-1}, x_{2n(k)})\}$ of $\{u(x_{2n-1}, x_{2n})\}$ such that

$$\lim_{k\to+\infty} u(x_{2n(k)-1}, x_{2n(k)}) = \rho_1 \leq \rho.$$

By the lower semi-continuity of φ we have

$$\varphi(\rho_1) \leq \liminf_{k\to+\infty} \varphi(u(x_{2n(k)-1}, x_{2n(k)})).$$

Now, by (2.19), we have

$$\begin{aligned}\psi(p(x_{2n(k)}, x_{2n(k)+1})) =& \psi(p(Tx_{2n(k)-1}, Sx_{2n(k)})) \\ \leq& \psi(u(x_{2n(k)-1}, x_{2n(k)})) - \varphi(u(x_{2n(k)-1}, x_{2n(k)})),\end{aligned}$$

and taking the upper limit as $k \to +\infty$, we have

$$\begin{aligned}\psi(\rho) \leq& \psi(\rho_1) - \liminf_{k\to+\infty} \varphi(u(x_{2n(k)-1}, x_{2n(k)})) \\ \leq& \psi(\rho_1) - \varphi(\rho_1) \\ \leq& \psi(\rho) - \varphi(\rho_1),\end{aligned}$$

so $\varphi(\rho_1) = 0$, which is a contradiction. Therefore, we have

$$\lim_{n\to+\infty} p(x_{n+1}, x_n) = \rho = 0. \tag{2.26}$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence in (X, p). Again, from Lemma 1.5, we need to check that $\{x_n\}$ is Cauchy in (X, p^s). To do this, it suffices to prove that $\{x_{2n}\}$ is Cauchy in (X, p^s). We proceed by contradiction. Then we can find an $\varepsilon > 0$ such that for each even integer $2k$ there exist even integers $2m(k) > 2n(k) > 2k$ such that

$$p^s(x_{2n(k)}, x_{2m(k)}) \geq \varepsilon. \tag{2.27}$$

By choosing $2m(k)$ to be smallest number exceeding $2n(k)$ for which (2.27) holds, we may also assume

$$p^s(x_{2m(k)-2}, x_{2n(k)}) < \varepsilon. \tag{2.28}$$

Now, (2.27) and (2.28) imply

$$\begin{aligned}0 < \varepsilon &\leq p^s(x_{2n(k)}, x_{2m(k)}) \\ &\leq p^s(x_{2n(k)}, x_{2m(k)-2}) + p^s(x_{2m(k)-2}, x_{2m(k)-1}) + p^s(x_{2m(k)-1}, x_{2m(k)}) \\ &< \varepsilon + p^s(x_{2m(k)-2}, x_{2m(k)-1}) + p^s(x_{2m(k)-1}, x_{2m(k)}),\end{aligned}$$

and so thanks to (2.26)

$$\lim_{k\to+\infty} p^s(x_{2n(k)}, x_{2m(k)}) = \varepsilon. \tag{2.29}$$

Also, by the triangular inequality,

$$|p^s(x_{2n(k)}, x_{2m(k)-1}) - p^s(x_{2n(k)}, x_{2m(k)})| \leq p^s(x_{2m(k)-1}, x_{2m(k)}),$$

and

$$|p^s(x_{2n(k)+1}, x_{2m(k)-1}) - p^s(x_{2n(k)}, x_{2m(k)})| \leq p^s(x_{2m(k)-1}, x_{2m(k)}) + p^s(x_{2n(k)}, x_{2n(k)+1}).$$

Therefore we get

$$\lim_{k\to+\infty} p^s(x_{2n(k)}, x_{2m(k)-1}) = \varepsilon, \tag{2.30}$$

and

$$\lim_{k\to+\infty} p^s(x_{2n(k)+1}, x_{2m(k)-1}) = \varepsilon. \tag{2.31}$$

On the other hand, by definition of p^s, as in (2.10) and (2.11), we get from (2.26), (2.29), (2.30) and (2.31)

$$\lim_{k\to+\infty} p(x_{2n(k)}, x_{2m(k)}) = \frac{\varepsilon}{2}, \tag{2.32}$$

$$\lim_{k\to+\infty} p(x_{2n(k)}, x_{2m(k)-1}) = \frac{\varepsilon}{2}, \tag{2.33}$$

$$\lim_{k\to+\infty} p(x_{2n(k)+1}, x_{2m(k)-1}) = \frac{\varepsilon}{2}. \tag{2.34}$$

On the other hand, since $x_{2n(k)}$ and $x_{2m(k)-1}$ are comparable, we can use the condition (2.19) for these points. By, (2.26), (2.32), (2.33) and (2.34)

$$\begin{aligned}
\lim_{k\to+\infty} u(x_{2m(k)-1}, x_{2n(k)}) &= \lim_{k\to+\infty} \Big(ap(x_{2m(k)-1}, x_{2n(k)}) + bp(x_{2m(k)-1}, Tx_{2m(k)-1}) \\
&\quad + cp(x_{2n(k)}, Sx_{2n(k)}) + e[p(x_{2n(k)}, Tx_{2m(k)-1}) \\
&\quad + p(x_{2m(k)-1}, Sx_{2n(k)})] \Big) \\
&= \lim_{k\to+\infty} \Big(ap(x_{2m(k)-1}, x_{2n(k)}) + bp(x_{2m(k)-1}, x_{2m(k)}) \\
&\quad + cp(x_{2n(k)}, x_{2n(k)+1}) + e[p(x_{2n(k)}, x_{2m(k)}) \\
&\quad + p(x_{2m(k)-1}, x_{2n(k)+1})] \Big) \\
&= (a+2e)\frac{\varepsilon}{2},
\end{aligned}$$

then we have

$$\begin{aligned}
\psi(\frac{\varepsilon}{2}) &= \limsup_{k\to+\infty} \psi(p(x_{2n(k)}, x_{2m(k)})) \\
&\le \limsup_{k\to+\infty} \psi(p(x_{2n(k)}, x_{2n(k)+1}) + p(x_{2n(k)+1}, x_{2m(k)})) \\
&\le \limsup_{k\to+\infty} \psi(p(x_{2n(k)}, x_{2n(k)+1}) + p(Sx_{2n(k)}, Tx_{2m(k)-1})) \\
&= \limsup_{k\to+\infty} \psi(p(Sx_{2n(k)}, Tx_{2m(k)-1})) \\
&\le \limsup_{k\to+\infty} [\psi(u(x_{2m(k)-1}, x_{2n(k)})) - \varphi(u(x_{2m(k)-1}, x_{2n(k)}))] \\
&= \psi((a+2e)\frac{\varepsilon}{2}) - \liminf_{k\to+\infty} \varphi(u(x_{2m(k)-1}, x_{2n(k)})) \\
&\le \psi((a+2e)\frac{\varepsilon}{2}) - \varphi((a+2e)\frac{\varepsilon}{2}) \\
&\le \psi(\frac{\varepsilon}{2}) - \varphi((a+2e)\frac{\varepsilon}{2}).
\end{aligned}$$

This is a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s), which is complete from Lemma 1.5. Then there is $z \in X$ such that

$$\lim_{n\to+\infty} p^s(x_n, z) = 0.$$

Again, from Lemma 1.5, we have thanks to (2.26) and the condition (p2)

$$p(z,z) = \lim_{n\to+\infty} p(x_n, z) = \lim_{n\to+\infty} p(x_n, x_n) = 0. \tag{2.35}$$

We will prove that $Tz = z$.
1. Assume that (i) holds, that is T is continuous in (X, p). In view of Remark 1.8, we have T is continuous in (X, p^s). Since the sequence $\{x_{2n+1}\}$ converges in (X, p^s) to z, hence $\{Tx_{2n+1}\}$ converges to Tz in (X, p^s), that is from Lemma 1.5

$$p(Tz, Tz) = \lim_{n\to+\infty} p(Tx_{2n+1}, Tz) = \lim_{n\to+\infty} p(Tx_{2n+1}, Tx_{2n+1}) = \lim_{n\to+\infty} p(x_{2n+2}, x_{2n+2}) = 0 \quad [\text{by } (2.26)]. \tag{2.36}$$

Again, thanks to (2.35)-(2.36),

$$p(z, Tz) = \lim_{n\to+\infty} p(x_{2n+2}, Tz) = \lim_{n\to+\infty} p(Tx_{2n+1}, Tz) = p(Tz, Tz). \tag{2.37}$$

It follows that $p(z, Tz) = 0$. Hence $z = Tz$, that is, z is a fixed point of T.
2. Assume that S is continuous. The proof of $Sz = z$ will be done similarly as in the first case (i).
3. Assume that (iii) holds. Then, we have $x_{2n} \leq z$ for all n. Therefore, we can use the inequality (2.19) for x_{2n} and z.

$$\begin{aligned}\psi(p(Tz, x_{2n+1})) &= \psi(p(Tz, Sx_{2n}))\\ &\leq \psi(u(z, x_{2n})) - \varphi(u(z, x_{2n})),\end{aligned}$$

where

$$\begin{aligned}u(z, x_{2n}) &= ap(z, x_{2n}) + bp(z, Tz) + cp(x_{2n}, Sx_{2n}) + e[p(x_{2n}, Tz) + p(z, Sx_{2n})]\\ &= ap(z, x_{2n}) + bp(z, Tz) + cp(x_{2n}, x_{2n+1}) + e[p(x_{2n}, Tz) + p(z, x_{2n+1})].\end{aligned}$$

Thanks to (2.35), we get

$$\lim_{n\to+\infty} u(z, x_{2n}) = (b + e)p(z, Tz) \leq p(z, Tz).$$

Therefore, taking the upper limit as $n \to +\infty$, we obtain using the properties of ψ and φ

$$\psi(p(Tz, z)) \leq \psi(p(z, Tz)) - \varphi((b + e)p(z, Tz)),$$

giving that $p(z, Tz) = 0$, so $Tz = z$.

We have proved that z is a fixed point of a one mapping in each precedent case. Now we show that, such z is also a common fixed point of S and T. Indeed, without loss of generality, we take z be a fixed point of S. Now assume that $p(z, Tz) > 0$. If we use the inequality (2.19), for $x = y = z$, we have

$$\begin{aligned}\psi(p(Tz, z)) &= \psi(p(Tz, Sz))\\ &\leq \psi(u(z, z)) - \varphi(u(z, z))\\ &\leq \psi(p(Tz, z)) - \varphi((a + c + e)p(z, z) + (b + e)p(Tz, z))\\ &= \psi(p(Tz, z)) - \varphi((b + e)p(Tz, z)).\end{aligned}$$

We have used (2.35) in the last identity, that is, $p(z, z) = 0$. It follows that $\varphi((b + e)p(Tz, z)) = 0$, so by a property of φ, we have $(b + e)p(Tz, z) = 0$ for $e > 0$, that is $p(Tz, z) = 0$, which is a contradiction because we assumed that $p(z, Tz) > 0$. Thus $p(z, Tz) = 0$ and so z is a common fixed point of S and T. The proof of Theorem 2.8 is completed. □

3. APPLICATION

In this section, we present some applications of previous results and we obtain some fixed point theorems for single mapping and pair of mappings satisfying a general contractive condition of integral type in ordered partial metric spaces. Take Γ to be the set of
$v : \mathbb{R}_+ \to \mathbb{R}_+$ which are Lebesgue integrable mappings, summable, nonnegative and satisfy

$$\int_0^{\varepsilon} v(t)dt > 0 \quad \text{for each} \quad \varepsilon > 0.$$

Theorem 3.1. *Let $(X, \leq)$ be a partially ordered set and (X, p) be a complete partial metric space. Suppose that $T : X \to X$ be a nondecreasing mapping such that for every two comparable elements $x, y \in X$*

$$\int_0^{\psi(p(Tx,Ty))} v(t)dt \leq \int_0^{\psi(\theta(x,y))} v(t)dt - \int_0^{\varphi(\theta(x,y))} v(t)dt, \tag{3.1}$$

where

$$\theta(x, y) = ap(x, y) + bp(x, Tx) + cp(y, Ty) + e[p(y, Tx) + p(x, Ty)], \tag{3.2}$$

with $a, e > 0$; b, $c \geq 0$, $a + b + c + 2e \leq 1$, and $\psi, \varphi : [0, +\infty) \to [0, +\infty)$, ψ is a continuous, nondecreasing, φ is a lower semi-continuous functions and $\psi(t) = 0 = \varphi(t)$ if and only if $t = 0$. Also suppose, there exists $x_0 \in X$ with $x_0 \leq Tx_0$. Assume that :
(i) T is continuous, or
(ii) if a nondecreasing sequence $\{x_n\}$ converges to x in (X, p), then $x_n \leq x$ for all n.
Then T has a fixed point.

Proof. Define $\triangle : \mathbb{R}_+ \to \mathbb{R}_+$ by $\triangle(x) = \int_0^x v(t)dt$, then $\triangle$ is continuous and nondecreasing with $\triangle(0) = 0$. Thus, equation (3.1) becomes

$$\triangle(\psi(p(Tx, Ty))) \leq \triangle(\psi(\theta(x, y))) - \triangle(\varphi(\theta(x, y)))$$

which further can be written as

$$\psi_1(p(Tx, Ty)) \leq \psi_1(\theta(x, y) - \varphi_1(\theta(x, y)),$$

where $\psi_1 = \triangle \circ \psi$ and $\varphi_1 = \triangle \circ \varphi$. Hence, Theorem 2.1 yields a fixed point. □

Theorem 3.2. *Let $(X, \leq)$ be a partially ordered set and (X, p) be a complete partial metric space. Suppose that $T, S : X \to X$ are weakly increasing such that for every two comparable elements $x, y \in X$*

$$\int_0^{\psi(p(Tx,Sy))} v(t)dt \leq \int_0^{\psi(u(x,y))} v(t)dt - \int_0^{\varphi(u(x,y))} v(t)dt, \tag{3.3}$$

where

$$u(x, y) = ap(x, y) + bp(x, Tx) + cp(y, Sy) + e[p(y, Tx) + p(x, Sy)], \tag{3.4}$$

with $a, e > 0$; b, $c \geq 0$, $a + b + c + 2e \leq 1$, and $\psi, \varphi : [0, +\infty) \to [0, +\infty)$, ψ is a continuous, nondecreasing, φ is a lower semi-continuous functions and $\psi(t) = 0 = \varphi(t)$ if and only if $t = 0$. Assume that :
(i) T is continuous, or
(ii) S is continuous, or
(iii) if a nondecreasing sequence $\{x_n\}$ converges to x in (X, p), then $x_n \leq x$ for all n.
Then T and S have a common fixed point.

Proof. Define $\triangle : \mathbb{R}_+ \to \mathbb{R}_+$ by $\triangle(x) = \int_0^x v(t)dt$, then $\triangle$ is continuous and nondecreasing with $\triangle(0) = 0$. Thus, equation (3.3) becomes

$$\triangle(\psi(p(Tx, Sy))) \leq \triangle(\psi(u(x, y))) - \triangle(\varphi(u(x, y)))$$

which further can be written as

$$\psi_1(p(Tx, Sy)) \leq \psi_1(u(x, y) - \varphi_1(u(x, y)),$$

where $\psi_1 = \triangle \circ \psi$ and $\varphi_1 = \triangle \circ \varphi$. Hence, Theorem 2.8 yields a common fixed point. □

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