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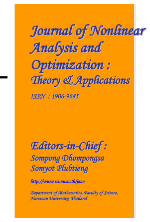

ON COMMON FIXED POINTS OF A NEW ITERATION FOR TWO NONSELF ASYMPTOTICALLY QUASI-NONEXPANSIVE-TYPE MAPPINGS IN BANACH SPACES

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ABSTRACT. Suppose that K is a nonempty closed convex subset of a real Banach space E which is also a nonexpansive retract of E. Let $T, S : K \to E$ be two nonself asymptotically quasi-nonexpansive-type mappings of E with $\mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in K$,

$$\begin{aligned} x_{n+1} &= P\left((1-a_n)x_n + a_n S(PS)^{n-1}\left((1-\beta_n)y_n + \beta_n S(PS)^{n-1} y_n\right)\right) \\ y_n &= P\left((1-b_n)x_n + b_n T(PT)^{n-1}\left((1-\gamma_n)x_n + \gamma_n T(PT)^{n-1} x_n\right)\right), \ n \geq 1, \end{aligned}$$

where $\{a_n\}, \{b_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0,1]$. In this paper, we study the strongly converges to a common fixed point of the a new iterative scheme for two nonself asymptotically quasi-nonexpansive-type mappings in Banach spaces. The results obtained in this paper extend and improve the recent ones announced by Tan and Xu [16], Shahzad [12], Thianwan [15], Kiziltunc et al. [17] and many others.

KEYWORDS : Nonself asymptotically quasi-nonexpansive-type mapping; strong convergence; common fixed points; iterative method; Banach space.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we assume that E is a real Banach space and K is a nonempty closed convex subset of E and $\mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$ denote the set of common fixed points of mappings T and S.

A mapping $T : K \to K$ is called *nonexpansive mapping* if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for all $x, y \in K$.

A mapping $T : K \to K$ is called $quasi-nonexpansive\ mapping$ if $F(T) \neq \varnothing$ and

$$\|Tx - x^*\| \leq \|x - x^*\| \tag{1.2}$$

for all $x \in K$ and $x^* \in F(T)$.

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A mapping $T : K \to K$ is called *asymptotically nonexpansive mapping* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\| \tag{1.3}$$

for all $x, y \in K$ and $n \ge 1$.

A mapping $T : K \to K$ is called *asymptotically quasi − nonexpansive mapping* if $F(T) \neq \varnothing$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - x^*\| \le k_n \|x - x^*\| \tag{1.4}$$

for all $x \in K, x^* \in F(T)$ and $n \ge 1$.

In [1], a mapping $T : K \to K$ is called *asymptotically nonexpansive − type mapping* if

$$\lim_{n\to\infty} \sup \left\{ \sup_{x,y\in K} \left\{ \|T^n x - T^n y\|^2 - \|x - y\|^2 \right\} \right\} \le 0,\ n \ge 1. \tag{1.5}$$

In [1], a mapping $T : K \to K$ is called *asymptotically quasi − nonexpansive − type mapping* if $F(T) \neq \varnothing$ and

$$\lim_{n\to\infty} \sup \left\{ \sup_{x\in K,\, x^*\in F(T)} \left\{ \|T^n x - x^*\|^2 - \|x - x^*\|^2 \right\} \right\} \le 0,\ n \ge 1. \tag{1.6}$$

From above definitions, if $F(T)$ is nonempty, a quasi-nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings and asymptotically nonexpansive-type mappings are all special cases of asymptotically quasi-nonexpansive-type mappings. But the converse does not hold.

The class of asymptotically nonexpansive mappings is a natural generalization of the important class of nonexpansive mappings. Goebel and Kirk [2] proved that if K is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self-mapping has a fixed point.

Iterative techniques for asymptotically nonexpansive self-mappings in Banach spaces including Mann type and Ishikawa type iteration processes have been studied extensively by various authors, see for example [[2]-[6]]. However, if the domain of T, $D(T)$, is a proper subset of E (and this is the case in several applications), and T map $D(T)$ into E, then the iteration processes of Mann type and Ishikawa type studied by these authors, and their modifications introduced may fail to be well defined.

A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to K$ such that $Px = x$, for all $x \in K$. Every closed convex set of a uniformly convex Banach space is a retract. A map $P : E \to K$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all $y \in R(P)$ in the range of P.

For nonself nonexpansive mappings, some authors (see, e.g., [[7],[8]]) have studied the strong and weak convergence theorems in Hilbert space or uniformly convex Banach spaces. The concept of nonself asymptotically nonexpansive mappings was introduced by Chidume [9] in 2003 as the generalization of asymptotically nonexpansive nonself-mappings. The nonself asymptotically nonexpansive mapping is defined as follows:

Definition 1.1. (see [9]) Let K a nonempty subset of real normed linear space E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A mapping $T : K \to E$ is called *nonself asymptotically nonexpansive* if there exists sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ $(n \to \infty)$ such that

$$\left\| T\left(PT\right)^{n-1} x - T\left(PT\right)^{n-1} y \right\| \leq k_n \left\| x - y \right\| \tag{1.7}$$

for all $x, y \in K$ and $n \geq 1$. If T is a self-mapping, then P becomes the identity mapping so that (1.7) reduces (1.1).

T is said to be *a nonself asymptotically quasi − nonexpansive mapping*, if $F(T) \neq \varnothing$ and there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ $(n \to \infty)$ such that

$$\left\| T\left(PT\right)^{n-1} x - x^* \right\| \leq k_n \left\| x - x^* \right\| \tag{1.8}$$

for all $x \in K$, $x^* \in F(T)$ and $n \geq 1$.

In [10], T is said to be *a nonself asymptotically nonexpansive − type mapping*, if

$$\lim_{n\to\infty} \sup \left\{ \sup_{x,\, y \in K} \left\{ \left\| T\left(PT\right)^{n-1} x - T\left(PT\right)^{n-1} y \right\| - \left\| x - y \right\| \right\} \right\} \leq 0, \;\; n \geq 1. \tag{1.9}$$

In [10], T is said to be *a nonself asymptotically quasi − nonexpansive − type mapping*, if $F(T) \neq \varnothing$ and

$$\lim_{n\to\infty} \sup \left\{ \sup_{x \in K,\, x^* \in F(T)} \left\{ \left\| T\left(PT\right)^{n-1} x - x^* \right\| - \left\| x - x^* \right\| \right\} \right\} \leq 0, \;\; n \geq 1. \tag{1.10}$$

Remark 1.2. It follows from above Definition 1.1 that,

i. If $T : K \to E$ is a nonself asymptotically nonexpansive mapping, then T is a nonself asymptotically nonexpansive-type mapping;

ii. If $F(T) \neq \varnothing$ and $T : K \to E$ is a nonself asymptotically quasi-nonexpansive mapping, then T is a nonself asymptotically quasi-nonexpansive-type mapping;

iii. If $F(T) \neq \varnothing$ and $T : K \to E$ is a nonself asymptotically nonexpansive-type mapping, then T is a nonself asymptotically quasi-nonexpansive-type mapping.

Remark 1.3. Observe again that

$$\limsup_{n\to\infty} \left(\sup_{x \in K,\, x^* \in F(T)} \left\{ \left\| T\left(PT\right)^{n-1} x - x^* \right\|^2 - \left\| x - x^* \right\|^2 \right\} \right) \leq 0$$

implies

$$\limsup_{n\to\infty} (\sup_{x \in K,\, x^* \in F(T)} \left\{ \left(\left\| T\left(PT\right)^{n-1} x - x^* \right\| - \left\| x - x^* \right\| \right) (\left\| T\left(PT\right)^{n-1} x - x^* \right\| + \left\| x - x^* \right\| \right) \right\}) \leq 0$$

which implies

$$\limsup_{n\to\infty} \left(\sup_{x \in K,\, x^* \in F(T)} \left\{ \left\| T\left(PT\right)^{n-1} x - x^* \right\| - \left\| x - x^* \right\| \right\} \right) \leq 0\,.$$

Similarly, we can show that

$$\limsup_{n\to\infty} \left(\sup_{x \in K,\, x^* \in F(S)} \left\{ \left\| S\left(PS\right)^{n-1} x - x^* \right\| - \left\| x - x^* \right\| \right\} \right) \leq 0.$$

Suantai [11] defined a new three-step iterations which is an extension of Noor iterations and gave some weak and strong convergence theorems of such iterations for asymptotically nonexpansive mappings in uniformly convex Banach spaces. Recently, Shahzad [12] extended Tan and Xu results [11, Theorem 1, p. 305] to the case of nonexpansive nonself-mapping in a uniformly convex Banach space. Peng [13] proved the convergence of finite steps iterative sequences with mean errors for asymptotically quasi-nonexpansive mappings in Banach spaces. In the same year, Yang [14] introduced a modified multistep iterative process for some common fixed point of a finite family of nonself asymptotically nonexpansive mappings on nonempty closed convex bounded subsets of a real uniformly convex Banach space. Thianwan [15] defined a weak and strong convergence theorems for new iterations with errors for nonexpansive nonself-mapping in a uniformly convex Banach space. In 2009, a new iterative scheme which is called the projection type Ishikawa iteration for two asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space was defined and constructed by Thianwan [18]. He gave some strong and weak convergence theorems of such iterations under some suitable conditions in a uniformly convex Banach space. In 2010, Wariam Chuayjan, Sornsak Thianwan and Boriboon Novaprateep [19] introduce and study a new type of multi-step iterative sequence with errors for a finite family of asymptotically quasi-nonexpansive-type nonself-mappings which can be viewed as an extension for Ishikawa type iterative schemes of Thianwan [18] and they proved strong convergence of a multi-step iterative scheme with errors to a common fixed point of a finite family of asymptotically quasi-nonexpansive-type nonself-mappings on nonempty closed convex subset of a real Banach space. In [1], Quan et al. proved approximation common fixed point of asymptotically quasi-nonexpansive-type mappings by the finite steps iterative sequences. In [10], Tian et al. introduced on the approximation problem of common fixed points for a finite family of nonself asymptotically quasi-nonexpansive-type mappings in Banach spaces.

Inspired and motivated by this facts, I define and study the convergence theorems of a new two steps iterative sequences for nonself asymptotically quasi-nonexpansive-type mappings in Banach spaces.

Let E be a Banach space and K be a nonempty closed convex subset of E, $P : E \to K$ the nonexpansive retraction of E onto K, and $T, S : K \to E$ be two nonself asymptotically quasi-nonexpansive-type mappings of E with sequences $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$ as $n \to \infty$, and $\mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in K$,

$$\begin{aligned} x_{n+1} &= P\left((1-a_n)x_n + a_n S(PS)^{n-1}\left((1-\beta_n)y_n + \beta_n S(PS)^{n-1}y_n\right)\right), \quad (1.11)\\ y_n &= P\left((1-b_n)x_n + b_n T(PT)^{n-1}\left((1-\gamma_n)x_n + \gamma_n T(PT)^{n-1}x_n\right)\right), \ n \geq 1, \end{aligned}$$

where $\{a_n\}, \{b_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0, 1]$.

The purpose of this paper is to study the convergence theorems of a new two steps iterative sequences for nonself asymptotically quasi-nonexpansive-type mappings in Banach spaces. The results of this paper can be viewed as an improve and extend the corresponding results of Tan and Xu [16], Shahzad [12], Thianwan [15], Kiziltunc et al. [17] and many others.

In the sequel, we need the following well known lemma to prove our main results.

Lemma 1.4. *(see [16]) Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)\, a_n + b_n, \qquad n \geq 1,$$

if $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, then

(i) $\lim\limits_{n\to\infty} a_n$ exists;

(ii) In particular, if $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim\limits_{n\to\infty} a_n = 0$.

2. MAIN RESULTS

In this section, we prove the convergence theorem of two steps iterative sequences of the a new iterative scheme (1.11) for nonself asymptotically quasi-nonexpansive-type mappings in Banach spaces.

Theorem 2.1. *Let E be a real Banach space and K a nonempty closed subset of E which is also a nonexpansive retract with retraction P. Let $T, S : K \to E$ be two nonself asymptotically quasi-nonexpansive-type mappings of E and $\mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$. Suppose that $\{a_n\}, \{b_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0,1]$. Starting from an arbitrary $x_1 \in K$, define the sequences $\{x_n\}$ and $\{y_n\}$ by the recursion (1.11). Then $\{x_n\}$ strongly converges to a common fixed point of T and S in E if and only if*

$$\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0. \tag{2.1}$$

Proof. We have that

$$\limsup_{n\to\infty} \left(\sup_{x\in K, x^*\in\mathcal{F}} \left\{ \left\| T(PT)^{n-1} x - x^* \right\| - \|x - x^*\| \right\} \right) \leq 0.$$

Similarly, we have that

$$\limsup_{n\to\infty} \left(\sup_{x\in K, x^*\in\mathcal{F}} \left\{ \left\| S(PS)^{n-1} x - x^* \right\| - \|x - x^*\| \right\} \right) \leq 0.$$

This implies that for any given $\varepsilon > 0$, there exists a positive integer n_0 such that for $n \geq n_0$ and $x^* \in \mathcal{F}$, we have that

$$\sup_{x\in K, x^*\in\mathcal{F}} \left\{ \left\| T(PT)^{n-1} x - x^* \right\|^2 - \|x - x^*\|^2 \right\} \leq \varepsilon$$

and similarly we have that

$$\sup_{x\in K, x^*\in\mathcal{F}} \left\{ \left\| S(PS)^{n-1} x - x^* \right\|^2 - \|x - x^*\|^2 \right\} \leq \varepsilon.$$

The necessity of (2.1) is obvious. Next we prove the sufficiency of (2.1). For $x^* \in \mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$. Since $\{x_n\}, \{y_n\} \subset E$, then we have

$$\left\| S(PS)^{n-1} y_n - x^* \right\| - \|y_n - x^*\| \leq \varepsilon,\ \forall x^* \in \mathcal{F},\ \forall n \geq n_0, \tag{2.2}$$

$$\left\| T(PT)^{n-1} x_n - x^* \right\| - \|x_n - x^*\| \leq \varepsilon,\ \forall x^* \in \mathcal{F},\ \forall n \geq n_0. \tag{2.3}$$

Set $\sigma_n = (1-\beta_n)\, y_n + \beta_n S(PS)^{n-1} y_n$ and $\xi_n = (1-\gamma_n)\, x_n + \gamma_n T(PT)^{n-1} x_n$. Thus for any $x^* \in \mathcal{F}$, using (1.11), (2.2) and (2.3) we have that

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \left\|P\left((1-a_n)x_n + a_n S(PS)^{n-1}(\sigma_n)\right) - x^*\right\| \\
&\leq \left\|(1-a_n)x_n + a_n S(PS)^{n-1}\left((1-\beta_n)y_n + \beta_n S(PS)^{n-1}y_n\right) - x^*\right\| \\
&\leq \left\|(1-a_n)x_n + a_n x^* - x^* + a_n\left(S(PS)^{n-1}(\sigma_n) - x^*\right)\right\| \\
&\leq a_n\varepsilon + a_n\|\sigma_n - x^*\| + (1-a_n)\|x_n - x^*\|.
\end{aligned} \tag{2.4}$$

Consider the second term in right-hand side of (2.4), using (1.11) and (2.2), we have that

$$\begin{aligned}
\|\sigma_n - x^*\| &= \left\|(1-\beta_n)y_n + \beta_n S(PS)^{n-1}y_n - x^*\right\| \\
&\leq (1-\beta_n)\|y_n - x^*\| + \beta_n\left\|S(PS)^{n-1}y_n - x^*\right\| \\
&\leq (1-\beta_n)\|y_n - x^*\| + \beta_n\varepsilon + \beta_n\|y_n - x^*\| \\
&= \|y_n - x^*\| + \beta_n\varepsilon.
\end{aligned} \tag{2.5}$$

Using a similar method, consider the first term in right-hand side of (2.5), together with (1.11) and (2.3), we have that

$$\begin{aligned}
\|y_n - x^*\| &= \left\|P\left((1-b_n)x_n + b_n T(PT)^{n-1}(\xi_n)\right) - x^*\right\| \\
&\leq \left\|(1-b_n)x_n + b_n T(PT)^{n-1}\left((1-\gamma_n)x_n + \gamma_n T(PT)^{n-1}x_n\right) - x^*\right\| \\
&\leq (1-b_n)\|x_n - x^*\| + b_n\left\|T(PT)^{n-1}\xi_n - x^*\right\| \\
&\leq b_n\varepsilon + b_n\|\xi_n - x^*\| + (1-b_n)\|x_n - x^*\|.
\end{aligned} \tag{2.6}$$

Consider the second term in right-hand side of (2.6), using (1.11) and (2.3), we have that

$$\begin{aligned}
\|\xi_n - x^*\| &= \left\|(1-\gamma_n)x_n + \gamma_n T(PT)^{n-1}x_n - x^*\right\| \\
&\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\left\|T(PT)^{n-1}x_n - x^*\right\| \\
&\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\varepsilon + \gamma_n\|x_n - x^*\| \\
&= \|x_n - x^*\| + \gamma_n\varepsilon.
\end{aligned} \tag{2.7}$$

From (2.4), (2.5), (2.6) and (2.7), we have that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq a_n\varepsilon + a_n\|\sigma_n - x^*\| + (1-a_n)\|x_n - x^*\| \\
&\leq a_n\left(\|y_n - x^*\| + \beta_n\varepsilon\right) + a_n\varepsilon + (1-a_n)\|x_n - x^*\| \\
&\leq a_n\left((b_n\varepsilon + b_n\|\xi_n - x^*\| + (1-b_n)\|x_n - x^*\|) + \beta_n\varepsilon\right) \\
&\quad + a_n\varepsilon + (1-a_n)\|x_n - x^*\| \\
&\leq a_n b_n\varepsilon + a_n b_n\|\xi_n - x^*\| + a_n(1-b_n)\|x_n - x^*\| + a_n\beta_n\varepsilon \\
&\quad + a_n\varepsilon + (1-a_n)\|x_n - x^*\| \\
&\leq a_n b_n\left(\|x_n - x^*\| + \gamma_n\varepsilon\right) + (1-a_n)\|x_n - x^*\| + a_n\beta_n\varepsilon \\
&\quad + a_n(1-b_n)\|x_n - x^*\| + a_n\varepsilon + a_n b_n\varepsilon \\
&\leq a_n b_n\|x_n - x^*\| + (1-a_n)\|x_n - x^*\| + a_n\beta_n\varepsilon \\
&\quad + a_n(1-b_n)\|x_n - x^*\| + a_n b_n\gamma_n\varepsilon + a_n\varepsilon + a_n b_n\varepsilon \\
&\leq \|x_n - x^*\| + a_n b_n\gamma_n\varepsilon + a_n\varepsilon + a_n b_n\varepsilon + a_n\beta_n\varepsilon.
\end{aligned} \tag{2.8}$$

Let $\varphi_n = a_n b_n \gamma_n \varepsilon + a_n \varepsilon + a_n b_n \varepsilon + a_n \beta_n \varepsilon$. Then $\sum_{n=1}^{\infty} \varphi_n < \infty$. Therefore, by (2.8), we have

$$\inf_{x^* \in \mathcal{F}} \|x_{n+1} - x^*\| \leq \inf_{x^* \in \mathcal{F}} \|x_n - x^*\| + \varphi_n, \quad \forall n \geq n_0. \tag{2.9}$$

It follows from (2.9) and $\sum_{n=1}^{\infty} \varphi_n < \infty$ that

$$d(x_{n+1}, \mathcal{F}) \leq d(x_n, \mathcal{F}) + \varphi_n. \tag{2.10}$$

By Lemma 1.4 and from (2.10), we know that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists. Because $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$, then we have that

$$\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0. \tag{2.11}$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence in E. In fact, from (2.9) that for any $n \geq n_0$, any $m \geq n_1$ and any $x^* \in \mathcal{F}$, we have that

$$\begin{aligned} \|x_{n+m} - x^*\| &\leq \|x_{n+m-1} - x^*\| + \varphi_{n+m-1} \\ &\leq \|x_{n+m-2} - x^*\| + (\varphi_{n+m-1} + \varphi_{n+m-2}) \\ &\leq \|x_{n+m-3} - x^*\| + (\varphi_{n+m-1} + \varphi_{n+m-2} + \varphi_{n+m-3}) \\ &\vdots \\ &\leq \|x_n - x^*\| + \sum_{k=n}^{n+m-1} \varphi_k. \end{aligned} \tag{2.12}$$

So by (2.12), we have that

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - x^*\| + \|x_n - x^*\| \\ &\leq 2\|x_n - x^*\| + \sum_{k=n}^{\infty} \varphi_k. \end{aligned} \tag{2.13}$$

By the arbitrariness of $x^* \in \mathcal{F}$ and from (2.13), we have

$$\|x_{n+m} - x_n\| \leq 2d(x_n, \mathcal{F}) + \sum_{k=n}^{\infty} \varphi_k, \quad \forall n \geq n_0. \tag{2.14}$$

For any given $\varepsilon > 0$, there exists a positive integer $n_1 \geq n_0$, such that for any $n \geq n_1$, $d(x_n, \mathcal{F}) < \frac{\varepsilon}{4}$ and $\sum_{k=n}^{\infty} \varphi_k < \frac{\varepsilon}{2}$, we have $\|x_{n+m} - x_n\| < \varepsilon$, and so for any $m \geq 1$

$$\lim_{n\to\infty} \|x_{n+m} - x_n\| = 0. \tag{2.15}$$

This show that $\{x_n\}$ is Cauchy sequence in K. Since K is a closed subset of E, and so it is complete. Hence, there exists a $q \in K$ such that $x_n \to q$ as $n \to \infty$.

Finally, we have to prove that $q \in \mathcal{F}$. By contradiction, we assume that q is not in $\mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$. Since $\mathcal{F}$ is a closed set, $d(q, \mathcal{F}) > 0$. Hence for all $q \in \mathcal{F}$, we have that

$$\|q - x^*\| \leq \|q - x_n\| + \|x_n - x^*\|. \tag{2.16}$$

This implies that

$$d(q, \mathcal{F}) \leq \|q - x_n\| + d(x_n, \mathcal{F}). \tag{2.17}$$

From (2.16) and (2.17) (as $n \to \infty$), we have that $d(q, \mathcal{F}) \leq 0$. This is a contradiction. Hence $q \in \mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$. This completes the proof of Theorem 2.1. $\square$

Theorem 2.2. *Let E be a real Banach space and K a nonempty closed subset of E which is also a nonexpansive retract with retraction P. Let T, $S : K \to E$ be two nonself asymptotically quasi-nonexpansive mappings of E and $\mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$. Suppose that $\{a_n\}, \{b_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0,1]$. Starting from an arbitrary $x_1 \in K$, define the sequences $\{x_n\}$ and $\{y_n\}$ by the recursion (1.11). Then $\{x_n\}$ strongly converges to a common fixed point of T and S in E if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.*

Proof. Since T, $S : K \to E$ are two nonself asymptotically quasi-nonexpansive mappings , by the definition they are nonself asymptotically quasi-nonexpansive-type mappings. The conclusion of Theorem 2.2 can be proved from Theorem 2.1 immediately. □

Theorem 2.3. *Let E be a real Banach space and K a nonempty closed subset of E which is also a nonexpansive retract with retraction P. Let T, $S : K \to E$ be two nonself asymptotically nonexpansive mappings of E and $\mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$. Suppose that $\{a_n\}, \{b_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0,1]$. Starting from an arbitrary $x_1 \in K$, define the sequences $\{x_n\}$ and $\{y_n\}$ by the recursion (1.11). Then $\{x_n\}$ strongly converges to a common fixed point of T and S in E if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.*

Proof. Since T, $S : K \to E$ are two nonself asymptotically nonexpansive mappings, taking $n = 1$ and $T = S$ in (1.11), we know that T, $S : K \to E$ are continuous nonself asymptotically nonexpansive mappings. Therefore, the conclusion of Theorem 2.3 can be proved from Theorem 2.1 immediately. □

Corollary 2.1. *Suppose the conditions in Theorem 2.1 are satisfied. Then the finite steps iterative sequence $\{x_n\}$ generated by the recursion (1.11) converges to common fixed point $x \in E$ iff there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to x.*

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