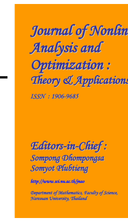
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 2, No. 2, (2011), 253-257
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



A VERSION OF KOLMOGOROV-ARNOLD REPRESENTATION THEOREM FOR DIFFERENTIABLE FUNCTIONS OF SEVERAL VARIABLES

SATOSHI KODAMA

Department of Information Sciences, Faculty of Science and Technology, Tokyo University of Science, 2641 Yamazaki Noda-city Chiba-prefecture, 278-8510, JAPAN

ABSTRACT. The most revised version of Kolmogorov-Arnold representation theorem shows that all continuous functions of several variables can be represented as superpositions which are constructed from the sums of continuous functions of one variables. In 2008, Kodama and Akashi show a relation between Kolmogorov-Arnold representation theorem and Vitušhkin theorem. In this paper, we discuss the problem asking what kind of differentiable functions of several variables can be represented as Kolmogorov-Arnold superpositions constructed from only several differentiable functions of one variable.

KEYWORDS : Kolmogorov-Arnold Representation; Differentiable Functions.

1. INTRODUCTION

Let $f(\cdot,\cdot,\cdot)$ be the function of three variable defined as

$$f(x,y,z) = xy + yz + zx, \quad x,y,z \in \mathbb{R},$$

where $\mathbb{R}$ is the set of all real numbers. Then, we can easily prove that there do not exist three functions $g(\cdot,\cdot)$, $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$ of two variables satisfying the following equality:

$$f(x,y,z) = g(u(x,y), v(x,z)), \quad x,y,z \in \mathbb{R}.$$

This fact shows us that $f(\cdot,\cdot,\cdot)$ cannot be represented as any 1-time nested superposition constructed from three real-valued functions of two variables. Actually, it is clear that the following equality holds:

$$f(x,y,z) = x(y+z) + yz, \quad x,y,z \in \mathbb{R}.$$

This result shows us that $f(\cdot,\cdot,\cdot)$ can be represented as a 2-time nested superposition.

In 1957, Kolmogorov and Arnold [**4**] solved Hilbert's 13th problem asking whether all continuous real-valued functions of several real variables can be represented as

Email address : j6309701@ed.noda.tus.ac.jp (S. Kodama).
Article history : Received 16 January 2011. Accepted 1 February 2011.

superpositions of functions of fewer variables or not. If the most revised version with respect to this problem is applied to the set of all real-valued functions of three variables, for any continuous real-valued function $f(\cdot,\cdot,\cdot)$ of three real variables, we can choose a continuous real-valued function $g_f(\cdot)$ of one variable, which is dependent only on $f(\cdot,\cdot,\cdot)$, and a family $\{\phi_{ij}(\cdot); 0 \le i \le 6, 1 \le j \le 3\}$ of twenty one real-valued functions of one variable, which is independent of $f(\cdot,\cdot,\cdot)$ satisfying

$$f(x_1, x_2, x_3) \;=\; \sum_{i=0}^{6} g_f\left(\sum_{j=1}^{3} \phi_{ij}(x_j)\right).$$

The above result, which is called Kolmogorov-Arnold representation theorem [**4**,**5**], implies that any continuous real-valued function of three real variables can be represented as a 7-time nested superposition of continuous real-valued functions of one real variables.

In 2008, Kodama and Akashi [**3**] show a relation between Kolmogorov-Arnold representation theorem and Vitušhkin theorem. In this paper, we discuss the problem asking what kind of differentiable functions of several variables can be represented as Kolmogorov-Arnold superpositions constructed from only several differentiable functions of one variable.

2. SUPERPOSITION REPRESENTATIONS DERIVED FROM KOLMOGOROV-ARNOLD REPERSENTATION

In this section, we consider a generalized version of Kolmogorov-Arnold theorem and a necessary condition enabling to discriminate functions of three variables being able to be represented as Kolmogorov-Arnold superposition from the other functions. Here, a relation between Kolmogorov-Arnold representation and Vitušhkin theorem, can be shown as the following:

Proposition 1. Not all differentiable functions of three variables can be represented as Kolmogorov-Arnold superpositions constructed from only differentiable functions of one variable.

Proof. Vitušhkin theorem [**6**] assures that there exists a differentiable function $v(\cdot,\cdot,\cdot)$ of three variables, which cannot be represented as any superposition constructed from several differentiable functions of two variables. Since Kolmogorov-Arnold theorem assures that all continuous functions of three variables can be represented as 7-time nested superposition constructed from several continuous functions of two variables. Therefore, if we apply Kolmogorov-Arnold representation to $v(\cdot,\cdot,\cdot)$, then we can obtain the following equality:

$$v(x_1, x_2, x_3) \;=\; \sum_{i=0}^{6} g_v\left(\sum_{j=1}^{3} \phi_{ij}(x_j)\right).$$

Here, if we can assume that all elements belonging to $\{g_v(\cdot); 0 \le i \le 6\}$ and $\{\phi_{ij}(\cdot); 0 \le i \le 6, 1 \le j \le 3\}$ are differentiable, then the above equality shows that $v(\cdot,\cdot,\cdot)$ can be represented as 7-time nested superposition. Now, we have a contradiction. □

Akashi [**2**] classified the concept of superposition irrepresentability, which plays an important role in Hilbert's 13th problem, into the following two concepts:

Strong superposition irrepresentability: There exists a function $s(\cdot,\cdot,\cdot)$ of three variables, which cannot be represented as any finite-time nested superposition constructed from several functions of two variables.

Weak superposition irrepresentability: For any positive integer k, there exists a function $s_k(\cdot,\cdot,\cdot)$ of three variables, which cannot be represented as any k-time nested superposition constructed from several functions of two variables.

For example, Vitušhkin proved that there exists a finitely differentiable function $s(\cdot,\cdot,\cdot)$ of three variables, which cannot be represented as any finite-time nested superposition constructed from several finitely differentiable functions of two variables, and Hilbert proved that, for any positive integer k, there exists a polynomial $s_k(\cdot,\cdot,\cdot)$ of three variables, which cannot be represented as any k-time nested superposition constructed from several polynomials of two variables. These results enables us to generalize Proposition 1 as the following:

Proposition 2. There exists a differentiable function $s(\cdot,\cdot,\cdot)$ of three variables, which cannot be represented as any finite-time nested superposition. Namely, for any positive integer n, for any family of differentiable functions $\{g_s^i(\cdot); 0 \le i \le n\}$ and for any family of differentiable functions $\{\phi_s^{ij}(\cdot); 0 \le i \le n, 1 \le j \le 3\}$, there exists a function $s(\cdot,\cdot,\cdot)$ of three variables satisfying the following inequality:

$$s(x_1,x_2,x_3) \neq \sum_{i=0}^{n} g_s^i\left(\sum_{j=1}^{3}\phi_s^{ij}(x_j)\right).$$

Proof. Let $v(\cdot,\cdot,\cdot)$ be one of the functions whose existence are proved by Vitušhkin **[6]** and assume that $v(\cdot,\cdot,\cdot)$ can be represented as the following:

$$v(x_1,x_2,x_3) = \sum_{i=0}^{n} g_v^i\left(\sum_{j=1}^{3}\phi_v^{ij}(x_j)\right).$$

Then, this equality shows that $v(\cdot,\cdot,\cdot)$ can be represented as a finite-time nested superposition, which contradicts Vitušhkin theorem. □

The above result shows that there exists a differentiable function $v(\cdot,\cdot,\cdot)$ of three variables, which cannot be represented as the Kolmogorov-Arnold superposition, if $g_v(\cdot)$ and $\phi_{01}(\cdot),\cdots,\phi_{63}(\cdot)$ are assumed to be differentiable. Actually, the above result cannot tell which functions of three variables can be represented as Kolmogorov-Arnold superpositions constructed from only differentiable functions of one variable. Therefore, we treat such a problem in the latter half of this paper.
Let $p(\cdot,\cdot,\cdot)$ and $q(\cdot,\cdot,\cdot)$ be differentiable functions of three variables which can be represented as n-time nested superpositions constructed from several differentiable functions of two variables, and let $r(\cdot,\cdot)$ be a differentiable functions of two variables. Then, the differentiable function $s(\cdot,\cdot,\cdot)$, which is defined as the following:

$$s(x_1,x_2,x_3) = r(p(x_1,x_2,x_3), q(x_1,x_2,x_3))$$

is said to be represented as an $n+1$-time nested superposition.

Proposition 3. Assume that there exists an element (x_1, x_2, x_3) belonging to $[0,1]^3$ satisfying $\prod_{i=1}^{3} s_i(x_1, x_2, x_3) \neq 0$ and that $s(\cdot,\cdot,\cdot)$ is differentiable and represented as the following superposition constructed from three differentiable functions $p(\cdot,\cdot)$, $q(\cdot,\cdot)$ and $r(\cdot)$:

$$s(\cdot,\cdot,\cdot) \;=\; r(p(\cdot,\cdot)+q(\cdot,\cdot)).$$

Then, for any constant c belonging to $[0,1]$, either $s_1(x_1,x_2,c)/s_2(x_1,x_2,c)$, $s_2(c,x_2,x_3)/s_3(c,x_2,x_3)$ or $s_3(x_1,c,x_3)/s_1(x_1,c,x_3)$ can be represented in the form of separation of two variables.

Proof. Without loss of generality, we can assume that $s(\cdot,\cdot,\cdot)$ is represented as the following:

$$s(x_1,x_2,x_3) \;=\; r(p(x_1,x_3)+q(x_2,x_3)).$$

Then, for any constant c belonging to $[0,1]$, we have

$$\frac{s_1(x_1,x_2,c)}{s_2(x_1,x_2,c)} \;=\; \frac{p_1(x_1,c)r'(p(x_1,c)+q(x_2,c))}{q_1(x_2,c)r'(p(x_1,c)+q(x_2,c))}.$$

This equality concludes the proof. □

Remark 1. Let $t(x_1,x_2,x_3)$ be the function of three variables defined as

$$t(x_1,x_2,x_3) \;=\; x_1^2 x_2^3 x_3^4.$$

This function can be represented as the following one-time nested superposition:

$$t(x_1,x_2,x_3) \;=\; \exp\left(\{\log x_1 + 3\log x_2\} + \{\log x_1 + 4\log x_3\}\right).$$

Let c be any constant belonging to $[0,1]$. Then, as for the function $t_1(\cdot,\cdot,c)/t_2(\cdot,\cdot,c)$, we have

$$\frac{t_1(x_1,x_2,c)}{t_2(x_1,x_2,c)} \;=\; \frac{2x_1}{3x_2}.$$

This equality shows that $t_1(\cdot,\cdot,c)/t_2(\cdot,\cdot,c)$ can be represented in the form of separation of two variables x_1 and x_2.

Remark 2. The condition presented in Proposition 3 is not a sufficient condition but a necessary one. For example, let $f_1(x_1,x_2,x_3)$ and $f_2(x_1,x_2,x_3)$ be the two functions of three variables defined as

$$f_1(x_1,x_2,x_3) = x_1x_2 + x_2x_3 + x_3x_1$$

and

$$f_2(x_1,x_2,x_3) = x_1x_2 + x_2x_3 + x_3x_1 + x_1^2 + x_2^2 + x_3^2\ ,$$

respectively. Then, the latter function satisfies the following representation:

$$f_2(x_1,x_2,x_3) = r(p(x_1,x_2)+q(x_1,x_3)),\quad 0 < x_1,x_2,x_3 < 1,$$

where $r(\cdot)$,$p(\cdot,\cdot)$ and $q(\cdot,\cdot)$ are defined as $r(x_1) = x_1^2$, $p(x_1,x_2) = x_1/2 + x_2$ and $q(x_1,x_3) = x_1/2 + x_3$, respectively, because we have

$$f_2(x_1,x_2,x_3) = \left\{\left(\frac{x_1}{2}+x_2\right)+\left(\frac{x_1}{2}+x_3\right)\right\}^2.$$

Nevertheless, it can be proved that we cannot find any functions $r(\cdot)$, $p(\cdot,\cdot)$ and $q(\cdot,\cdot)$ enabling the former function to be represented as

$$f_1(\cdot,\cdot,\cdot) = r(p(\cdot,\cdot) + q(\cdot,\cdot)).$$

Acknowledgments. The author would like to express his hearty thanks to Prof. S.Dhompongsa, Prof. S.Plubtieng, Prof. S.Suantai and the other members belonging to the organizing committee of the Second Asian Conference on Nonlinear Analysis and Optimization for kind hospitality while this conference was in session.

References

1. S. Akashi, A version of Hilbert's 13th problem for analytic functions, Bull. London Math., Soc.35(2003), 8-14.
2. S. Akashi, A version of Hilbert's 13th problem for entire functions, Taiwanese Mathematical Journal, vol.38(2008), 15-27.
3. S. Kodama and S. Akashi, Compressibility of Multidimensional Numerical Data Constructed from Smooth Functions of one variable, Proceedings of Asian Conference on Nonlinear Analysis and Optimization,vol.1(2008),173-177.
4. A. N. Kolmogorov, On the representation of continuous functions of several variables by superpositions of continuous functions of one variable and addition, Doklady, 114(1957), 679-681. (Russian)
5. G. G. Lorentz, Approximation of functions, Holt, Rinehart and Winston Inc., New York, 1966.
6. A. G. Vitušhkin, Some properties of linear superpositions of smooth functions, Doklady, 156(1964), 1003-1006. (Russian)