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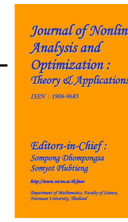

WEAK CONVERGENCE FOR A SEQUENCE OF NONEXPANSIVE MAPPINGS IN BANACH SPACES WITH GAUGE FUNCTIONS

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ABSTRACT. We investigate the convergence of Mann-type iteration for a sequence of nonexpansive mappings in the framework of a uniformly convex Banach space having the duality mapping j_φ, where φ is a gauge function on $[0,\infty)$. Our results improve and extend some well-known results in the literature.

KEYWORDS : Banach space; Common fixed point; Gauge function; Mann-type iteration.

1. INTRODUCTION AND PRELIMINARIES

Let K be a nonempty, closed and convex subset of a Banach space E. Let $T : K \to K$ be a mapping. Then T is said to be *nonexpansive* if $\|Tx-Ty\| \le \|x-y\|$ for all $x, y \in K$. We denote $F(T) = \{x \in K : x = Tx\}$ by the fixed points set of T.

One classical way to approximate a fixed point of a nonexpansive mapping was introduced, in 1953, by Mann [1] as follows: a sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \ge 1, \tag{1.1}$$

where $\alpha_n \in (0,1)$. Such a process is known as *Mann's iteration process*. Reich [2] showed that the sequence $\{x_n\}$ generated by (1.1) converges weakly to a fixed point of T if a real sequence $\{\alpha_n\}$ satisfies the condition $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n) = \infty$. This is valid in a uniformly convex Banach space with a Fréchet differentiable norm. Since 1953, the convergence of nonlinear mappings by Mann iteration process has been extensively studied by many authors (see also [3, 4]). However, we note that

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the convergence results of the process (1.1) have been widely established in spaces having the normalized duality mappings. By the way, Browder [5] initiated the study of certain classes of nonlinear operators by means of the duality mapping associated to a gauge function which includes the generalized and the normalized duality mappings as special cases.

Motivated by Browder [5] and Reich [2], we continue the work to study the convergence of Mann-type iteration in a much more general setting, a uniformly convex Banach space having the duality mapping associated to a gauge function.

Let E be a Banach space which admits the duality mapping associated to a gauge function and K a nonempty, closed and convex subset of E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings of K. We consider the following Mann-type iteration: $x_1 \in K$ and

$$x_{n+1} \quad = \quad \alpha_n x_n + (1-\alpha_n)T_n x_n, \quad n \geq 1, \tag{1.2}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$.

We recall that a Banach space E is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. E is called *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1-\delta)$ holds. The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf\Big\{1 - \Big\|\frac{1}{2}(x+y)\Big\| : \ \|x\|, \|y\| \leq 1, \ \|x-y\| \geq \epsilon\Big\},$$

for all $\epsilon \in [0,2]$. E is uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. It is known that every uniformly convex Banach space is strictly convex and reflexive. Let $S(E) = \{x \in E : \|x\| = 1\}$. E is said to be *smooth* if $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in S(E)$. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit is attained uniformly for $y \in S(E)$. See [6, 7, 8].

We need the following definitions and results which can be found in [5, 6, 7].

Definition 1.1. *A continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ is said to be gauge function if $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.*

Definition 1.2. *Let E be a normed space and φ a gauge function. Then the mapping $J_\varphi : E \to 2^{E^*}$ defined by*

$$J_\varphi(x) = \big\{f^* \in E^* : \ \langle x, f^*\rangle = \|x\|\varphi(\|x\|), \ \|f^*\| = \varphi(\|x\|)\big\}, \quad x \in E$$

is called the duality mapping with gauge function φ.

In the case $\varphi(t) = t^{q-1}$, $q > 1$, the duality mapping $J_\varphi = J_q$ is called the generalized duality mapping.

Remark 1.3. *For the gauge function φ, the function $\Phi : [0,\infty) \to [0,\infty)$ defined by $\Phi(t) = \int_0^t \varphi(s)ds$ is a continuous convex and strictly increasing function on $[0,\infty)$. Therefore, Φ has a continuous inverse function Φ^{-1}.*

Remark 1.4. *For each x in a Banach space E, $J_\varphi(x) = \partial\Phi(\|x\|)$, where ∂ denotes the sub-differential.*

Remark 1.5. *For each $x, y \in E$, the following inequality holds:*

$$\Phi\big(\|x+y\|\big) \leq \Phi\big(\|x\|\big) + \big\langle y, j_\varphi(x+y)\big\rangle, \quad j_\varphi(x+y) \in J_\varphi(x+y). \tag{1.3}$$

Remark 1.6. A Banach space E is smooth if and only if each duality mapping J_φ with gauge function φ is single-valued; in this case

$$\frac{d}{dt}\Phi(\|x+ty\|)\Big|_{t=0} = \lim_{t\to 0}\frac{\Phi(\|x+ty\|) - \Phi(\|x\|)}{t} = \big\langle y, J_\varphi(x)\big\rangle, \quad \forall x, y \in E.$$

Lemma 1.7. [9] *Assume that a Banach space E has a weakly continuous duality mapping with gauge φ. Then for any sequence $\{x_n\}$ that converges weakly to x, we have for any $y \in E$, $\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|x - y\|)$.*

Let K be a subset of a real Banach space E and let $\{T_n\}$ be a family of mappings of K such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}$ is said to satisfy *the AKTT-condition* [10] if for each bounded subset B of K, $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$.

Lemma 1.8. [10] *Let K be a nonempty and closed subset of a Banach space E and let $\{T_n\}$ be a family of mappings of K into itself which satisfies the AKTT-condition, then the mapping $T : K \to K$ defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in K$ satisfies*

$$\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in B\} = 0$$

for each bounded subset B of K.

In the sequel, we will write $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}$ satisfies the AKTT-condition and T is defined by Lemma 1.8 with $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

2. MAIN RESULTS

Proposition 2.1. *Let E be a reflexive Banach space having a weakly continuous duality mapping j_φ. Let $\{x_n\}$ be a sequence in E and $p, q \in \omega_w(\{x_n\})$. If $\lim_{n\to\infty} \|x_n - p\|$ and $\lim_{n\to\infty} \|x_n - q\|$ exist. Then $p = q$.*

Proof. Suppose $p, q \in \omega_w(\{x_n\})$ and $p \neq q$. Then there exist subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p$ and $x_{n_j} \rightharpoonup q$. By Lemma 1.7 and the continuity of Φ, we have

$$\begin{aligned}\lim_{n\to\infty} \Phi(\|x_n - p\|) &= \lim_{k\to\infty} \Phi(\|x_{n_k} - p\|) < \lim_{k\to\infty} \Phi(\|x_{n_k} - q\|) \\ &= \lim_{j\to\infty} \Phi(\|x_{n_j} - q\|) < \lim_{j\to\infty} \Phi(\|x_{n_j} - p\|) = \lim_{n\to\infty} \Phi(\|x_n - p\|),\end{aligned}$$

which is a contradiction. Thus $p = q$. □

Proposition 2.2. *Let E be a Banach space having a Fréchet differentiable norm and a duality mapping j_φ. Then there exists an increasing function $b : [0, \infty) \to [0, \infty)$ with $\lim_{t\to 0} \frac{b(t)}{t} = 0$ and*

$$\Phi(\|x\|) + \langle h, j_\varphi(x)\rangle \le \Phi(\|x + h\|) \le \Phi(\|x\|) + \langle h, j_\varphi(x)\rangle + b(\|h\|) \quad \forall x, h \in E. \tag{2.1}$$

Proof. Let $x \in E$ and define $b : [0, \infty) \to [0, \infty)$ by $b(0) = 0$ and

$$b(t) = \sup_{y \in S(E)} \left|\Phi(\|x + ty\|) - \Phi(\|x\|) - t\langle y, j_\varphi(x)\rangle\right|, \quad t > 0.$$

Then b is an increasing function. Since E has a Fréchet differentiable norm,

$$\lim_{t\to 0} \frac{b(t)}{t} = \lim_{t\to 0} \sup_{y\in S(E)} \left|\frac{\Phi(\|x + ty\|) - \Phi(\|x\|)}{t} - \langle y, j_\varphi(x)\rangle\right| = 0$$

and

$$\left|\Phi(\|x + ty\|) - \Phi(\|x\|) - t\langle y, j_\varphi(x)\rangle\right| \le b(t) \quad \forall y \in S(E)$$

which implies

$$\Phi(\|x + ty\|) \le \Phi(\|x\|) + t\langle y, j_\varphi(x)\rangle + b(t) \quad \forall y \in S(E). \tag{2.2}$$

Suppose $h \neq 0$. Put $y = \frac{h}{\|h\|}$ and $t = \|h\|$. By (2.2), we have

$$\Phi(\|x + h\|) \le \Phi(\|x\|) + \langle h, j_\varphi(x)\rangle + b(\|h\|). \tag{2.3}$$

On the other hand, by (1.3), we have

$$\Phi(\|x\|) = \Phi(\|x+h-h\|) \le \Phi(\|x+h\|) - \langle h, j_\varphi(x)\rangle \tag{2.4}$$

for each $h \in E$. Combining (2.3) and (2.4), we get the desired result. □

Following [2, 11], we can prove Proposition 2.3.

Proposition 2.3. *Let E be a uniformly convex Banach space having a Fréchet differentiable norm and a duality mapping j_φ and K a nonempty, closed and convex subset of E. Let $\{S_n\}_{n=1}^\infty$ be a sequence of L_n-Lipschitzian self-mappings of K with $L_n \ge 1,\ n \ge 1$ and $\sum_{n=1}^\infty (L_n - 1) < \infty$. Assume that $F = \bigcap_{n=1}^\infty F(S_n) \ne \emptyset$. If $x_1 \in E$ and $x_{n+1} = S_n x_n,\ n \ge 1$, then $\forall f_1, f_2 \in F$, $\lim_{n\to\infty}\langle x_n, j_\varphi(f_1 - f_2)\rangle$ exists.*

Proof. For each $f \in F$, we see that

$$\|x_{n+1} - f\| = \|S_n x_n - f\| \le \big(1 + (L_n - 1)\big)\|x_n - f\|.$$

Hence $\lim_{n\to\infty}\|x_n - f\|$ exists for all $f \in F$. Now, taking $x = f_1 - f_2, h = t(x_n - f_1)$ in (2.1) and setting $a_n(t) = \|tx_n + (1-t)f_1 - f_2\|$, we obtain

$$\begin{aligned}\Phi(\|f_1 - f_2\|) \quad + \quad & t\langle x_n - f_1, j_\varphi(f_1 - f_2)\rangle \le \Phi(a_n(t)) \\ \le \quad & \Phi(\|f_1 - f_2\|) + t\langle x_n - f_1, j_\varphi(f_1 - f_2)\rangle + b(t\|x_n - f_1\|) \\ \le \quad & \Phi(\|f_1 - f_2\|) + t\langle x_n - f_1, j_\varphi(f_1 - f_2)\rangle + b(tM)\end{aligned}$$

for some $M > 0$. Since E is uniformly convex, by Lemma 2.2 of [11], we know that $\lim_{n\to\infty} a_n(t)$ exists. Hence $\lim_{n\to\infty}\Phi(a_n(t))$ also exists since Φ is continuous. Thus

$$\limsup_{n\to\infty}\langle x_n - f_1, j_\varphi(f_1 - f_2)\rangle \le \liminf_{n\to\infty}\langle x_n - f_1, j_\varphi(f_1 - f_2)\rangle + b(tM)/t.$$

Since $b(tM)/t \to 0$ as $t \to 0$, $\lim_{n\to\infty}\langle x_n - f_1, j_\varphi(f_1 - f_2)\rangle$ exists. □

Theorem 2.4. *Let E be a Banach space having a duality mapping j_φ and K a nonempty, closed and convex subset of E. Let $\{T_n\}_{n=1}^\infty$ be a sequence of nonexpansive self-mappings of K such that $\bigcap_{n=1}^\infty F(T_n) \ne \emptyset$. Suppose that $(\{T_n\}, T)$ satisfies the AKTT-condition and let $\{x_n\}$ be defined by (1.2). Then,*
(i) For each $f \in \bigcap_{n=1}^\infty F(T_n)$, $\lim_{n\to\infty}\|x_n - f\|$ exists.
(ii) If E is uniformly convex and $\sum_{n=1}^\infty \min\{\alpha_n, 1-\alpha_n\} = \infty$, then

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

Proof. (i) For any $f \in \bigcap_{n=1}^\infty F(T_n)$, we have

$$\|x_{n+1} - f\| \le \alpha_n\|x_n - f\| + (1-\alpha_n)\|T_n x_n - f\| \le \|x_n - f\|.$$

Hence $\{\|x_n - f\|\}$ is nonincreasing; consequently, $\lim_{n\to\infty}\|x_n - f\|$ exists.

(ii) Let $f \in \bigcap_{n=1}^\infty F(T_n)$ and assume $\|x_n - f\| > 0$. Since $\|T_n x_n - f\| \le \|x_n - f\|$ and E is uniformly convex, it follows (see, for example, [12]) that

$$\|x_{n+1} - f\| \le \|x_n - f\|\left\{1 - 2\min\{\alpha_n, 1-\alpha_n\}\delta_E\left(\frac{\|x_n - T_n x_n\|}{\|x_n - f\|}\right)\right\}.$$

Therefore

$$2\min\{\alpha_n, 1-\alpha_n\}\|x_n - f\|\delta_E\left(\frac{\|x_n - T_n x_n\|}{\|x_n - f\|}\right) \le \|x_n - f\| - \|x_{n+1} - f\|.$$

Since $\lim_{n\to\infty}\|x_n - f\|$ exists and $\sum_{n=1}^\infty \min\{\alpha_n, 1-\alpha_n\} = \infty$, by the continuity of δ_E, we conclude that $\liminf_{n\to\infty}\|x_n - T_n x_n\| = 0$. Observe that

$$\|x_{n+1} - T_{n+1}x_{n+1}\| \quad \le \quad \alpha_n\|x_n - T_{n+1}x_{n+1}\| + (1-\alpha_n)\|T_n x_n - T_{n+1}x_{n+1}\|$$

$$\begin{aligned}
&\leq \alpha_n\|x_n - x_{n+1}\| + \alpha_n\|x_{n+1} - T_{n+1}x_{n+1}\| \\
&\quad + (1-\alpha_n)\|T_nx_n - T_{n+1}x_n\| + (1-\alpha_n)\|T_{n+1}x_n - T_{n+1}x_{n+1}\| \\
&\leq \alpha_n(1-\alpha_n)\|x_n - T_nx_n\| + \alpha_n\|x_{n+1} - T_{n+1}x_{n+1}\| \\
&\quad + (1-\alpha_n)\sup_{z\in\{x_n\}}\|T_nz - T_{n+1}z\| + (1-\alpha_n)^2\|x_n - T_nx_n\| \\
&= (1-\alpha_n)\|x_n - T_nx_n\| + \alpha_n\|x_{n+1} - T_{n+1}x_{n+1}\| \\
&\quad + (1-\alpha_n)\sup_{z\in\{x_n\}}\|T_nz - T_{n+1}z\|,
\end{aligned}$$

which implies

$$\|x_{n+1} - T_{n+1}x_{n+1}\| \leq \|x_n - T_nx_n\| + \sup_{z\in\{x_n\}}\|T_nz - T_{n+1}z\|.$$

Since $\{T_n\}$ satisfies the AKTT-condition, $\lim_{n\to\infty}\|x_n - T_nx_n\|$ exists; consequently, $\lim_{n\to\infty}\|x_n - T_nx_n\| = 0$. On the other hand, we see that

$$\|x_n - Tx_n\| \leq \|x_n - T_nx_n\| + \|T_nx_n - Tx_n\| \leq \|x_n - T_nx_n\| + \sup_{z\in\{x_n\}}\|T_nz - Tz\|.$$

Since $(\{T_n\}, T)$ satisfies the AKTT-condition, we have $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ by Lemma 1.8. This completes the proof. □

Theorem 2.5. *Let E be a uniformly convex Banach space having a duality mapping j_φ and K a nonempty, closed and convex subset of E. Let $\{T_n\}_{n=1}^\infty$ be a sequence of nonexpansive self-mappings of K such that $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Suppose that $(\{T_n\}, T)$ satisfies the AKTT-condition and let $\{x_n\}$ be defined by (1.2) with $\sum_{n=1}^\infty \min\{\alpha_n, 1-\alpha_n\} = \infty$. If one of the following statements holds:*
(i) E has a weakly continuous duality mapping j_φ;
(ii) E has a Fréchet differentiable norm.
Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_n\}_{n=1}^\infty$.

Proof. Set $S_n = \alpha_nI + (1-\alpha_n)T_n$, $n \geq 1$. Then $x_{n+1} = S_nx_n$ and $F(T_n) = F(S_n)$ for all $n \geq 1$. By Theorem 2.4 (i) and (ii), we get that $\omega_w(\{x_n\}) \subset F(T)$ by the demiclosedness principle. Next, we show that $\omega_w(\{x_n\})$ is singleton. To this end, let $p, q \in \omega_w(\{x_n\})$. If E has a weakly continuous duality mapping j_φ, then $p = q$ by Proposition 2.1. Suppose that E has a Fréchet differentiable norm, and $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p$ and $x_{m_k} \rightharpoonup q$. Then

$$\|p-q\|\varphi(\|p-q\|) = \langle p-q, j_\varphi(p-q)\rangle = \lim_{k\to\infty}\langle x_{n_k} - x_{m_k}, j_\varphi(p-q)\rangle.$$

By Proposition 2.3, we conclude that $\|p-q\|\varphi(\|p-q\|) = 0$ and $p = q$. Therefore $\{x_n\}$ converges weakly to a common fixed point of $\{T_n\}_{n=1}^\infty$. □

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