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CONVERGENCE OF NONLINEAR PROJECTIONS AND SHRINKING PROJECTION METHODS FOR COMMON FIXED POINT PROBLEMS

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ABSTRACT. In this paper, we first study some properties of Mosco convergence for a sequence of nonempty sunny generalized nonexpansive retracts in Banach spaces. Next, motivated by the result of Kimura and Takahashi and that of Plubtieng and Ungchittrakool, we prove a strong convergence theorem for finding a common fixed point of generalized nonexpansive mappings in Banach spaces by using the shrinking projection method.

KEYWORDS : Shrinking projection method; Sunny generalized nonexpansive retraction; Generalized nonexpansive; Mosco convergence; Fixed point.

1. INTRODUCTION

Let E be a real Banach space and C a nonempty closed convex subset of E. A mapping $T : C \longrightarrow C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Iterative methods for approximation of fixed points of nonexpansive mappings have been studied by many researchers; see [5, 21, 25, 28, 29, 31, 34, 36] and others. In particular, Takahashi, Takeuchi and Kubota [34] established strong convergence of an iterative scheme with new type of hybrid method as follows:

Theorem 1.1 (Takahashi-Takeuchi-Kubota [34]). *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that the set $F(T)$ of fixed points of T is nonempty. Let $\{\alpha_n\}$ be a*

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sequence $[0, a]$, *where* $0 < a < 1$. *For a point* $x \in H$ *chosen arbitrarily, generate a sequence* $\{x_n\}$ *by the following iterative scheme:* $x_1 \in C$, $C_1 = C$, *and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x \end{cases}$$

for every $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to* $P_{F(T)}x \in C$, *where* P_K *is the metric projection of* H *onto a nonempty closed convex subset* K *of* H.

This iterative method is also known as the shrinking projection method. We note that the original result is a convergence theorem to a common fixed point of a family of nonexpansive mappings with certain conditions.

On the other hand, relatively nonexpansive mappings and generalized nonexpansive mappings, which are generalizations of a nonexpansive mappings in Hilbert spaces, have been considered recently. Their properties and iterative schemes have been studied in [3, 8, 9, 10, 11, 12, 13, 14, 30, 22, 23, 18, 19, 20, 27] and others. Recently, Kimura and Takahashi [18] obtain a strong convergence theorem for finding a common fixed point of relatively nonexpansive mappings in a Banach space by using the shrinking projection method. The method for its proof is different from the original one; they use the concept of Mosco convergence of sequences of nonempty closed convex subsets of a Banach space. They also succeed in making conditions of the coefficients and the underlying space weaker.

In this paper, we study the shrinking projection method for generalized nonexpansive mappings in a Banach space. We first prove convergence theorems for a sequence of sunny generalized nonexpansive retractions, which is a generalization of the metric projections in Hilbert spaces. Next, using the technique developed by Kimura and Takahashi [18], we prove a strong convergence theorem for finding a common fixed point of a finite family of generalized nonexpansive mappings in Banach space by using the iterative scheme of [27]; see also [1, 11, 13, 16, 20] and others.

2. Preliminaries

Let E be a real Banach space with its dual E^*. We denote strong convergence and weak convergence of a sequence $\{x_n\}$ to x in E by $x_n \longrightarrow x$ and $x_n \rightharpoonup x$, respectively. A Banach space E is said to be strictly convex if $\|x + y\|/2 < 1$ whenever $x, y \in E$ satisfies $\|x\| = \|y\| = 1$ and $x \neq y$. E is said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$ implies $\|x + y\|/2 \leq 1 - \delta$. The following lemma holds.

Lemma 2.1 (Plubtieng-Ungchittrakool [27]). *Let* E *be a uniformly convex Banach space and* $\rho > 0$. *Then, there exists a continuous, strictly increasing, and convex function* $g : [0, \infty) \longrightarrow [0, \infty)$ *with* $g(0) = 0$ *such that*

$$\left\|\sum_{i=1}^{r} \delta_i x_i\right\|^2 \leq \sum_{i=1}^{r} \delta_i \|x_i\|^2 - \delta_j \delta_k g(\|x_j - x_k\|) \tag{2.1}$$

for each $j, k \in \{1, 2, \ldots, r\}$, *where* $\{x_1, x_2, \ldots, x_r\} \subset E$ *satisfy* $\|x_i\| \leq \rho$ *for each* $i = 1, 2, \ldots, r$ *and* $\{\delta_1, \delta_2, \ldots, \delta_r\} \subset [0, 1]$ *satisfy* $\sum_{i=1}^{r} \delta_i = 1$.

A Banach space E is said to be smooth if

$$\lim_{t \longrightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for each $x, y \in B = \{z \in E : \|z\| = 1\}$. In this case, the norm of E is said to be Gâteaux differentiable. The norm of E is said to be Fréchet differentiable if for each $x \in B$, the limit (2.2) is attained uniformly for $y \in B$. See [32] for more details. A Banach space E is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of E converges strongly to x_0 whenever it satisfies $x_n \rightharpoonup x_0$ and $\|x_n\| \longrightarrow \|x_0\|$.

The normalized duality mapping J from E into E^* is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for each $x \in E$. We also know the following properties; see [4, 32, 33] for more details.

(i) $Jx \neq \emptyset$ for each $x \in E$;
(ii) if E is reflexive, then J is surjective;
(iii) if E is strictly convex, then J is one-to-one and satisfies that $\langle x - y, x^* - y^* \rangle > 0$ for each $x, y \in E$ with $x \neq y$, $x^* \in Jx$ and $y^* \in Jy$;
(iv) if E is smooth, then J is single-valued and norm-to-weak* continuous;
(v) if E is reflexive, smooth and strictly convex, then the duality mapping $J_* : E^* \longrightarrow E$ is the inverse of J, that is, $J_* = J^{-1}$;
(vi) if E has a Fréchet differentiable norm, then J is norm-to-norm continuous;
(vii) E is reflexive, strictly convex and has the Kadec-Klee property if and only if E^* has a Fréchet differentiable norm.

Let E be a reflexive Banach space and $\{C_n\}$ a sequence of nonempty closed convex subsets of E. We denote by $\text{s-Li}_n C_n$ the set of limit points of $\{C_n\}$, that is, $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $x_n \in C_n$ for each $n \in \mathbb{N}$ and $x_n \longrightarrow x$ as $n \longrightarrow \infty$. Similarly, we denote by $\text{w-Ls}_n C_n$ the set of weak cluster points of $\{C_n\}$; $y \in \text{w-Ls}_n C_n$ if and only if there exists $\{y_{n_i}\} \subset E$ such that $y_{n_i} \in C_{n_i}$ for each $i \in \mathbb{N}$ and $y_{n_i} \rightharpoonup y$ as $i \longrightarrow \infty$. Using these definitions, we define Mosco convergence[24] of $\{C_n\}$. If C_0 satisfies

$$\operatorname*{s-Li}_n C_n = C_0 = \operatorname*{w-Ls}_n C_n,$$

then we say that $\{C_n\}$ is a Mosco convergent sequence to C_0. In this case, we denote it by

$$C_0 = \operatorname*{M-lim}_n C_n.$$

Notice that the inclusion $\text{s-Li}_n C_n \subset \text{w-Ls}_n C_n$ is always true. Thus, to prove $C_0 = \text{M-lim}_n C_n$ we only need to show $\text{w-Ls}_n C_n \subset C_0 \subset \text{s-Li}_n C_n$.

Let E be a smooth Banach space and consider the following function $V : E \times E \longrightarrow \mathbb{R}$ defined by

$$V(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for each $x, y \in E$. We know the following properties; see [2, 7, 9, 15, 23] for more details:

(i) $(\|x\| - \|y\|)^2 \leq V(x, y) \leq (\|x\| + \|y\|)^2$ for each $x, y \in E$;
(ii) $V(x, y) + V(y, x) = 2\langle x - y, Jx - Jy \rangle$ for each $x, y \in E$;
(iii) $V(x, y) = V(x, z) + V(z, y) + 2\langle x - z, Jz - Jy \rangle$ for each $x, y, z \in E$;
(iv) if E is additionally assumed to be strictly convex, then $V(x, y) = 0$ if and only if $x = y$.

The following lemma is due to [15].

Lemma 2.2 (Kamimura-Takahashi [15]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n \longrightarrow \infty} V(x_n, y_n) = 0$, then $\lim_{n \longrightarrow \infty} \|x_n - y_n\| = 0$.*

Let C be a nonempty closed subset of a smooth Banach space E. A mapping $T : C \longrightarrow C$ is said to be generalized nonexpansive [8, 9] if $F(T) \neq \emptyset$ and

$$V(Tx, p) \leq V(x, p)$$

for each $x \in C$ and $p \in F(T)$. Let D be a nonempty subset of a Banach space E. A mapping $R : E \longrightarrow D$ is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx$$

for all $x \in E$ and $t \geq 0$. A mapping $R : E \longrightarrow D$ is said to be a retraction if $Rx = x$ for each $x \in D$. If E is smooth and strictly convex, then a sunny generalized nonexpansive retraction of E onto D is uniquely determined if it exists; see [9]. Then, such a sunny generalized nonexpansive retraction of E onto D is denoted by R_D. A nonempty subset D of E is called a sunny generalized nonexpansive retract of E if there exists a sunny generalized nonexpansive retraction of E onto D. Obviously, the set of fixed points of a sunny generalized nonexpansive retraction of E onto D is D; see [8, 9] for more details. We recall the following results for sunny generalized nonexpansive retractions and sunny generalized nonexpansive retracts.

Lemma 2.3 (Ibaraki-Takahashi [8, 9]). *Let D be a nonempty subset of a smooth and strictly convex Banach space E. Let R_D be a retraction of E onto D. Then R_D is sunny and generalized nonexpansive if and only if*

$$\langle x - R_D x, JR_D x - Jy \rangle \geq 0.$$

for each $y \in D$.

Theorem 2.4 (Ibaraki-Takahashi [12], Inthakon-Dhompongsa-Takahashi [14]). *Let E be a reflexive, smooth and strictly convex Banach space and C a nonempty closed subset of E such that JC is closed and convex. Let T be a generalized nonexpansive mapping of C into itself. Then $F(T)$ is a sunny generalized nonexpansive retract of E.*

Lemma 2.5 (Kohsaka-Takahashi [19]). *Let E be a smooth, reflexive, and strictly convex Banach space and D a nonempty sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction of E onto D, $x \in E$, and $z \in D$. Then, the following conditions are equivalent:*

(i) *$z = Rx$;*
(ii) *$V(x, z) = \min_{y \in D} V(x, y)$.*

Theorem 2.6 (Kohsaka-Takahashi [19]). *Let E be a smooth, reflexive, and strictly convex Banach space and D a nonempty subset of E. Then, the following conditions are equivalent:*

(i) *D is a sunny generalized nonexpansive retract of E;*
(ii) *D is a generalized nonexpansive retract of E;*
(iii) *JD is closed and convex.*

In this case, D is closed.

3. Convergence theorem for sunny generalized nonexpansive retractions

In this section, we prove weak and strong convergence theorems for a sequence of sunny generalized nonexpansive retractions. The sequence of ranges of these nonlinear projections is assumed to converge in the sense of Mosco; see [7, 13, 17, 35] for related results.

We remark that, using the relation between generalized nonexpansive retractions and generalized projections shown in [19] and the methods used in [7], we can prove the essential parts of the theorems in this section; see also [9]. For the sake of completeness, we will give a proof for all results.

Theorem 3.1. *Let E be a reflexive, smooth and strictly convex Banach space and let $\{D_n\}$ be a sequence of nonempty sunny generalized nonexpansive retracts of E. Let $u \in E$ and $\{u_n\}$ be a sequence of E converging strongly to u. If $D_0^* = \text{M-}\lim_n JD_n$ exists and is nonempty, then $\{JR_{D_n}u_n\}$ converges weakly to $JR_{D_0}u$, where $D_0 = J^{-1}D_0^*$.*

Proof. It is easy to prove that D_0^* is closed and convex if JD_n is a closed convex subset of E for each $n \in \mathbb{N}$. Let $x_n = R_{D_n}u_n$ for each $n \in \mathbb{N}$. Since $D_0^* = \text{M-}\lim_n JD_n$, we have, for each $y \in D_0$, there exists $\{y_n^*\} \subset E^*$ such that $y_n^* \longrightarrow Jy$ as $n \longrightarrow \infty$ and that $y_n^* \in JD_n$ for each $n \in \mathbb{N}$. From Lemma 2.3, we have

$$\langle u_n - x_n, Jx_n - y_n^* \rangle \geq 0.$$

Hence, we obtain that

$$\begin{aligned} 0 &\leq \langle u_n - x_n, Jx_n - Ju_n \rangle + \langle u_n - x_n, Ju_n - y_n^* \rangle \\ &\leq -\big(\|u_n\| - \|x_n\|\big)^2 + \big(\|u_n\| + \|x_n\|\big)\|Ju_n - y_n^*\| \end{aligned}$$

and thus

$$\big(\|u_n\| - \|x_n\|\big)^2 \leq \big(\|u_n\| + \|x_n\|\big)\|Ju_n - y_n^*\|. \tag{3.1}$$

Assume that $\{x_n\}$ is unbounded. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i\longrightarrow\infty} \|x_{n_i}\| = \infty$. Since $y_n^* \longrightarrow Jy$ and $u_n \longrightarrow u$, by (3.1) we get contradiction. Hence $\{x_n\}$ is bounded and so is $\{Jx_n\}$. Let $\{Jx_{n_i}\}$ be a subsequence of $\{Jx_n\}$ converging weakly to some $x_0^* \in E^*$. From the definition of D_0^*, we get $x_0^* \in \text{M-}\lim_n JD_n = D_0^*$.

Now we let $x_0 = J^{-1}x_0^*$ and prove that $R_{D_0}u = x_0$. From lower semicontinuity of the norm, we have

$$\begin{aligned} \liminf_{i\longrightarrow\infty} V(u_{n_i}, x_{n_i}) &= \liminf_{i\longrightarrow\infty}\big(\|u_{n_i}\|^2 - 2\langle u_{n_i}, Jx_{n_i}\rangle + \|Jx_{n_i}\|^2\big) \\ &\geq \|u\|^2 - 2\langle u, x_0^*\rangle + \|x_0^*\|^2 \\ &= \|u\|^2 - 2\langle u, Jx_0\rangle + \|x_0\|^2 \\ &= V(u, x_0). \end{aligned}$$

On the other hand, we get from Lemma 2.5 that

$$\begin{aligned} \liminf_{n\longrightarrow\infty} V(u_n, x_n) \leq \liminf_{n\longrightarrow\infty} V(u_n, J^{-1}y_n^*) &= \lim_{n\longrightarrow\infty}\big(\|u_n\|^2 - 2\langle u_n, y_n^*\rangle + \|y_n^*\|^2\big) \\ &= \|u\|^2 - 2\langle u, Jy\rangle + \|Jy\|^2 = V(u, y), \end{aligned}$$

that is,

$$V(u, x_0) = \min_{y\in D_0} V(u, y).$$

Hence we get $R_{D_0}u = x_0$.

According to our consideration above, each subsequence $\{Jx_{n_i}\}$ of $\{Jx_n\}$ which converges weakly has the unique limit $JR_{D_0}u$. Therefore, the sequence $\{Jx_n\}$ itself converges weakly to $JR_{D_0}u$. □

Theorem 3.2. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm. Let $\{D_n\}$ be a sequence of nonempty sunny generalized nonexpansive retracts of E. Let $u \in E$ and let $\{u_n\}$ be a sequence of E converging strongly to u. If $D_0^* = \text{M-}\lim_n JD_n$ exists and is nonempty, then $\{JR_{D_n}u_n\}$*

converges strongly to $JR_{D_0}u$, where $D_0 = J^{-1}D_0^*$. Moreover, $\{R_{D_n}u_n\}$ converges weakly to $R_{D_0}u$.

Proof. We write $x_n = R_{D_n}u_n$ and $x_0 = R_{D_0}u$. By Theorem 3.1, we obtain $Jx_n \rightharpoonup Jx_0$ as $n \longrightarrow \infty$. We first prove that $Jx_n \longrightarrow Jx_0$ as $n \longrightarrow \infty$. Since E has a Fréchet differential norm, E^* has the Kadec-Klee property. Therefore, it is sufficient to prove that $\|Jx_n\| \longrightarrow \|Jx_0\|$ as $n \longrightarrow \infty$. Since $Jx_0 \in D_0^*$, there exists a sequence $\{y_n^*\} \subset E^*$ such that $y_n^* \longrightarrow Jx_0$ as $n \longrightarrow \infty$ and $y_n^* \in JD_n$ for each $n \in \mathbb{N}$. It follows that

$$\begin{aligned} V(u, x_0) &\leq \liminf_{n\longrightarrow\infty} V(u_n, x_n) \\ &\leq \limsup_{n\longrightarrow\infty} V(u_n, x_n) \\ &\leq \lim_{n\longrightarrow\infty} V(u_n, J^{-1}y_n^*) \\ &\leq V(u, x_0). \end{aligned}$$

Hence we obtain $V(u, x_0) = \lim_{n\longrightarrow\infty} V(u_n, x_n)$. Since $\lim_{n\longrightarrow\infty}\langle u_n, Jx_n\rangle = \langle u, Jx_0\rangle$, we get

$$\lim_{n\longrightarrow\infty} \|Jx_n\| = \|Jx_0\|.$$

Therefore we obtain that $\{Jx_n\}$ converges strongly to Jx_0.

Since E is reflexive and strictly convex, E^* is smooth and thus the duality mapping J^{-1} of E^* is norm-to-weak continuous. Hence, we obtain that

$$x_n = J^{-1}Jx_n \rightharpoonup J^{-1}Jx_0 = x_0,$$

which completes the proof. □

Theorem 3.3. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let $\{D_n\}$ be a sequence of nonempty sunny generalized nonexpansive retracts of E. Let $u \in E$ and let $\{u_n\}$ be a sequence of E converging strongly to u. If $D_0^* = \text{M-}\lim_n JD_n$ exists and is nonempty, then $\{R_{D_n}u_n\}$ converges strongly to $R_{D_0}u$, where $D_0 = J^{-1}D_0^*$.*

Proof. By Theorem 3.2, we obtain that $\{JR_{D_n}u_n\}$ converges strongly to $JR_{D_0}u$. Since E is reflexive, strictly convex and has the Kadec-Klee property, E^* has a Fréchet differentiable norm. Therefore the duality mapping J^{-1} of E^* is norm-to-norm continuous. Hence, we have that

$$R_{D_n}u_n = J^{-1}JR_{D_n}u_n \longrightarrow J^{-1}JR_{D_0}u = R_{D_0}u,$$

which completes the proof. □

On the other hand, the following theorem shows that the strong convergence of sequences $\{R_{D_n}u\}$ implies the Mosco convergence of $\{JD_n\}$ under certain conditions.

Theorem 3.4. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm. Let $D_0, D_1, D_2, D_3, \ldots$ be nonempty sunny generalized nonexpansive retracts of E. Suppose that $\{R_{D_n}u\}$ converges strongly to $R_{D_0}u$ for each $u \in E$, where R_{D_n} is the sunny generalized nonexpansive retractions of E onto D_n for each $n \in \mathbb{N} \cup \{0\}$. Then*

$$JD_0 = \text{M-}\lim_n JD_n.$$

Proof. For an arbitrary $u^* \in JD_0$, put $u = J^{-1}u^* \in D_0$. Since E has a Fréchet differentiable norm, we have

$$JR_{D_n}u \longrightarrow JR_{D_0}u = Ju = u^*$$

and that $JR_{D_n}u \in JD_n$ for all $n \in \mathbb{N}$. This means that $u^* \in$ s-Li$_n\, JD_n$ and hence we have $JD_0 \subset$ s-Li$_n\, JD_n$. Next we show that w-Ls$_n\, JD_n \subset JD_0$. For any $z^* \in$ w-Ls$_n\, JD_n$, there exists $\{z_i^*\}$ such that $\{z_i^*\}$ converges weakly to z^* as $i \longrightarrow \infty$ and that $z_i^* \in JD_{n_i}$ for each $i \in \mathbb{N}$. Using Lemma 2.3, we have that

$$\langle z - R_{D_{n_i}}z, JR_{D_{n_i}}z - z_i^*\rangle \geq 0.$$

where $z = J^{-1}z^*$. Tending $i \longrightarrow \infty$, we get

$$\langle z - R_{D_0}z, JR_{D_0}z - Jz\rangle \geq 0.$$

By the strict convexity of E, we have that J is strictly monotone. Hence we have $z^* = JR_{D_0}z \in JD_0$. This means that w-Ls$_n\, JD_n \subset JD_0$, and consequently we obtain $JD_0 =$ M-lim$_{n\longrightarrow\infty}\, JD_n$. □

Using Theorem 3.3 and 3.4, we obtain the following theorem.

Theorem 3.5. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let $D_0, D_1, D_2, D_3, \ldots$ be nonempty sunny generalized nonexpansive retracts of E. If each R_{D_n} is the sunny generalized nonexpansive retractions of E onto D_n for each $n \in \mathbb{N} \cup \{0\}$, then*

$$JD_0 = \text{M-}\lim_n JD_n.$$

if and only if $\{R_{D_n}u_n\}$ converges strongly to $R_{D_0}u$ for every strongly convergent sequence $\{u_n\} \subset E$ having a limit $u \in E$.

4. Strong convergence theorem for generalized nonexpansive mappings

In this section, using the technique developed by Kimura and Takahashi [18], we prove a strong convergence theorem for finding a common fixed point of finite family of generalized nonexpansive mappings in Banach space. In this result, we adopt the iterative scheme used in [27].

Theorem 4.1. *Let E be a uniformly convex Banach space having a Fréchet differentiable norm, C a nonempty closed subset of E such that JC is closed and convex, and $T_1, T_2, \ldots, T_r$ generalized nonexpansive mappings of C into itself such that $\bigcap_{i=1}^r F(T_i)$ is nonempty. Suppose that $z \in \bigcap_{i=1}^r F(T_i)$ whenever both $\{z_n\}$ and $\{T_i z_n\}$ converge strongly to z for each $i = 1, 2, \ldots, r$. For a point $x \in E$ chosen arbitrarily, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$, and*

$$\begin{cases} y_n = \gamma_n x_n + (1-\gamma_n)\displaystyle\sum_{i=1}^r \delta_n^{(i)} T_i x_n, \\ C_{n+1} = \{z \in C_n : V(y_n, z) \leq V(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\gamma_n\}, \{\delta_n^{(i)}\} \subset [0,1]$ satisfy the following conditions:

(i) $\liminf_{n\longrightarrow\infty} \gamma_n < 1$,
(ii) $\liminf_{n\longrightarrow\infty} \delta_n^{(i)} > 0$ *for each* $i = 1, 2, \ldots, r$,
(iii) $\sum_{i=1}^r \delta_n^{(i)} = 1$ *for each* $n \in \mathbb{N}$.

Then $\{x_n\}$ converges strongly to $R_{\bigcap_{i=1}^r F(T_i)}x$.

Proof. We first show that $\{x_n\}$ is well defined. Let $p \in \bigcap_{i=1}^r F(T_i)$. Then for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}
V(y_n, p) &= V\left(\gamma_n x_n + (1-\gamma_n)\sum_{i=1}^r \delta_n^{(i)} T_i x_n, p\right) \\
&\leq \gamma_n V(x_n, p) + (1-\gamma_n) V\left(\sum_{i=1}^r \delta_n^{(i)} T_i x_n, p\right) \\
&\leq \gamma_n V(x_n, p) + (1-\gamma_n) \sum_{i=1}^r \delta_n^{(i)} V(T_i x_n, p) \\
&\leq \gamma_n V(x_n, p) + (1-\gamma_n) \sum_{i=1}^r \delta_n^{(i)} V(x_n, p) \\
&\leq \gamma_n V(x_n, p) + (1-\gamma_n) V(x_n, p) = V(x_n, p).
\end{aligned}$$

Therefore $p \in C_n$ for all $n \in \mathbb{N}$ and hence $\bigcap_{i=1}^r F(T_i) \subset C_n$ for all $n \in \mathbb{N}$. This implies that C_n is nonempty for all $n \in \mathbb{N}$. Next, we show that JC_n is closed and convex for all $n \in \mathbb{N}$. From the definition of V, we may show that

$$\begin{aligned}
C_{n+1} &= \{z \in C_n : V(y_n, z) \leq V(x_n, z)\} \\
&= \{z \in C : 2\langle x_n - y_n, Jz\rangle + \|y_n\|^2 - \|x_n\|^2 \leq 0\} \cap C_n.
\end{aligned}$$

for all $n \in \mathbb{N}$. The injectivity of J implies that

$$\begin{aligned}
JC_{n+1} &= J\Big(\{z \in C : 2\langle x_n - y_n, Jz\rangle + \|y_n\|^2 - \|x_n\|^2 \leq 0\} \cap C_n\Big) \\
&= J\{z \in C : 2\langle x_n - y_n, Jz\rangle + \|y_n\|^2 - \|x_n\|^2 \leq 0\} \cap JC_n \\
&= \{z^* \in JC : 2\langle x_n - y_n, z^*\rangle + \|y_n\|^2 - \|x_n\|^2 \leq 0\} \cap JC_n
\end{aligned}$$

for all $n \in \mathbb{N}$. From the assumption for C, JC_1 is closed and convex. Suppose that JC_k is closed and convex for some $k \in \mathbb{N}$. Then, letting

$$D_k^* = \{z^* \in JC : 2\langle x_k - y_k, z^*\rangle + \|y_k\|^2 - \|x_k\|^2 \leq 0\},$$

we have that D_k^* is obviously closed and convex and thus $JC_{k+1} = D_k^* \cap JC_k$ is also closed and convex. By Theorem 2.6, there exists a unique sunny generalized nonexpansive retraction of E onto C_n for each $n \in \mathbb{N}$ and hence $\{x_n\}$ is well defined.

Since $\{JC_n\}$ is a decreasing sequence of closed convex subsets of E^* such that $C_0^* = \bigcap_{n=1}^\infty JC_n$ is nonempty, it follows that

$$\text{M-}\lim_n JC_n = C_0^* = \bigcap_{n=1}^\infty JC_n \neq \emptyset.$$

By Theorem 3.3, $\{x_n\}$ converges strongly to $R_{C_0}x \in C_0$, where $C_0 = J^{-1}C_0^*$. The injectivity of J implies

$$JC_0 = C_0^* = \bigcap_{n=1}^\infty JC_n = J\bigcap_{n=1}^\infty C_n.$$

Therefore, we obtain that $x_0 = R_{C_0}x \in C_0 = J^{-1}C_0^* = \bigcap_{n=1}^\infty C_n$. From the definition of C_n, we have that

$$0 \leq \limsup_{n\longrightarrow\infty} V(y_n, x_0) \leq \lim_{n\longrightarrow\infty} V(x_n, x_0) = 0$$

and hence $\lim_{n\longrightarrow\infty} V(y_n, x_0) = 0$. From the property of the mapping V, we have that

$$0 \le \limsup_{n\longrightarrow\infty} (\|y_n\| - \|x_0\|)^2 \le \lim_{n\longrightarrow\infty} V(y_n, x_0) = 0,$$

and hence

$$\lim_{n\longrightarrow\infty} \|y_n\| = \|x_0\|. \tag{4.1}$$

Therefore, we also have that

$$\lim_{n\longrightarrow\infty} \langle y_n, Jx_0\rangle = \lim_{n\longrightarrow\infty} \frac{1}{2}\Big\{\|y_n\|^2 + \|x_0\|^2 - V(y_n, x_0)\Big\} = \|x_0\|^2.$$

Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_i}\}$ converging weakly to some $y_0 \in E$. Using weak lower semicontinuity of the norm, we get from (4.1) that

$$\begin{aligned}\|x_0\|^2 &= \lim_{i\longrightarrow\infty} \langle y_{n_i}, Jx_0\rangle = \langle y_0, Jx_0\rangle \\ &\le \|y_0\|\|Jx_0\| = \|y_0\|\|x_0\| \\ &\le \|x_0\| \liminf_{i\longrightarrow\infty} \|y_{n_i}\| \\ &= \|x_0\| \lim_{i\longrightarrow\infty} \|y_{n_i}\| = \|x_0\|^2.\end{aligned}$$

Therefore, we have that $\|y_0\|^2 = \langle y_0, Jx_0\rangle = \|x_0\|^2 = \|Jx_0\|^2$ and hence $Jy_0 = Jx_0$. This implies that $y_0 = x_0$. Thus we have that $\{y_n\}$ converges weakly to x_0. Since (4.1) holds and E has the Kadec-Klee property, we have that $\{y_n\}$ converges strongly to x_0.

Put $S_n = \sum_{i=1}^r \delta_n^{(i)} T_i$. From the assumption that $\liminf_{n\longrightarrow\infty} \gamma_n < 1$, we may take a subsequence $\{\gamma_{n_j}\}$ of $\{\gamma_n\}$ such that $\lim_{j\longrightarrow\infty} \gamma_{n_j} = \gamma_0$ with $0 \le \gamma_0 < 1$. Then we have that

$$\begin{aligned}\|y_{n_j} - x_0\| &= \|\gamma_{n_j} x_{n_j} + (1-\gamma_{n_j}) S_{n_j} x_{n_j} - x_0\| \\ &\ge (1-\gamma_{n_j})\|S_{n_j} x_{n_j} - x_0\| - \gamma_{n_j}\|x_{n_j} - x_0\| \\ &\ge (1-\gamma_{n_j})\|S_{n_j} x_{n_j} - x_0\|\end{aligned}$$

for each $j \in \mathbb{N}$ and hence

$$\lim_{j\longrightarrow\infty} (1-\gamma_{n_j})\|S_{n_j} x_{n_j} - x_0\| = \lim_{j\longrightarrow\infty} (1-\gamma_0)\|S_{n_j} x_{n_j} - x_0\| = 0.$$

Since $\gamma_0 < 1$, we have that

$$\lim_{j\longrightarrow\infty} \|S_{n_j} x_{n_j} - x_0\| = 0.$$

Since $\{T_i x_{n_j}\}$ are bounded for each $i = 1, 2, \dots, r$, there exists $\rho > 0$ such that $\{T_i x_{n_j}\} \subset B_\rho$ for each $i = 1, 2, \dots, r$. Therefore, Lemma 2.1 is applicable. Then we obtain that, for each $k, l = 1, 2, \dots, r$,

$$\begin{aligned}&V(S_{n_j} x_{n_j}, x_0) \\ &= \left\|\sum_{i=1}^r \delta_{n_j}^{(i)} T_i x_{n_j}\right\|^2 - 2\left\langle \sum_{i=1}^r \delta_{n_j}^{(i)} T_i x_{n_j}, Jx_0 \right\rangle + \|x_0\|^2 \\ &\le \sum_{i=1}^r \delta_{n_j}^{(i)} \|T_i x_{n_j}\|^2 - \delta_{n_j}^{(k)} \delta_{n_j}^{(l)} g(\|T_k x_{n_j} - T_l x_{n_j}\|) \\ &\quad - 2\left\langle \sum_{i=1}^r \delta_{n_j}^{(i)} T_i x_{n_j}, Jx_0 \right\rangle + \|x_0\|^2\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{r} \delta_{n_j}^{(i)} \left(\|T_i x_{n_j}\|^2 - 2\langle T_i x_{n_j}, Jx_0 \rangle + \|x_0\|^2 \right) - \delta_{n_j}^{(k)} \delta_{n_j}^{(l)} g(\|T_k x_{n_j} - T_l x_{n_j}\|) \\
&= \sum_{i=1}^{r} \delta_{n_j}^{(i)} V(T_i x_{n_j}, x_0) - \delta_{n_j}^{(k)} \delta_{n_j}^{(l)} g(\|T_k x_{n_j} - T_l x_{n_j}\|) \\
&\le \sum_{i=1}^{r} \delta_{n_j}^{(i)} V(x_{n_j}, x_0) - \delta_{n_j}^{(k)} \delta_{n_j}^{(l)} g(\|T_k x_{n_j} - T_l x_{n_j}\|) \\
&= V(x_{n_j}, x_0) - \delta_{n_j}^{(k)} \delta_{n_j}^{(l)} g(\|T_k x_{n_j} - T_l x_{n_j}\|)
\end{aligned}
$$

and hence

$$
\delta_{n_j}^{(k)} \delta_{n_j}^{(l)} g(\|T_k x_{n_j} - T_l x_{n_j}\|) \le V(x_{n_j}, x_0) - V(S_{n_j} x_{n_j}, x_0). \tag{4.2}
$$

for each $k, l = 1, 2, \ldots, r$. From the assumption that $\liminf_{n \longrightarrow \infty} \delta_n^{(k)} > 0$ for $k = 1, 2, \ldots, r$, we may take subsequences, again denoted by $\{\delta_{n_j}^{(k)}\}$, such that $\lim_{j \longrightarrow \infty} \delta_{n_j}^{(k)} > 0$ for every $k = 1, 2, \ldots, r$. Since

$$
\lim_{j \longrightarrow \infty} \|x_{n_j} - x_0\| = \lim_{j \longrightarrow \infty} \|S_{n_j} x_{n_j} - x_0\| = 0,
$$

we obtain from (4.2) that

$$
\lim_{j \longrightarrow \infty} g(\|T_k x_{n_j} - T_l x_{n_j}\|) = 0
$$

for $k, l = 1, 2, \ldots, r$. Then the properties of g yield that

$$
\lim_{j \longrightarrow \infty} \|T_k x_{n_j} - T_l x_{n_j}\| = 0
$$

for each $k, l = 1, 2, \ldots, r$. Therefore we obtain that

$$
\begin{aligned}
V(S_{n_j} x_{n_j}, T_k x_{n_j}) &= V\left(\sum_{i=1}^{r} \delta_{n_j}^{(i)} T_i x_{n_j}, T_k x_{n_j} \right) \\
&\le \sum_{i=1}^{r} \delta_{n_j}^{(i)} V(T_i x_{n_j}, T_k x_{n_j}) \\
&\le \sum_{i=1}^{r} \delta_{n_j}^{(i)} (V(T_i x_{n_j}, T_k x_{n_j}) + V(T_k x_{n_j}, T_i x_{n_j})) \\
&= 2 \sum_{i=1}^{r} \delta_{n_j}^{(i)} \langle T_i x_{n_j} - T_k x_{n_j}, JT_i x_{n_j} - JT_k x_{n_j} \rangle \\
&\le 2 \sum_{i=1}^{r} \delta_{n_j}^{(i)} \|T_i x_{n_j} - T_k x_{n_j}\| \|JT_i x_{n_j} - JT_k x_{n_j}\|
\end{aligned}
$$

for each $k = 1, 2, \ldots, r$ and hence we have $\lim_{j \longrightarrow \infty} V(S_{n_j} x_{n_j}, T_k x_{n_j}) = 0$. From Lemma 2.2, we have $\lim_{j \longrightarrow \infty} \|S_{n_j} x_{n_j} - T_k x_{n_j}\| = 0$ for each $k = 1, 2, \ldots, r$ and hence

$$
\lim_{j \longrightarrow \infty} \|T_k x_{n_j} - x_{n_j}\| \le \lim_{j \longrightarrow \infty} \|T_k x_{n_j} - S_{n_j} x_{n_j}\| + \lim_{j \longrightarrow \infty} \|S_{n_j} x_{n_j} - x_{n_j}\| = 0.
$$

for all $k = 1, 2, \ldots, r$. By the assumption of T_k, it follows that $x_0 \in \bigcap_{i=1}^{r} F(T_i)$. Therefore we have

$$
x_0 \in \bigcap_{i=1}^{r} F(T_i) \subset \bigcap_{n=1}^{\infty} C_n
$$

and hence $x_0 = R_{\bigcap_{i=1}^{r} F(T_i)} x$, which completes the proof. $\square$

In the end of this section, we will discuss the assumptions for the coefficients used in Theorem 4.1. Plubtieng and Ungchittrakool [27] proved the following theorem:

Theorem 4.2 (Plubtieng-Ungchittrakool [27]). *Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty closed convex subset of E, and $T_1, T_2, \dots, T_r$ relatively nonexpansive mappings of C into itself such that $\bigcap_{i=1}^r F(T_i)$ is nonempty. For a point $x \in E$ chosen arbitrarily, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$, and*

$$\begin{cases} y_n = J^{-1}\left(\alpha_n J x_n + (1-\alpha_n)\displaystyle\sum_{i=0}^{r} \beta_n^{(i)} J T_i x_n\right), \\ C_{n+1} = \{z \in C_n : V(z, y_n) \le V(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x \end{cases}$$

for all $n \in \mathbb{N}$, where T_0 is the identity mapping on C, and $\{\alpha_n\}, \{\beta_n^{(i)}\} \subset [0,1]$ are real sequences for $i = 0, 1, 2, \dots, r$ satisfying the following conditions:

(i) $\sup_{n\in\mathbb{N}} \alpha_n < 1$,

(ii) $\sum_{i=0}^r \beta_n^{(i)} = 1$ *for all* $n \in \mathbb{N}$, *and either*

  (a) $\liminf_{n\longrightarrow\infty} \beta_n^{(0)} \beta_n^{(i)} > 0$ *for all* $i = 1, 2, \dots, r$, *or*

  (b) $\lim_{n\longrightarrow\infty} \beta_n^{(0)} = 0$ *and* $\liminf_{n\longrightarrow\infty} \beta_n^{(k)} \beta_n^{(l)} > 0$ *for all* $k, l = 1, 2, \dots, r$ *with* $k \ne l$.

Then $\{x_n\}$ converges strongly to $\Pi_{\bigcap_{i=1}^r F(T_i)} x$, where Π_K is a generalized projection of E onto a nonempty closed convex subset K of E.

In order to compare our main result with the theorem above, we consider an analogous scheme to that in Theorem 4.2. For a given sequence $\{x_n\}$ in E, let $\{y_n\}$ be such that

$$y_n = \alpha_n x_n + (1-\alpha_n)\sum_{i=0}^{r} \beta_n^{(i)} T_i x_n$$

for $n \in \mathbb{N}$, where $T_1, T_2, \dots, T_r$ are mappings of C into itself, T_0 is the identity mapping on C, and $\{\alpha_n\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for $i = 0, 1, 2, \dots, r$. Suppose that

(i) $\liminf_{n\longrightarrow\infty} \alpha_n < 1$,

(ii) $\liminf_{n\longrightarrow\infty} \beta_n^{(i)} > 0$ for each $i = 1, 2, \dots r$,

(iii) $\sum_{i=0}^r \beta_n^{(i)} = 1$ for each $n \in \mathbb{N}$.

Then, letting $\gamma_n = \alpha_n + (1-\alpha_n)\beta_n^{(0)}$ and

$$\delta_n^{(i)} = \begin{cases} 0 & (\beta_n^{(0)} = 1) \\ \beta_n^{(i)}/(1-\beta_n^{(0)}) & (\beta_n^{(0)} < 1) \end{cases}$$

for $i = 1, 2, \dots, r$ and $n \in \mathbb{N}$, we have that

$$\begin{aligned} y_n &= \alpha_n x_n + (1-\alpha_n)\sum_{i=0}^{r} \beta_n^{(i)} T_i x_n \\ &= \gamma_n x_n + (1-\gamma_n)\sum_{i=1}^{r} \delta_n^{(i)} T_i x_n \end{aligned}$$

for every $n \in \mathbb{N}$. Then we can prove that the coefficients $\{\gamma_n\}$ and $\{\delta_n^{(i)}\}$ satisfy the conditions assumed in Theorem 4.1 such as

(i) $\liminf_{n\longrightarrow\infty} \gamma_n < 1$,
(ii) $\liminf_{n\longrightarrow\infty} \delta_n^{(i)} > 0$ for each $i = 1, 2, \dots r$,
(iii) $\sum_{i=1}^{r} \delta_n^{(i)} = 1$ for each $n \in \mathbb{N}$.

Indeed, since $\liminf_{n\longrightarrow\infty} \beta_n^{(i)} > 0$ for each $i = 1, 2, \dots, r$ and $\sum_{i=0}^{r} \beta_n^{(i)} = 1$, we have that

$$\liminf_{n\longrightarrow\infty}(1-\beta_n^{(0)}) = \liminf_{n\longrightarrow\infty} \sum_{i=1}^{r} \beta_n^{(i)} \geq \sum_{i=1}^{r} \liminf_{n\longrightarrow\infty} \beta_n^{(i)} > 0,$$

and thus $\limsup_{n\longrightarrow\infty} \beta_n^{(0)} < 1$. It follows that $\beta_n^{(0)} < 1$ for sufficiently large n and hence

$$\liminf_{n\longrightarrow\infty} \delta_n^{(i)} = \liminf_{n\longrightarrow\infty} \frac{\beta_n^{(i)}}{1-\beta_n^{(0)}} \geq \liminf_{n\longrightarrow\infty} \beta_n^{(i)} > 0$$

for each $i = 1, 2, \dots, r$. It is obvious that $\sum_{i=1}^{r} \delta_n^{(i)} = 1$ for each $n \in \mathbb{N}$ from the definition of $\{\delta_n^{(i)}\}$.

On the other hand, since $\liminf_{n\longrightarrow\infty} \alpha_n < 1$ and $\limsup_{n\longrightarrow\infty} \beta_n^{(0)} < 1$, there exist subsequences $\{\alpha_{n_j}\} \subset \{\alpha_n\}$ and $\{\beta_{n_j}^{(0)}\} \subset \{\beta_n^{(0)}\}$ converging $\alpha < 1$ and $\beta < 1$, respectively. Then we obtain that

$$\lim_{j\longrightarrow\infty} \gamma_{n_j} = \lim_{j\longrightarrow\infty} \alpha_{n_j} + (1-\alpha_{n_j})\beta_{n_j}^{(0)} = \alpha + (1-\alpha)\beta < 1$$

and hence $\liminf_{n\longrightarrow\infty} \gamma_n < 1$.

From the argument above, we obtain the following result.

Theorem 4.3. *Let E, C, and mappings $T_1, T_2, \dots, T_r$ be the same as in Theorem 4.1. Let T_0 be the identity mapping on C. For a point $x \in E$ chosen arbitrarily, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$, and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) \sum_{i=0}^{r} \beta_n^{(i)} T_i x_n, \\ C_{n+1} = \{z \in C_n : V(y_n, z) \leq V(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n^{(i)}\} \subset [0,1]$ are sequences for $i = 0, 1, 2, \dots, r$ satisfying the following conditions:

(i) $\liminf_{n\longrightarrow\infty} \alpha_n < 1$,
(ii) $\liminf_{n\longrightarrow\infty} \beta_n^{(i)} > 0$ *for each* $i = 1, 2, \dots, r$,
(iii) $\sum_{i=0}^{r} \beta_n^{(i)} = 1$ *for each* $n \in \mathbb{N}$.

Then $\{x_n\}$ converges strongly to $R_{\bigcap_{i=1}^{r} F(T_i)} x$.

Using the relations between generalized nonexpansive mappings and relatively nonexpansive mappings, and between sunny generalized nonexpansive retractions and generalized projections, we can see that the setting and the assumptions in Theorem 4.3 are more general than that of Theorem 4.2; see [19, 6, 26] for more details.

In particular, in the case where the underlying space E is a Hilbert space, the iterative schemes in Theorems 4.3 and 4.2 coincide with each other, except for the assumptions of mappings and the control coefficients. It is easy to see that the assumptions in Theorem 4.3 are milder than that of Theorem 4.2.

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