

A STRUCTURE THEOREM ON NON-HOMOGENEOUS LINEAR EQUATIONS IN HILBERT SPACES

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ABSTRACT. A very particular by-product of the result announced in the title reads as follows: Let $(X, \langle \cdot, \cdot \rangle)$ be a real Hilbert space, $T : X \longrightarrow X$ a compact and symmetric linear operator, and $z \in X$ such that the equation $T(x) - \|T\|x = z$ has no solution in X . For each $r > 0$, set $\gamma(r) = \sup_{x \in S_r} J(x)$, where $J(x) = \langle T(x) - 2z, x \rangle$ and $S_r = \{x \in X : \|x\|^2 = r\}$. Then, the function γ is C^1 , increasing and strictly concave in $]0, +\infty[$, with $\gamma'(\cdot) = \|T\|$; moreover, for each $r > 0$, the problem of maximizing J over S_r is well-posed, and one has

$$T(\hat{x}_r) - \gamma'(r)\hat{x}_r = z$$

where \hat{x}_r is the only global maximum of $J|_{S_r}$.

KEYWORDS : Linear equation; Hilbert space; Eigenvalue; Well-posedness.

1. INTRODUCTION AND PRELIMINARIES

Here and in the sequel, $(X, \langle \cdot, \cdot \rangle)$ is real Hilbert space. For each $r > 0$, set

$$S_r = \{x \in X : \|x\|^2 = r\}.$$

In [1], we established the following result (with the usual conventions $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$):

Theorem A ([1], Theorem 1). *Let $J : X \longrightarrow \mathbf{R}$ be a sequentially weakly continuous C^1 functional, with $J(0) = 0$. Set*

$$\rho = \limsup_{\|x\| \longrightarrow +\infty} \frac{J(x)}{\|x\|^2}$$

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and

$$\sigma = \sup_{x \in X \setminus \{0\}} \frac{J(x)}{\|x\|^2}.$$

Let a, b satisfy

$$\max\{0, \rho\} \leq a < b \leq \sigma.$$

Assume that, for each $\lambda \in]a, b[$, the functional $x \rightarrow \lambda\|x\|^2 - J(x)$ has a unique global minimum, say \hat{y}_λ . Let M_a (resp. M_b if $b < +\infty$ or $M_b = \emptyset$ if $b = +\infty$) be the set of all global minima of the functional $x \rightarrow a\|x\|^2 - J(x)$ (resp. $x \rightarrow b\|x\|^2 - J(x)$ if $b < +\infty$). Set

$$\alpha = \max \left\{ 0, \sup_{x \in M_b} \|x\|^2 \right\},$$

$$\beta = \inf_{x \in M_a} \|x\|^2$$

and, for each $r > 0$,

$$\gamma(r) = \sup_{x \in S_r} J(x).$$

Finally, assume that J has no local maximum with norm less than β .

Then, the following assertions hold:

- (a₁) the function $\lambda \rightarrow g(\lambda) := \|\hat{y}_\lambda\|^2$ is decreasing in $]a, b[$ and its range is $]\alpha, \beta[$;
- (a₂) for each $r \in]\alpha, \beta[$, the point $\hat{x}_r := \hat{y}_{g^{-1}(r)}$ is the unique global maximum of $J|_{S_r}$ and every maximizing sequence for $J|_{S_r}$ converges to \hat{x}_r ;
- (a₃) the function $r \rightarrow \hat{x}_r$ is continuous in $]\alpha, \beta[$;
- (a₄) the function γ is C^1 , increasing and strictly concave in $]\alpha, \beta[$;
- (a₅) one has

$$J'(\hat{x}_r) = 2\gamma'(r)\hat{x}_r$$

for all $r \in]\alpha, \beta[$;

(a₆) one has

$$\gamma'(r) = g^{-1}(r)$$

for all $r \in]\alpha, \beta[$.

We want to remark that, in the original statement of [1], one assumes that X is infinite-dimensional and that J has no local maxima in $X \setminus \{0\}$. These assumptions come from [2] whose results are applied to get (a₃), (a₄) and (a₅). The validity of the current formulation just comes from the proofs themselves given in [2] (see also [3]).

The aim of this very short paper is to show the impact of Theorem A in the theory of non-homogeneous linear equations in X .

2. MAIN RESULTS

Throughout the sequel, z is a non-zero point of X and $T : X \rightarrow X$ is a continuous linear operator.

We are interested in the study of the equation

$$T(x) - \lambda x = z$$

for $\lambda > \|T\|$. In this case, by the contraction mapping theorem, the equation has a unique non-zero solution, say \hat{v}_λ . Our structure result just concerns such solutions.

As usual, we say that:

- T is compact if, for each bounded set $A \subset X$, the set $\overline{T(A)}$ is compact ;

- T is symmetric if

$$\langle T(x), u \rangle = \langle T(u), x \rangle$$

for all $x, u \in X$.

We also denote by V the set (possibly empty) of all solutions of the equation

$$T(x) - \|T\|x = z$$

and set

$$\theta = \inf_{x \in V} \|x\|^2.$$

Of course, $\theta > 0$. Our result reads as follows:

Theorem 1. - Assume that T is compact and symmetric.

For each $\lambda > \|T\|$ and $r > 0$, set

$$g(\lambda) = \|\hat{v}_\lambda\|^2$$

and

$$\gamma(r) = \sup_{x \in S_r} J(x)$$

where

$$J(x) = \langle T(x) - 2z, x \rangle.$$

Then, the following assertions hold:

(b₁) the function g is decreasing in $] \|T\|, +\infty[$ and

$$g(] \|T\|, +\infty[) =]0, \theta[;$$

(b₂) for each $r \in]0, \theta[$, the point $\hat{x}_r := \hat{v}_{g^{-1}(r)}$ is the unique global maximum of $J|_{S_r}$ and every maximizing sequence for $J|_{S_r}$ converges to \hat{x}_r ;

(b₃) the function $r \longrightarrow \hat{x}_r$ is continuous in $]0, \theta[$;

(b₄) the function γ is C^1 , increasing and strictly concave in $]0, \theta[$;

(b₅) one has

$$T(\hat{x}_r) - \gamma'(r)\hat{x}_r = z$$

for all $r \in]0, \theta[$;

(b₆) one has

$$\gamma'(r) = g^{-1}(r)$$

for all $r \in]0, \theta[$.

Before giving the proof of Theorem 1, we establish the following

Proposition 1. - Let T be symmetric and let J be defined as in Theorem 1. Then, for $\tilde{x} \in X$, the following are equivalent:

(i) \tilde{x} is a local maximum of J .

(ii) \tilde{x} is a global maximum of J .

(iii) $T(\tilde{x}) = z$ and $\sup_{x \in X} \langle T(x), x \rangle \leq 0$.

Proof. First, observe that, since T is symmetric, the functional J is Gâteaux differentiable and its derivative, J' , is given by

$$J'(x) = 2(T(x) - z)$$

for all $x \in X$ ([4], p. 235). By the symmetry of T again, it is easy to check that, for each $x \in X$, the inequality

$$J(\tilde{x} + x) \leq J(\tilde{x}) \quad (1)$$

is equivalent to

$$\langle 2(T(\tilde{x}) - z) + T(x), x \rangle \leq 0. \quad (2)$$

Now, if (i) holds, then $J'(\tilde{x}) = 0$ (that is $T(\tilde{x}) = z$) and there is $\rho > 0$ such that (1) holds for all $x \in X$ with $\|x\| \leq \rho$. So, from (2), we have $\langle T(x), x \rangle \leq 0$ for the same

x and then, by linearity, for all $x \in X$, getting (iii). Vice versa, if (iii) holds, then (2) is satisfied for all $x \in X$ and so, by (1), \tilde{x} is a global maximum of J , and the proof is complete. \triangle

Proof of Theorem 1. For each $x \in X$, we clearly have

$$J(x) \leq \|T(x) - 2z\|\|x\| \leq \|T\|\|x\|^2 + 2\|z\|\|x\|$$

and so

$$\limsup_{\|x\| \rightarrow +\infty} \frac{J(x)}{\|x\|^2} \leq \|T\|. \quad (3)$$

Moreover, if $v \in X \setminus \{0\}$ and $\mu \in \mathbf{R} \setminus \{0\}$, we have

$$\frac{J(\mu v)}{\|\mu v\|^2} \geq -2 \frac{\langle z, v \rangle}{\mu \|v\|^2} - \|T\|$$

and so

$$\limsup_{x \rightarrow 0} \frac{J(x)}{\|x\|^2} = +\infty. \quad (4)$$

Moreover, the compactness of T implies that J is sequentially weakly continuous ([4], Corollary 41.9). Now, let $\lambda \geq \|T\|$. For each $x \in X$, set

$$\Phi(x) = \|x\|^2.$$

Then, for each $x, v \in X$, we have

$$\begin{aligned} \langle \lambda \Phi'(x) - J'(x) - (\lambda \Phi'(v) - J'(v)), x - v \rangle &= \langle 2\lambda(x - v) - 2(T(x) - T(v)), x - v \rangle \geq \\ &2\lambda\|x - v\|^2 - 2\|T(x) - T(v)\|\|x - v\| \geq 2(\lambda - \|T\|)\|x - v\|^2. \end{aligned} \quad (5)$$

From (5) we infer that the derivative of the functional $\lambda\Phi - J$ is monotone, and so the functional is convex. As a consequence, the critical points of $\lambda\Phi - J$ are exactly its global minima. So, \hat{v}_λ is the only global minimum of $\lambda\Phi - J$ if $\lambda > \|T\|$ and V is the set of all global minima of $\|T\|\Phi - J$. Now, assume that J has a local maximum, say w . Then, by Proposition 1, w is a global minimum of $-J$ and $\sup_{x \in X} \langle T(x), x \rangle \leq 0$. Since T is symmetric, this implies, in particular, that $\|T\|$ is not in the spectrum of T . So, V is a singleton. By Proposition 1 of [1], we have

$$\|w\|^2 \geq \theta.$$

In other words, J has no local maximum with norm less than θ . At this point, taking (3) and (4) into account, we see that the assumptions of Theorem A are satisfied (with $a = \|T\|$ and $b = +\infty$, and so $\alpha = 0$ and $\beta = \theta$), and the conclusion follows directly from that result. \triangle

Some remarks on Theorem 1 are now in order.

Remark 1. - Each of the two properties assumed on T cannot be dropped. Indeed, consider the following two counter-examples.

Take $X = \mathbf{R}^2$, $z = (1, 0)$ and $T(t, s) = (t + s, s - t)$ for all $(t, s) \in \mathbf{R}^2$. So, T is compact but not symmetric. In this case, we have

$$\begin{aligned} \hat{x}_r &= (-\sqrt{r}, 0), \\ \gamma(r) &= r + 2\sqrt{r} \end{aligned}$$

for all $r > 0$. Hence, in particular, we have

$$T(\hat{x}_r) - \gamma'(r)\hat{x}_r = (1, \sqrt{r}) \neq z.$$

That is, (b_5) is not satisfied.

Now, take $X = l_2$, $z = \{w_n\}$, where $w_2 = 1$ and $w_n = 0$ for all $n \neq 2$, and $T(\{x_n\}) = \{v_n\}$ for all $\{x_n\} \in l_2$, where $v_1 = 0$ and $v_n = x_n$ for all $n \geq 2$.

So, T is symmetric but not compact. In this case, we have $\theta = +\infty$ and

$$\gamma(r) = r - 2\sqrt{r}$$

for all $r \geq 4$. Hence, γ is not strictly concave in $]0, +\infty[$.

Remark 2. - Note that the compactness of T serves only to guarantee that the functional $x \longrightarrow \langle T(x), x \rangle$ is sequentially weakly continuous. So, Theorem 1 actually holds under such a weaker condition.

Remark 3. - A natural question is: if assertions $(b_1) - (b_6)$ hold, must the operator T be symmetric and the functional $x \longrightarrow \langle T(x), x \rangle$ sequentially weakly continuous?

Remark 4. - Note that if T , besides to be compact and symmetric, is also positive (i.e. $\inf_{x \in X} \langle T(x), x \rangle \geq 0$), then, by classical results, the operator $x \longrightarrow T(x) - \|T\|x$ is not surjective, and so there are $z \in X$ for which the conclusion of Theorem 1 holds with $\theta = +\infty$.

We conclude with an application of Theorem 1 to a classical Dirichlet problem.

So, let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary. Let λ_1 be the first eigenvalue of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Fix a non-zero continuous function $\varphi : \bar{\Omega} \longrightarrow \mathbf{R}$.

For each $\mu \in]0, \lambda_1[$, let u_μ be the unique classical solution of the problem

$$\begin{cases} -\Delta u = \mu(u + \varphi(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Also, set

$$\psi(\mu) = \int_{\Omega} |\nabla u_\mu(x)|^2 dx$$

and

$$\eta(r) = \sup_{u \in U_r} \Phi(u)$$

where

$$\Phi(u) = \int_{\Omega} |u(x)|^2 dx + 2 \int_{\Omega} \varphi(x)u(x) dx$$

and

$$U_r = \left\{ u \in H_0^1(\Omega) : \int_{\Omega} |\nabla u(x)|^2 dx = r \right\}.$$

Finally, denote by A the set of all classical solutions of the problem

$$\begin{cases} -\Delta u = \lambda_1(u + \varphi(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and set

$$\delta = \inf_{u \in A} \int_{\Omega} |\nabla u(x)|^2 dx.$$

Then, by using standard variational methods, we can directly draw the following result from Theorem 1 :

Theorem 2. - The following assertions hold:

(c_1) the function ψ is increasing in $]0, \lambda_1[$ and one has

$$\psi(]0, \lambda_1[) =]0, \delta[;$$

(c₂) for each $r \in]0, \delta[$, the function $w_r := u_{\psi^{-1}(r)}$ is the unique global maximum of $\Phi|_{U_r}$ and each maximizing sequence for $\Phi|_{U_r}$ converges to w_r with respect to the topology of $H_0^1(\Omega)$;

(c₃) the function $r \longrightarrow w_r$ is continuous in $]0, \delta[$ with respect to the topology of $H_0^1(\Omega)$;

(c₄) the function η is C^1 , increasing and strictly concave in $]0, \delta[$;

(c₅) for each $r \in]0, \delta[$, the function w_r is the unique classical solution of the problem

$$\begin{cases} -\Delta u = \frac{1}{\eta'(r)}(u + \varphi(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$

(c₆) one has

$$\eta'(r) = \frac{1}{\psi^{-1}(r)}$$

for all $r \in]0, \delta[$.

REFERENCES

1. B. Ricceri, *On a theory by Schechter and Tintarev*, Taiwanese J. Math., **12** (2008), 1303-1312.
2. M. Schechter and K. Tintarev, *Spherical maxima in Hilbert space and semilinear elliptic eigenvalue problems*, Differential Integral Equations, **3** (1990), 889-899.
3. K. Tintarev, *Level set maxima and quasilinear elliptic problems*, Pacific J. Math., **153** (1992), 185-200.
4. E. Zeidler, *Nonlinear functional analysis and its applications*, vol. III, Springer-Verlag, 1985.