

EXISTENCE OF SYSTEMS OF GENERALIZED VECTOR QUASI-EQUILIBRIUM PROBLEMS IN PRODUCT FC-SPACES[◇]

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ABSTRACT. In this paper, three classes of systems of generalized vector quasi-equilibrium problems are introduced and studied in product FC-spaces without convexity structure. We prove some new equilibrium existence theorems for three classes of systems of generalized vector quasi-equilibrium problems in noncompact product FC-spaces. These results improve and generalize some recent results in literature to product FC-spaces without any convexity structure.

KEYWORDS : Systems of generalized vector quasi-equilibrium problem; $C_i(x)$ – FC – diagonal quasi-convex; $C_i(x)$ – FC – quasi-convex; $C_i(x)$ – FC – quasi-convex-like; FC-spaces.

1. INTRODUCTION

Let X be a convex subset of a real topological vector space E (in short t.v.s.) and $F : X \times X \rightarrow R$ be a given function with $F(x, x) \geq 0$ for all $x \in X$. By equilibrium problem, Blum and Oettli [1] considered the problem of finding $u \in X$ such that $F(u, y) \geq 0$ for all $y \in X$. This problem contains optimization problems, Nash type equilibria problems, variational inequality problems, complementary problems and fixed point problems as special case. In 1980, Giannessi [2] introduced the vector variational inequality problem in finite dimensional Euclidean spaces. From the above applications, generalized vector quasi-equilibrium problems, and system of generalized vector quasi-equilibrium problems have become important developed directions of vector variational inequality theory, for example, see [4-26].

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In 2000, Ansari et al. [4] introduced the system of vector equilibrium problems (for short, SVEP), that is, a family of equilibrium problems for vector-valued bifunctions defined on a product set, with applications in vector optimization problems and Nash equilibrium problem [5] for vector valued functions. Recently, Ansari et al. [6] introduced the following concept of system of vector quasi-equilibrium problems (in short, SVQEP) as follows (see also in [7-13]; Let I be any index set and for each $i \in I$, let X_i be a topological vector space. Consider a family of nonempty convex subsets $\{K_i\}_{i \in I}$ with $K_i \subset X_i$. We denote by $K = \prod_{i \in I} K_i$ and $X = \prod_{i \in I} X_i$. For each $i \in I$, let Y_i be a topological vector space and let $C_i : K \rightarrow 2^{Y_i}$, $S_i : K \rightarrow 2^{K_i}$ and $F_i : K \times K_i \rightarrow 2^{Y_i}$ be multi-valued mappings. The system of vector quasi-equilibrium problems (in short, SVQEP), that is, to find $x \in K$ such that for each $i \in I$,

$$x_i \in S_i(x) : F_i(x, y_i) \not\subseteq -\text{int}C_i(x) \quad \forall y_i \in S_i(x). \quad (1.1)$$

If $S_i(x) = K_i$ for all $x \in K$, then (SVQEP) reduces to (SVEP) (see [4]) and if the index set I is singleton, then (SVQEP) becomes the vector quasi-equilibrium problem which contains vector quasi-optimization problem and vector quasi-variational inequality problem as special cases (see [3]).

In 2010, Li and Li [14] considered three following problems. Let X, Y and Z be three real topological spaces, let X and Y be Hausdorff spaces, $E \subset X$ and $D \subset Z$ be two nonempty subsets. Let $C : X \rightarrow 2^Y$ be set-valued mapping such that $C(x)$ be a proper, closed and convex cone of Y with nonempty interior. Let $S : E \rightarrow 2^E$, $T : E \rightarrow 2^D$ and $F : E \times D \times E \rightarrow 2^Y$ be three set-valued maps. Three classes of generalized vector quasi-equilibrium problems: Find $\bar{x} \in E$ and $\bar{z} \in T(\bar{x})$ such that

$$(i) \text{ (GVQEP I) } \bar{x} \in S(\bar{x}) \text{ and } F(\bar{x}, \bar{z}) \not\subseteq -\text{int}C(\bar{x}), \forall y \in S(\bar{x}).$$

$$(ii) \text{ (GVQEP II) } \bar{x} \in S(\bar{x}) \text{ and } F(\bar{x}, \bar{z}) \cap -\text{int}C(\bar{x}) = \emptyset, \forall y \in S(\bar{x}).$$

$$(iii) \text{ (GVQEP III) } \bar{x} \in S(\bar{x}) \text{ and } F(\bar{x}, \bar{z}) \subset -C(\bar{x}), \forall y \in S(\bar{x}).$$

Moreover, they obtained some existence results by using the well know Fan-KKM theorem without the compact assumption and unless otherwise specified.

On the other hand, it is well known that many existence theorems of maximal elements for set-valued mappings have been established in topological vector spaces, H-spaces and G-convex spaces by many authors. The notion of generalized convex (in short, G-convex) spaces was introduced by Park and Kim in [15, 16]. In 2005, Ding [17] was introduced the notion of a finitely continuous topological space (in short, FC-space). It is clear that the class of G-convex spaces is a subclass of FC-spaces. We emphasize that FC-space is a topological space without any convexity structure.

Motivated and inspired by research works mentioned above, in this paper, we introduce three classes of systems of generalized vector quasi-equilibrium problems in product FC-spaces. Let X and Y be two nonempty sets. We denote by 2^Y and $\langle X \rangle$ the family of all subsets of Y and the family of all nonempty finite subsets of X , respectively. Let I be any index set. For each $i \in I$, let X_i and Y_i be topological spaces and Z_i be a topological vector space. Let $X = \prod_{i \in I} X_i$, $Y = \prod_{i \in I} Y_i$ and for $i \in I$ and $z \in X$, $z_i = \pi_i(z)$ be the projection of z onto X_i . For each $i \in I$, let $A_i : Y \times X \rightarrow 2^{X_i}$, $T_i : Y \times X \rightarrow 2^{Y_i}$, $C_i : X \rightarrow 2^{Z_i}$ such that for each $z \in X$, $C_i(z)$ be a closed convex cone with nonempty interior, and $\Psi_i : X_i \times Y \times X \rightarrow 2^{Z_i}$ be set-valued mappings.

We consider the following three classes of systems of generalized vector quasi-equilibrium problems:

- (i) (SGVQEP 1) Find $(\hat{y}, \hat{z}) \in Y \times X$ such that for each $i \in I$,
 $\hat{y}_i \in T_i(\hat{y}, \hat{z}), \hat{z}_i \in A_i(\hat{y}, \hat{z})$ and $\Psi_i(x_i, \hat{y}, \hat{z}) \not\subseteq -\text{int}C_i(\hat{z}), \forall x_i \in A_i(\hat{y}, \hat{z})$.
- (ii) (SGVQEP 2) Find $(\hat{y}, \hat{z}) \in Y \times X$ such that for each $i \in I$,
 $\hat{y}_i \in T_i(\hat{y}, \hat{z}), \hat{z}_i \in A_i(\hat{y}, \hat{z})$ and $\Psi_i(x_i, \hat{y}, \hat{z}) \cap -\text{int}C_i(\hat{z}) = \emptyset, \forall x_i \in A_i(\hat{y}, \hat{z})$.
- (iii) (SGVQEP 3) Find $(\hat{y}, \hat{z}) \in Y \times X$ such that for each $i \in I$,
 $\hat{y}_i \in T_i(\hat{y}, \hat{z}), \hat{z}_i \in A_i(\hat{y}, \hat{z})$ and $\Psi_i(x_i, \hat{y}, \hat{z}) \subset -C_i(\hat{z}), \forall x_i \in A_i(\hat{y}, \hat{z})$.

Let V_0 be a topological vector space ordered by a proper closed convex cone D in V_0 and let $h : Y \times X \rightarrow 2^{V_0}$ is a set-valued mapping. Moreover, we introduce the notations of $C_i(z) - FC$ -diagonal quasi-convex, $C_i(x) - FC$ -quasi-convex and $C_i(x) - FC$ -quasiconvex-like for set-valued mappings in FC-space. By using these notions and an existence theorem of maximal elements for a family of set-valued mappings, we prove some new existence theorems of solutions for the SGVQEP (1), SGVQEP (2) and SGVQEP (3) in noncompact product FC-spaces without convexity structure. These results improve and generalize some recent known results in literature to noncompact FC-spaces.

2. PRELIMINARIES

Let Δ_n be the standard n -dimensional simplex with vertices $\{e_0, e_1, \dots, e_n\}$. If J is a nonempty subset of $\{0, 1, \dots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j : j \in J\}$. The following notion was introduced by Ben-El-Mechaiekh et al. [18].

Definition 2.1. (X, Γ) is said to be a L -convex space if X is a topological space and $\Gamma : \langle X \rangle \rightarrow 2^X$ is a mapping such that for each $N \in \langle X \rangle$ with $|N| = n + 1$, there exists a continuous mapping $\varphi_N : \Delta_n \rightarrow \Gamma(N)$ satisfying $A \in \langle N \rangle$ with $|A| = J + 1$ implies $\varphi_N(\Delta_J) \subset \Gamma(A)$, where Δ_J is the face of Δ_N corresponding to A .

The following notion of a finitely continuous topological space (in short, FC-space) was introduced by Ding [17].

Definition 2.2. (X, φ_N) is said to be a FC-space if X is a topological space and for each $N = \{x_0, \dots, x_n\} \in \langle X \rangle$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \rightarrow X$. A subset D of (X, φ_N) is said to be a FC-subspace of X if for each $N = \{x_0, \dots, x_n\} \in \langle X \rangle$ and for each $\{x_{i_0}, \dots, x_{i_k}\} \subset N \cap D$, $\varphi_N(\Delta_k) \subset D$, where $\Delta_k = \text{co}(\{e_{i_j} : j = 0, \dots, k\})$.

It is clear that any convex subset of a topological vector space, any H-space introduced by Horvath [19], any G-convex space introduced by Park and Kim [15, 16], and any L-convex spaces introduced by Ben-El-Mechaiekh et al. [18] are all FC-space.

By the definition of FC-subspaces of a FC-space, it is easy to see that if $\{B_i\} \in I$ is a family of FC-subspaces of a FC-space (Y, φ_N) and $\bigcap_{i \in I} B_i \neq \emptyset$, then $\bigcap_{i \in I} B_i$ is also a FC-subspace of (Y, φ_N) where I is any index set. For a subset A of (Y, φ_N) , we can define the FC-hull of A as follows:

$$FC(A) = \bigcap \{B \subset Y : A \subset B \text{ and } B \text{ is } FC\text{-subspace of } Y\}.$$

Clearly, $FC(A)$ is the smallest FC-subspace of Y containing A and each FC-subspace of a FC-space is also a FC-space.

Lemma 2.3. [20] Let (Y, φ_N) be a FC-space and A be a nonempty subset of Y . Then

$$FC(A) = \bigcup \{FC(N) : N \in \langle A \rangle\}.$$

Lemma 2.4. [20] Let X be a topological space, (Y, φ_N) be a FC-space and $G : X \rightarrow 2^Y$ be such that $G^{-1}(y) = \{x \in X : y \in G(x)\}$ is compactly open in X for each $y \in Y$. Then the mapping $FC(G) : X \rightarrow 2^Y$ defined by $FC(G)(x) = FC(G(x))$ for each $x \in X$ satisfies that $(FC(G))^{-1}(y)$ is also compactly open in X for each $y \in Y$.

Lemma 2.5. [17] Let I be any index set. For each $i \in I$, let (Y_i, φ_{N_i}) be a FC-space. Let $Y = \prod_{i \in I} Y_i$ and $\varphi_N = \prod_{i \in I} \varphi_{N_i}$. Then (Y, φ_N) is also a FC-space.

Lemma 2.6. [20] Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) be a FC-space, $X = \prod_{i \in I} X_i$ and K be a compact subset of X . For each $i \in I$, let $G_i : X \rightarrow 2^{X_i}$ be such that

- (i) for each $i \in I$ and $x \in X$, $G_i(x)$ is a FC-subspace of X_i ,
- (ii) for each $x \in X$, $\pi_i(x) \notin G_i(x)$ for all $i \in I$,
- (iii) for each $y_i \in X_i$, $G_i^{-1}(y_i)$ is compactly open in X
- (iv) for each $N_i \in \langle X_i \rangle$, there exists a nonempty compact FC-subspace L_{N_i} of X_i containing N_i and for each $x \in X \setminus K$, there exists $i \in I$ satisfying $L_{N_i} \cap G_i(x) \neq \emptyset$.

Then there exists $\hat{x} \in K$ such that $G_i(\hat{x}) = \emptyset$, for each $i \in I$.

Lemma 2.7. [21] Let X and Y be topological spaces and $G : X \rightarrow 2^Y$ be a set-valued mapping. Then G is lower semicontinuous in $x \in X$ if and only if for any $y \in G(x)$ and any net $\{x_\alpha\} \subset X$ satisfying $x_\alpha \rightarrow x$, there exists a net $\{y_\alpha\}$ such that $y_\alpha \in G(x_\alpha)$ and $y_\alpha \rightarrow y$.

Lemma 2.8. [22] Let X, Y and Z be topological spaces. Let $F : X \times Y \rightarrow 2^Z$ and $C : X \rightarrow 2^Z$ be set-valued mappings such that

- (i) C has closed (resp., open) graph,
- (ii) for each $y \in Y$, $F(\cdot, y)$ is lower semicontinuous on each compact subset of X .

Then the mapping $F^* : Y \rightarrow 2^X$ defined by $F^*(y) = \{x \in X : F(x, y) \subset C(x)\}$ (resp., $F^*(y) = \{x \in X : F(x, y) \cap C(x) = \emptyset\}$) has compactly closed values.

Lemma 2.9. [22] Let X, Y and Z be topological spaces. Let $F : X \times Y \rightarrow 2^Z$ and $C : X \rightarrow 2^Z$ be set-valued mappings such that

- (i) C has closed (resp., open) graph in $X \times Z$,
- (ii) for each $y \in Y$, $F(\cdot, y)$ is upper semicontinuous on each compact subset of X with nonempty compactly closed values.

Then the mapping $F^* : Y \rightarrow 2^X$ defined by $F^*(y) = \{x \in X : F(x, y) \not\subset C(x)\}$ (resp., $F^*(y) = \{x \in X : F(x, y) \cap C(x) \neq \emptyset\}$) has compactly closed values.

3. THE EXISTENCE OF THE SYSTEM OF GENERALIZED VARIATIONAL INEQUALITY

Throughout this section, unless otherwise specified, we assume the following notations and assumptions. Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) and (Y_i, φ_{N_i}) be FC-spaces, and Z_i be a nonempty set. Let $X = \prod_{i \in I} X_i$ and $Y = \prod_{i \in I} Y_i$. For each $i \in I$, let $A_i : Y \times X \rightarrow 2^{X_i}$, $T_i : Y \times X \rightarrow 2^{Y_i}$ and $C_i : Y \times X \rightarrow 2^{Z_i}$ such that for each $z \in X$, $C_i(z)$ be a closed convex cone with

nonempty interior, $\Psi_i : X_i \times Y \times X \rightarrow 2^{Z_i}$ be set valued mappings. From Li and Li [14] and Ding [22], we first propose the following generalized convexity definitions.

Definition 3.1. Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) and (Y_i, φ'_{N_i}) be FC -spaces, and Z_i be a nonempty set. Let $X = \prod_{i \in I} X_i$ and $Y = \prod_{i \in I} Y_i$. For each $i \in I$, let $A_i : Y \times X \rightarrow 2^{X_i}$, $T_i : Y \times X \rightarrow 2^{Y_i}$ and $C_i : Y \times X \rightarrow 2^{Z_i}$ such that for each $z \in X$, $C_i(z)$ be a closed convex cone with nonempty interior, $\Psi_i : X_i \times Y \times X \rightarrow 2^{Z_i}$ be set valued mappings. For each $i \in I$ and $y \in Y$, $\Psi_i : X_i \times Y \times X \rightarrow 2^{Z_i}$ is said to be

- (i) Ψ_i is said to be $C_i(z) - FC$ - diagonal quasi-convex of weak type 1 in first argument if each $N_i = \{x_{i,0}, \dots, x_{i,n}\} \in \langle X_i \rangle$ and for each $z \in X$ with $z_i \in FC(N_i)$, there exists $j \in \{0, \dots, k\}$ such that

$$\Psi_i(x_{i,j}, y, z) \not\subseteq -\text{int}C_i(z),$$

- (ii) Ψ_i is said to be $C_i(z) - FC$ - diagonal quasi-convex of weak type 2 in first argument if each $N_i = \{x_{i,0}, \dots, x_{i,n}\} \in \langle X_i \rangle$ and for each $z \in X$ with $z_i \in FC(N_i)$, there exists $j \in \{0, \dots, k\}$ such that

$$\Psi_i(x_{i,j}, y, z) \cap -\text{int}C_i(z) = \emptyset,$$

- (iii) Ψ_i is said to be $C_i(z) - FC$ - diagonal quasi-convex of strong type 1 in first argument if each $N_i = \{x_{i,0}, \dots, x_{i,n}\} \in \langle X_i \rangle$ and for each $z \in X$ with $z_i \in FC(N_i)$, there exists $j \in \{0, \dots, k\}$ such that

$$\Psi_i(x_{i,j}, y, z) \subset C_i(z).$$

- (iii) Ψ_i is said to be $C_i(z) - FC$ - diagonal quasi-convex of strong type 2 in first argument if each $N_i = \{x_{i,0}, \dots, x_{i,n}\} \in \langle X_i \rangle$ and for each $z \in X$ with $z_i \in FC(N_i)$, there exists $j \in \{0, \dots, k\}$ such that

$$\Psi_i(x_{i,j}, y, z) \cap C_i(z) \neq \emptyset.$$

Now, we establish an existence result for a solution of systems of generalized vector quasi-equilibrium problems (SGVQEP) as follows :

Theorem 3.2. Let K and H be nonempty compact subsets of X and Y , respectively, and for each $i \in I$, $C_i(z)$ be closed convex cone with nonempty interior. Suppose that for each $i \in I$, the following condition are satisfied;

- (i) for each $(y, z) \in Y \times X$, $T_i(y, z)$ and $A_i(y, z)$ are nonempty FC -subspaces of Y_i and X_i , respectively;
- (ii) for each $(u_i, v_i) \in Y_i \times X_i$, $T_i^{-1}(u_i)$ and $A_i^{-1}(v_i)$ are compactly open in $Y \times X$;
- (iii) the mapping $z \mapsto \text{int}C_i(z)$ has an open graph and for each $x_i \in X_i$, the mapping $(y, z) \mapsto \Psi_i(x_i, y, z)$ is upper semicontinuous on each compact subsets of $Y \times X$ with nonempty compactly closed values;
- (iv) for each $z \in X$, $\Psi_i : X_i \times Y \times X \rightarrow 2^{Z_i}$ is $C_i(z) - FC$ -diagonal quasi-convex of weak type 1 in first argument;
- (v) the set $W_i = \{(y, z) \in Y \times X : y_i \in T_i(y, z), z_i \in A_i(y, z)\}$ is compactly closed in $Y \times X$;
- (vi) there exists nonempty compact subset $H \times K$ of $Y \times X$, and for each $M_i \times N_i \in \langle Y_i \times X_i \rangle$, there exists compact FC -subspace $L_{M_i} \times L_{N_i}$ containing $M_i \times N_i$ such that for each $(y, z) \in Y \times X \setminus H \times K$, there exist $i \in I$, $\bar{u}_i \in T_i(y, z) \cap L_{M_i}$ and $\bar{v}_i \in A_i(y, z) \cap L_{N_i}$ satisfying $\Psi_i(\bar{v}_i, y, z) \subset -\text{int}C_i(z)$.

Then the solution set M_1 of SGVQEP(1) is nonempty and compact in $H \times K$, where $M_1 = \{(y, z) \in H \times K : y_i \in T_i(y, z), z_i \in A_i(y, z) \text{ and } \Psi_i(x_i, y, z) \not\subseteq -\text{int}C_i(z), \forall x_i \in A_i(y, z), i \in I\}$.

Proof. Step I. Show that M_1 is nonempty.

For each $i \in I$, we define a set-valued mapping $P_i : Y \times X \rightarrow 2^{X_i}$ by

$$P_i(y, z) = \{x_i \in X_i : \Psi_i(x_i, y, z) \subset -\text{int}C_i(z)\}, \forall (y, z) \in Y \times X.$$

We now, show that for each $i \in I$ and $(y, z) \in Y \times X$,

$$z_i = \pi_i(z) \notin FC(P_i(y, z)). \quad (3.1)$$

If it is false, then there exist $i \in I$ and $(\bar{y}, \bar{z}) \in Y \times X$ such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(P_i(\bar{y}, \bar{z}))$. Hence by Lemma 2.3, there exists $N_i = \{x_{i,o}, \dots, x_{i,n}\} \in \langle P_i(\bar{y}, \bar{z}) \rangle$ such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(N_i)$. Thus, we have

$$\Psi_i(x_{i,j}, \bar{y}, \bar{z}) \subset -\text{int}C_i(\bar{z}), \forall j = 0, \dots, n.$$

By (iv) and Definition 3.1(i), there exists $j \in \{0, \dots, n\}$ such that

$$\Psi_i(x_{i,j}, \bar{y}, \bar{z}) \not\subseteq -\text{int}C_i(\bar{z}), \forall j = 0, \dots, n,$$

which is a contradiction. Then, for each $i \in I$ and $(y, z) \in Y \times X$, $z_i = \pi_i(z) \notin FC(P_i(y, z))$. Hence by condition (iii) and Lemma 2.9, for each $i \in I$ and $x_i \in X_i$, the set

$$P_i^{-1}(x_i) = \{(y, z) \in Y \times X : \Psi_i(x_i, y, z) \subset -\text{int}C_i(z)\}$$

is compactly open in $Y \times X$. From Lemma 2.4 that $(FC(P_i))^{-1}(x_i)$ is also compactly open in $Y \times X$ for each $x_i \in X_i$. By Lemma 2.5, it follows $i \in I, Y_i \times X_i$ is a FC-space and $Y \times X$ is also a FC-space for all $i \in I$.

Next, for each $i \in I$, we define a set-valued mapping $G_i : Y \times X \rightarrow 2^{Y_i \times X_i}$ by

$$G_i(y, z) = \begin{cases} T_i(y, z) \times [A_i(y, z) \cap FC(P_i(y, z))], & \text{if } (y, z) \in W_i, \\ T_i(y, z) \times A_i(y, z), & \text{if } (y, z) \notin W_i. \end{cases}$$

By condition (i), for each $i \in I$ and $(y, z) \in Y \times X$, $G_i(y, z)$ is an FC-subspace of $Y_i \times X_i$. From the definition of W_i and (3.2), $(y_i, z_i) \notin G_i(y, z)$, for each $i \in I$ and $(y, z) \in Y \times X$.

Then, for each $i \in I$ and $(u_i, v_i) \in Y_i \times X_i$, we have

$$\begin{aligned} G_i^{-1}(u_i, v_i) &= [T_i^{-1}(u_i) \cap A_i^{-1}(v_i) \cap (FC(P_i))^{-1}(v_i)] \\ &\quad \cup [(Y \times X \setminus W_i) \cap T_i^{-1}(u_i) \cap A_i^{-1}(v_i)]. \end{aligned}$$

Since $(FC(P_i))^{-1}(v_i)$ is compactly open in $Y \times X$ for each $v_i \in X_i$, it follows by the condition (ii) that $G_i^{-1}(u_i, v_i)$ is also compactly open in $Y \times X$. By (vi), for each $H_i = M_i \times N_i \in \langle Y_i \times X_i \rangle$ there exists compact FC-subspace $L_{H_i} = L_{M_i} \times L_{N_i}$ of $Y_i \times X_i$ containing H_i such that

$$G_i(y, z) \cap L_{H_i} \neq \emptyset.$$

Thus all conditions of Lemma 2.6 are satisfied. Hence by Lemma 2.6, there exists $(\hat{y}, \hat{z}) \in H \times K$ such that $G_i(\hat{y}, \hat{z}) = \emptyset$ for each $i \in I$. If $(\hat{y}, \hat{z}) \notin W_j$ for some $j \in I$, then either $T_j(\hat{y}, \hat{z}) = \emptyset$ or $A_j(\hat{y}, \hat{z}) = \emptyset$, which contradicts the condition (i). Therefore, $(\hat{y}, \hat{z}) \in W_i$ for all $i \in I$. This implies that for each $i \in I$, $\hat{y}_i \in T_i(\hat{y}, \hat{z})$, $\hat{z}_i \in A_i(\hat{y}, \hat{z})$ and $A_i(\hat{y}, \hat{z}) \cap FC(P_i(\hat{y}, \hat{z})) = \emptyset$ and hence $A_i(\hat{y}, \hat{z}) \cap P_i(\hat{y}, \hat{z}) = \emptyset$. Therefore, for each $i \in I$, $\hat{y}_i \in T_i(\hat{y}, \hat{z})$, $\hat{z}_i \in A_i(\hat{y}, \hat{z})$ and

$$\Psi_i(x_i, \hat{y}, \hat{z}) \not\subseteq -\text{int}C_i(\hat{z}), \forall x_i \in A_i(\hat{y}, \hat{z}).$$

Hence, $(\hat{y}, \hat{z}) \in M_1$, and M_1 is nonempty.

Step II. Show that M_1 is compact.

By condition (iii) and Lemma 2.9, we note that, for each $i \in I$ and $v_i \in X_i$, the set

$$\{(y, z) \in Y \times X : \Psi_i(v_i, y, z) \not\subseteq -\text{int}C_i(z)\}$$

is compactly closed in $Y \times X$. This implies that the set

$$\{(y, z) \in H \times K : \Psi_i(x_i, y, z) \not\subseteq -\text{int}C_i(z), \forall x_i \in A_i(y, z)\}$$

is closed in $H \times K$, for all $i \in I$. By condition (v), the set

$$W_i = \{(y, z) \in H \times K : y_i \in T_i(y, z), z_i \in A_i(y, z)\}$$

is also closed in $H \times K$ for each $i \in I$. It follows that M_1 is closed in $H \times K$. Hence $H \times K$ is compact in $Y \times X$ and therefore M_1 is nonempty and compact. This completes the proof. \square

Theorem 3.3. Let K and H be nonempty compact subsets of X and Y , respectively, and for each $i \in I$, $C_i(z)$ be closed convex cone with nonempty interior. Suppose that for each $i \in I$, the following condition are satisfied;

- (i) for each $(y, z) \in Y \times X$, $T_i(y, z)$ and $A_i(y, z)$ are nonempty FC-subspaces of Y_i and X_i , respectively;
- (ii) for each $(u_i, v_i) \in Y_i \times X_i$, $T_i^{-1}(u_i)$ and $A_i^{-1}(v_i)$ are compactly open in $Y \times X$;
- (iii) the mapping $z \mapsto \text{int}C_i(z)$ has an open graph and for each $x_i \in X_i$, the mapping $(y, z) \mapsto \Psi_i(x_i, y, z)$ is lower semicontinuous on each compact subsets of $Y \times X$ with nonempty compactly closed values;
- (iv) for each $z \in X$, $\Psi_i : X_i \times Y \times X \rightarrow 2^{Z_i}$ is $C_i(z)$ -FC-diagonal quasi-convex of weak type 2 in first argument;
- (v) the set $W_i = \{(y, z) \in Y \times X : y_i \in T_i(y, z), z_i \in A_i(y, z)\}$ is compactly closed in $Y \times X$;
- (vi) there exists nonempty compact subset $H \times K$ of $Y \times X$, and for each $M_i \times N_i \in \langle Y_i \times X_i \rangle$, there exists compact FC-subspace $L_{M_i} \times L_{N_i}$ containing $M_i \times N_i$ such that for each $(y, z) \in Y \times X \setminus H \times K$, there exist $i \in I$, $\bar{u}_i \in T_i(y, z) \cap L_{M_i}$ and $\bar{v}_i \in A_i(y, z) \cap L_{N_i}$ satisfying $\Psi_i(\bar{v}_i, y, z) \cap (-\text{int}C_i(z)) \neq \emptyset$.

Then the solution set M_2 of SGVQEP(2) is nonempty and compact in $H \times K$, where $M_2 = \{(y, z) \in H \times K : y_i \in T_i(y, z), z_i \in A_i(y, z) \text{ and } \Psi_i(x_i, y, z) \cap (-\text{int}C_i(z)) = \emptyset, \forall x_i \in A_i(y, z), i \in I\}$.

Proof. For each $i \in I$, we define a set-valued mapping $P_i : Y \times X \rightarrow 2^{X_i}$ by

$$P_i(y, z) = \{x_i \in X_i : \Psi_i(x_i, y, z) \cap (-\text{int}C_i(z)) \neq \emptyset\}, \forall (y, z) \in Y \times X.$$

We now show that, for each $i \in I$ and $(y, z) \in Y \times X$,

$$z_i = \pi_i(z) \notin FC(P_i(y, z)). \quad (3.2)$$

If it is false, then there exist $i \in I$ and $(\bar{y}, \bar{z}) \in Y \times X$ such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(P_i(\bar{y}, \bar{z}))$. Hence by Lemma 2.3, there exists $N_i = \{x_{i,o}, \dots, x_{i,n}\} \in \langle P_i(\bar{y}, \bar{z}) \rangle$, such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(N_i)$. Thus, we note that

$$\Psi_i(x_{i_j}, \bar{y}, \bar{z}) \subset -\text{int}C_i(\bar{z}), \forall j = 0, \dots, n.$$

It follows by (iv) and Definition 3.1.(ii) that there exists $j \in \{0, \dots, n\}$ such that

$$\Psi_i(x_{i_j}, \bar{y}, \bar{z}) \not\subseteq -\text{int}C_i(\bar{z}), \forall j = 0, \dots, n,$$

which is a contradiction. Hence, for each $i \in I$ and $(y, z) \in Y \times X, z_i = \pi_i(z) \notin FC(P_i(y, z))$. By condition (iii) and Lemma 2.8, we note that, the set

$$P_i^{-1}(x_i) = \{(y, z) \in Y \times X : \Psi_i(x_i, y, z) \cap (-\text{int}C_i(z)) \neq \emptyset\}$$

is compactly open in $Y \times X$ for all $i \in I$ and $x_i \in X_i$. It follows from Lemma 2.4 that $(FC(P_i))^{-1}(x_i)$ is also compactly open in $Y \times X$ for each $x_i \in X_i$. Hence, by Lemma 2.5, for each $i \in I, Y_i \times X_i$ is a FC-space and $Y \times X$ is also an FC-space. By the similar argument as in the proof of Theorem 3.2, we obtain the desired result. \square

Theorem 3.4. *Let K and H be nonempty compact subsets of X and Y , respectively, and for each $i \in I, C_i(z)$ be closed convex cone with nonempty interior. Suppose that for each $i \in I$,*

- (i) *for each $(y, z) \in Y \times X, T_i(y, z)$ and $A_i(y, z)$ are nonempty FC-subspaces of Y_i and X_i , respectively;*
- (ii) *for each $(u_i, v_i) \in Y_i \times X_i, T_i^{-1}(u_i)$ and $A_i^{-1}(v_i)$ are compactly open in $Y \times X$;*
- (iii) *the mapping $z \mapsto \text{int}C_i(z)$ has an open graph and for each $x_i \in X_i$, the mapping $(y, z) \mapsto \Psi_i(x_i, y, z)$ is lower semicontinuous on each compact subsets of $Y \times X$ with nonempty compactly closed values;*
- (iv) *for each $z \in X, \Psi_i : X_i \times Y \times X \rightarrow 2^{Z_i}$ is $C_i(z)$ -FC-diagonal quasi-convex of strong type 1 in first argument;*
- (v) *the set $W_i = \{(y, z) \in Y \times X : y_i \in T_i(y, z), z_i \in A_i(y, z)\}$ is compactly closed in $Y \times X$;*
- (vi) *there exists nonempty compact subset $H \times K$ of $Y \times X$, and for each $M_i \times N_i \in \langle Y_i \times X_i \rangle$, there exists compact FC-subspace $L_{M_i} \times L_{N_i}$ containing $M_i \times N_i$ such that for each $(y, z) \in Y \times X \setminus H \times K$, there exist $i \in I, \bar{u}_i \in T_i(y, z) \cap L_{M_i}$ and $\bar{v}_i \in A_i(y, z) \cap L_{N_i}$ satisfying $\Psi_i(\bar{v}_i, y, z) \not\subseteq -C_i(z)$.*

Then the solution set M_3 of SGVQEP(3) is nonempty and compact in $H \times K$, where $M_3 = \{(y, z) \in H \times K : y_i \in T_i(y, z), z_i \in A_i(y, z) \text{ and } \Psi_i(x_i, y, z) \subset -C_i(z), \forall x_i \in A_i(y, z), i \in I\}$.

Proof. For each $i \in I$, we define a set-valued mapping $P_i : Y \times X \rightarrow 2^{X_i}$ by

$$P_i(y, z) = \{x_i \in X_i : \Psi_i(x_i, y, z) \not\subseteq -C_i(z)\}, \forall (y, z) \in Y \times X.$$

We now show that, for each $i \in I$ and $(y, z) \in Y \times X$,

$$z_i = \pi_i(z) \notin FC(P_i(y, z)). \quad (3.3)$$

If it is false, then there exist $i \in I$ and $(\bar{y}, \bar{z}) \in Y \times X$ such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(P_i(\bar{y}, \bar{z}))$. Hence by Lemma 2.3, there exists $N_i = \{x_{i,0}, \dots, x_{i,n}\} \in \langle P_i(\bar{y}, \bar{z}) \rangle$, such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(N_i)$. Thus, we note that

$$\Psi_i(x_{i,j}, \bar{y}, \bar{z}) \not\subseteq -C_i(\bar{z}), \forall j = 0, \dots, n.$$

It follows (iv) and Definition 3.1.(iii) that there exists $j \in \{0, \dots, n\}$ such that

$$\Psi_i(x_{i,j}, \bar{y}, \bar{z}) \subset -C_i(\bar{z}), \forall j = 0, \dots, n,$$

which is a contradiction. Hence, for each $i \in I$ and $(y, z) \in Y \times X, z_i = \pi_i(z) \notin FC(P_i(y, z))$. Then by condition (iii) and Lemma 2.8, for each $i \in I$ and $x_i \in X_i$, the set

$$P_i^{-1}(x_i) = \{(y, z) \in Y \times X : \Psi_i(x_i, y, z) \not\subseteq -C_i(z)\}$$

is compactly open in $Y \times X$. It follows from Lemma 2.4 that $(FC(P_i))^{-1}(x_i)$ is also compactly open in $Y \times X$ for each $x_i \in X_i$. Hence, by Lemma 2.5, for

each $i \in I$, $Y_i \times X_i$ is a FC-space and $Y \times X$ is also an FC-space. By the similar argument as in the proof of Theorem 3.2, we obtain the desired result. \square

Definition 3.5. For each $i \in I$ and $(y, z) \in Y \times X$, $\Psi_i : X_i \times Y \times X \rightarrow 2^{Z_i}$ is said to be

- (i) $C_i(z) - FC$ -quasi-convex in first argument if each $(y, z) \in Y \times X$, $N_i = \{x_{i,0}, \dots, x_{i,n}\} \in \langle X_i \rangle$, $\{x_{i,i_0}, \dots, x_{i,i_k}\} \subset N_i$ and $x_i^* \in \varphi'_{N_i}(\Delta_k)$, there exists $j \in \{0, \dots, k\}$ such that

$$\Psi_i(x_{i,i_j}, y, z) \subset \Psi_i(x_i^*, y, z) + C_i(z),$$

- (ii) $C_i(z) - FC$ -quasi-convex-like in first argument if each $(y, z) \in Y \times X$, $N_i = \{x_{i,0}, \dots, x_{i,n}\} \in \langle X_i \rangle$, $\{x_{i,i_0}, \dots, x_{i,i_k}\} \subset N_i$ and $x_i^* \in \varphi'_{N_i}(\Delta_k)$, there exists $j \in \{0, \dots, k\}$ such that

$$\Psi_i(x_i^*, y, z) \subset \Psi_i(x_{i,i_j}, y, z) - C_i(z).$$

Form Ding [23], we give the following lemma:

Lemma 3.6. [23] For each $i \in I$, let Z_i be a topological vector space and $C_i : X \rightarrow 2^{Z_i}$ be a set-valued mapping, such that for each $z \in X$, $C_i(z)$ is closed convex cone in Z_i with nonempty interior. If for each $i \in I$, $(y, z) \in Y \times X$, $\Psi_i : X_i \times Y \times X \rightarrow 2^{Z_i}$ is $C_i(z) - FC$ -quasiconvex-like in first argument, then, the set

$$\{x_i \in X_i : \Psi_i(x_i, y, z) \not\subseteq -\text{int}C_i(z)\}$$

is FC-subspace of X_i .

Lemma 3.7. [23] For each $i \in I$, let Z_i be a topological vector space and $C_i : X \rightarrow 2^{Z_i}$ be a set-valued mapping, such that for each $z \in X$, $C_i(z)$ is closed convex cone in Z_i with nonempty interior. If for each $i \in I$, $(y, z) \in Y \times X$, $\Psi_i : X_i \times Y \times X \rightarrow 2^{Z_i}$ is $C_i(z) - FC$ -quasiconvex in first argument, then, the set

$$\{x_i \in X_i : \Psi_i(x_i, y, z) \cap -\text{int}C_i(z) = \emptyset\}$$

is FC-subspace of X_i .

Lemma 3.8. [23] For each $i \in I$, let Z_i be a topological vector space and $C_i : X \rightarrow 2^{Z_i}$ be a set-valued mapping, such that for each $z \in X$, $C_i(z)$ is closed convex cone in Z_i with nonempty interior. If for each $i \in I$, $(y, z) \in Y \times X$, $\Psi_i : X_i \times Y \times X \rightarrow 2^{Z_i}$ is $C_i(z) - FC$ -quasiconvex-like in first argument, then, the set

$$\{x_i \in X_i : \Psi_i(x_i, y, z) \subset -C_i(z)\}$$

is FC-subspace of X_i .

Theorem 3.9. Let K and H be nonempty compact subsets of X and Y , respectively, and for each $i \in I$, $C_i(z)$ be closed convex cone with nonempty interior. Suppose that for each $i \in I$, the following condition are satisfied;

- (i) for each $(y, z) \in Y \times X$, $T_i(y, z)$ and $A_i(y, z)$ are nonempty FC-subspaces of Y_i and X_i , respectively;
- (ii) for each $(u_i, v_i) \in Y_i \times X_i$, $T_i^{-1}(u_i)$ and $A_i^{-1}(v_i)$ are compactly open in $Y \times X$;
- (iii) the mapping $z \mapsto \text{int}C_i(z)$ has an open graph and for each $x_i \in X_i$, the mapping $(y, z) \mapsto \Psi_i(x_i, y, z)$ is upper semicontinuous on each compact subsets of $Y \times X$ with nonempty compactly closed values;
- (iv) for each $z \in X$, $\Psi_i : X_i \times Y \times X \rightarrow 2^{Z_i}$ is $C_i(z) - FC$ -quasi-convex-like in first argument and for each $(y, z) \in Y \times X$, $\Psi_i(x_i, y, z) \not\subseteq -\text{int}C_i(z)$;

- (v) the set $W_i = \{(y, z) \in Y \times X : y_i \in T_i(y, z), z_i \in A_i(y, z)\}$ is compactly closed in $Y \times X$;
- (vi) there exists nonempty compact subset $H \times K$ of $Y \times X$, and for each $M_i \times N_i \in \langle Y_i \times X_i \rangle$, there exists compact FC-subspace $L_{M_i} \times L_{N_i}$ containing $M_i \times N_i$ such that for each $(y, z) \in Y \times X \setminus H \times K$, there exist $i \in I, \bar{u}_i \in T_i(y, z) \cap L_{M_i}$ and $\bar{v}_i \in A_i(y, z) \cap L_{N_i}$ satisfying $\Psi_i(\bar{v}_i, y, z) \subset -\text{int}C_i(z)$.

Then the solution set M_1 of SGVQEP(1) is nonempty and compact in $H \times K$, where $M_1 = \{(y, z) \in H \times K : y_i \in T_i(y, z), z_i \in A_i(y, z) \text{ and } \Psi_i(x_i, y, z) \not\subset -\text{int}C_i(z), \forall x_i \in A_i(y, z), i \in I\}$.

Proof. We first show that, Ψ_i is $C_i(z) - FC$ -diagonal quasi-convex of SK-type 1 in first argument for all $i \in I$ and $z \in X$. If it is false, then there exist $i \in I, N_i = \{x_{i,0}, \dots, x_{i,n}\} \in \langle X_i \rangle$ and for each $\bar{z} \in X$ with $\bar{z}_i \in FC(N_i)$, such that $\Psi_i(x_{i,j}, y, z) \subset -\text{int}C_i(z)$, for all $j \in \{0, \dots, k\}$. Hence, we obtain $N_i \subset P_i(y, \bar{z})$. It follows from (iv) and Lemma 3.6 that for each $i \in I$ and $(y, \bar{z}) \in Y \times X$, $P_i(y, \bar{z})$ is an FC-subspace of X_i . Then we have $\bar{z}_i \in FC(N_i) \subset P_i(y, \bar{z})$, which contradicts the fact that for each $(y, z) \in Y \times X, z_i \notin P_i(y, z)$. Therefore, for each $i \in I$ and $z \in X$, Ψ_i is $C_i(z) - FC$ -diagonal quasi-convex of SK-type 1 in first argument. Thus all conditions of Theorem 3.2 for the SGVQEP(1) are satisfied. Hence the conclusion of Theorem 3.9 hold from Theorem 3.2. This completes the proof. \square

Theorem 3.10. Let K and H be nonempty compact subsets of X and Y , respectively, and for each $i \in I, C_i(z)$ be closed convex cone with nonempty interior. Suppose that for each $i \in I$, the following condition are satisfied;

- (i) for each $(y, z) \in Y \times X, T_i(y, z)$ and $A_i(y, z)$ are nonempty FC-subspaces of Y_i and X_i , respectively;
- (ii) for each $(u_i, v_i) \in Y_i \times X_i, T_i^{-1}(u_i)$ and $A_i^{-1}(v_i)$ are compactly open in $Y \times X$;
- (iii) the mapping $z \mapsto \text{int}C_i(z)$ has an open graph and for each $x_i \in X_i$, the mapping $(y, z) \mapsto \Psi_i(x_i, y, z)$ is lower semicontinuous on each compact subsets of $Y \times X$ with nonempty compactly closed values;
- (iv) for each $z \in X, \Psi_i : X_i \times Y \times X \rightarrow 2^{Z_i}$ is $C_i(z) - FC$ -quasi-convex in first argument and for each $(y, z) \in Y \times X, \Psi_i(x_i, y, z) \cap (-\text{int}C_i(z)) = \emptyset$;
- (v) the set $W_i = \{(y, z) \in Y \times X : y_i \in T_i(y, z), z_i \in A_i(y, z)\}$ is compactly closed in $Y \times X$;
- (vi) there exists nonempty compact subset $H \times K$ of $Y \times X$, and for each $M_i \times N_i \in \langle Y_i \times X_i \rangle$, there exists compact FC-subspace $L_{M_i} \times L_{N_i}$ containing $M_i \times N_i$ such that for each $(y, z) \in Y \times X \setminus H \times K$, there exist $i \in I, \bar{u}_i \in T_i(y, z) \cap L_{M_i}$ and $\bar{v}_i \in A_i(y, z) \cap L_{N_i}$ satisfying $\Psi_i(\bar{v}_i, y, z) \cap (-\text{int}C_i(z)) \neq \emptyset$.

Then the solution set M_2 of SGVQEP(2) is nonempty and compact in $H \times K$, where $M_2 = \{(y, z) \in H \times K : y_i \in T_i(y, z), z_i \in A_i(y, z) \text{ and } \Psi_i(x_i, y, z) \cap (-\text{int}C_i(z)) = \emptyset, \forall x_i \in A_i(y, z), i \in I\}$.

Proof. We first show that, Ψ_i is $C_i(z) - FC$ -diagonal quasi-convex of SK-type 2 in first argument, for each $i \in I$ and $z \in X$. If it is false, then, there exist $i \in I, N_i = \{x_{i,0}, \dots, x_{i,n}\} \in \langle X_i \rangle$ and for each $\bar{z} \in X$ with $\bar{z}_i \in FC(N_i)$, such that $\Psi_i(x_{i,j}, y, z) \cap -\text{int}C_i(z) \neq \emptyset$ for all $j \in \{0, \dots, k\}$. Hence, we obtain $N_i \subset P_i(y, \bar{z})$. It follows from (iv) and Lemma 3.7 that for each $i \in I$ and $(y, \bar{z}) \in Y \times X$, $P_i(y, \bar{z})$ is an FC-subspace of X_i . Then we have $\bar{z}_i \in FC(N_i) \subset P_i(y, \bar{z})$, which contradicts the fact that for each $(y, z) \in Y \times X, z_i \notin P_i(y, z)$. Therefore, for each $i \in I$ and

$z \in X$, Ψ_i is $C_i(z)$ – FC –diagonal quasi-convex of SK-type 2 in first argument. Thus all conditions of Theorem 3.3 for the SGVQEP(2) are satisfied. Hence the conclusion of Theorem 3.10 hold from Theorem 3.3. This completes the proof. \square

By applying Lemma 2.6, Lemma 3.8 and the similar argument as in the proof of Theorem 3.9-3.10, we can easily prove the following results.

Theorem 3.11. *Let K and H be nonempty compact subsets of X and Y , respectively, and for each $i \in I$, $C_i(z)$ be closed convex cone with nonempty interior. Suppose that for each $i \in I$,*

- (i) *for each $(y, z) \in Y \times X$, $T_i(y, z)$ and $A_i(y, z)$ are nonempty FC –subspaces of Y_i and X_i , respectively;*
- (ii) *for each $(u_i, v_i) \in Y_i \times X_i$, $T_i^{-1}(u_i)$ and $A_i^{-1}(v_i)$ are compactly open in $Y \times X$;*
- (iii) *the mapping $z \mapsto \text{int}C_i(z)$ has an open graph and for each $x_i \in X_i$, the mapping $(y, z) \mapsto \Psi_i(x_i, y, z)$ is lower semicontinuous on each compact subsets of $Y \times X$ with nonempty compactly closed values;*
- (iv) *for each $z \in X$, $\Psi_i : X_i \times Y \times X \rightarrow 2^{Z_i}$ is $C_i(z)$ – FC –quasi-convex-like in first argument and for each $(y, z) \in Y \times X$, $\Psi_i(x_i, y, z) \subset -C_i(z)$;*
- (v) *the set $W_i = \{(y, z) \in Y \times X : y_i \in T_i(y, z), z_i \in A_i(y, z)\}$ is compactly closed in $Y \times X$;*
- (vi) *there exists nonempty compact subset $H \times K$ of $Y \times X$, and for each $M_i \times N_i \in \langle Y_i \times X_i \rangle$, there exists compact FC –subspace $L_{M_i} \times L_{N_i}$ containing $M_i \times N_i$ such that for each $(y, z) \in Y \times X \setminus H \times K$, there exist $i \in I$, $\bar{u}_i \in T_i(y, z) \cap L_{M_i}$ and $\bar{v}_i \in A_i(y, z) \cap L_{N_i}$ satisfying $\Psi_i(\bar{v}_i, y, z) \not\subset -C_i(z)$.*

Then the solution set M_3 of SGVQEP(3) is nonempty and compact in $H \times K$, where $M_3 = \{(y, z) \in H \times K : y_i \in T_i(y, z), z_i \in A_i(y, z) \text{ and } \Psi_i(x_i, y, z) \subset -C_i(z), \forall x_i \in A_i(y, z), i \in I\}$.

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