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WEAK CONVERGENCE THEOREMS FOR GENERALIZED HYBRID MAPPINGS IN BANACH SPACES

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ABSTRACT. Let E be a real Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called generalized hybrid if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. In this paper, we first deal with some properties for generalized hybrid mappings in a Banach space. Then, we prove weak convergence theorems of Mann's type for such mappings in a Banach space satisfying Opial's condition.

KEYWORDS : Banach space; Nonexpansive mapping; Nonspreading mapping; Hybrid mapping; Fixed point; Weak convergence.

1. INTRODUCTION

Let H be a real Hilbert space and let C be a nonempty subset of H. Then a mapping $T : C \longrightarrow H$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping F is said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \le \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$; see, for instance, Browder [3] and Goebel and Kirk [7]. It is known that a firmly nonexpansive mapping F can be deduced from an equilibrium

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problem in a Hilbert space; see, for instance, [2] and [6]. Recently, Kohsaka and Takahashi [15], and Takahashi [21] introduced the following nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping $T : C \longrightarrow H$ is called nonspreading [15] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \tag{1.1}$$

for all $x, y \in C$. A mapping $T : C \longrightarrow H$ is called hybrid [21] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2 \tag{1.2}$$

for all $x, y \in C$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [14], Iemoto and Takahashi [10] and Takahashi and Yao [23]. Motivated by these mappings and results, Aoyama, Iemoto, Kohsaka and Takahashi [1] introduced a broad class of nonlinear mappings in a Hilbert space called λ-hybrid which contains the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings. Furthermore, Kocourek, Takahashi and Yao [12] introduced a more broad class of nonlinear mappings than the class of λ-hybrid mappings in a Hilbert space. They called such a class the class of generalized hybrid mappings and then proved general fixed point theorems and some convergence theorems in a Hilbert space; see also [25] and [8]. Hsu, Takahashi and Yao [9] extended this class of generalized hybrid mappings in a Hilbert space to Banach spaces and they also called such a class the class of generalized hybrid mappings. Further, they proved general fixed point theorems in a Banach space; see also [13].

In this paper, we first deal with some properties for generalized hybrid mappings in a Banach space. Then, we prove weak convergence theorems for such mappings in a Banach space satisfying Opial's condition.

2. PRELIMINARIES

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \longrightarrow x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \longrightarrow E$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \longrightarrow E$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T. If C is a nonempty closed convex subset of a strictly convex Banach space E and $T : C \longrightarrow C$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [11]. Let E be a Banach space. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \longrightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is norm to weak* uniformly continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm to norm continuous. If E is uniformly smooth, J is norm to norm uniformly continuous on each bounded subset of E. For more details, see [19, 20]. The following results are also in [19, 20].

Theorem 2.1. *Let E be a Banach space and let J be the duality mapping on E. Then, for any $x, y \in E$,*

$$\|x\|^2 - \|y\|^2 \geq 2\langle x - y, j\rangle,$$

where $j \in Jy$.

Theorem 2.2. *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy\rangle \geq 0$ for all $x, y \in E$. Further, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

The following result was proved by Xu [26].

Theorem 2.3 (Xu [26])**.** *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \longrightarrow [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a Banach space. Then, E satisfies Opial's condition [17] if for any $\{x_n\}$ of E such that $x_n \rightharpoonup x$ and $x \neq y$,

$$\liminf_{n \longrightarrow \infty} \|x_n - x\| < \liminf_{n \longrightarrow \infty} \|x_n - y\|.$$

Let E be a Banach space and let $A \subset E \times E$. Then, A is accretive if for $(x_1, y_1), (x_2, y_2) \in A$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$, where J is the duality mapping of E. An accretive operator $A \subset E \times E$ is called m-accretive if $R(I + rA) = E$ for all $r > 0$, where I is the identity operator and $R(I + rA)$ is the range of $I + rA$. An accretive operator $A \subset E \times E$ is said to satisfy the range condition if $\overline{D(A)} \subset R(I + rA)$ for all $r > 0$, where $\overline{D(A)}$ is the closure of the domain $D(A)$ of A. An m-accretive operator satisfies the range condition.

3. GENERALIZED HYBRID MAPPINGS IN BANACH SPACES

Let E be a Banach space and let C be a nonempty subset of E. Then, a mapping $T : C \longrightarrow E$ is said to be firmly nonexpansive [4] if

$$\|Tx - Ty\|^2 \leq \langle x - y, j\rangle,$$

for all $x, y \in C$, where $j \in J(Tx-Ty)$. It is known that the resolvent of an accretive operator satisfying the range condition in a Banach space is a firmly nonexpansive mapping. In fact, let $C = \overline{D(A)}$ and $r > 0$. Define the resolvent J_r of A as follows:

$$J_r x = \{z \in D(A) : x \in z + rAz\}$$

for all $x \in \overline{D(A)}$. It is known that such $J_r x$ is a singleton; see [19]. We have that for $x_1, x_2 \in \overline{D(A)}$, $x_1 = z_1 + ry_1,\ y_1 \in Az_1$ and $x_2 = z_2 + ry_2,\ y_2 \in Az_2$. Since A is accretive, we have that $\langle y_1 - y_2, j\rangle \geq 0$, where $j \in J(z_1 - z_2)$. So, we have

$$\langle \frac{x_1 - z_1}{r} - \frac{x_2 - z_2}{r}, j\rangle \geq 0.$$

Furthermore, we have that

$$\begin{aligned} &\langle \frac{x_1 - z_1}{r} - \frac{x_2 - z_2}{r}, j\rangle \geq 0 \\ &\qquad \Longleftrightarrow \langle x_1 - z_1 - (x_2 - z_2), j\rangle \geq 0 \\ &\qquad \Longleftrightarrow \langle x_1 - x_2, j\rangle \geq \|z_1 - z_2\|^2. \end{aligned}$$

From $z_1 = J_r x_1$ and $z_2 = J_r x_2$, we have that J_r is a firmly nonexpansive mapping; see also [4], [5] and [24]. From [9] we know that the classes of nonexpansive mappings, nonspreading mappings and hybrid mappings are deduced from the class of firmly nonexpansive mappings in a Banach space. In general, Hsu, Takahashi and Yao [9] defined a class of nonlinear mappings in a Banach space containing the classes of nonexpansive mappings, nonspreading mappings and hybrid mappings as follows: Let E be a Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called generalized hybrid if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{3.1}$$

for all $x, y \in C$. They also called such a mapping an (α, β)-generalized hybrid mapping in a Banach space. We note that an (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. As in [9], we have the following result in a Banach space; see [9] for the proof.

Theorem 3.1. *Let C be a nonempty subset of a Banach space E and let T be a generalized hybrid mapping of C into E, i.e., there are $\alpha, \beta \in \mathbb{R}$ such that*

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{3.2}$$

for all $x, y \in C$. Then, the following hold:

(i) *If $\alpha + \beta < 1$, then $T = I$, where $Ix = x$ for all $x \in C$;*
(ii) *if $\alpha = 0$ and $\beta = 1$, then T satisfies that $\|Tx - y\| = \|Ty - x\|$ for all $x, y \in C$;*
(iii) *if $\alpha = 0$ and $\beta > 1$, then T satisfies that*

$$2\|x - y\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$;
(iv) *if $\beta = t\alpha + 1$, $-1 \leq t < \infty$ and $\alpha > 0$, then T satisfies that*

$$2\|Tx - Ty\|^2 + 2t\|x - y\|^2 \leq (t+1)\|Tx - y\|^2 + (t+1)\|Ty - x\|^2$$

for all $x, y \in C$. In particular, T is nonexpansive for $t = -1$, nonspreading for $t = 0$, and hybrid for $t = -\frac{1}{2}$;

(v) *if* $\beta = t\alpha + 1$, $-\infty < t < -1$ *and* $\alpha < 0$, *then* T *satisfies that*

$$2\|Tx - Ty\|^2 + 2t\|x - y\|^2 \geq (t+1)\|Tx - y\|^2 + (t+1)\|Ty - x\|^2$$

for all $x, y \in C$.

Furthermore, we have the following result.

Theorem 3.2. *Let* E *be a Banach space, let* C *be a nonempty subset of* E *and let* $\lambda \in [0,1]$. *Then the following hold:*

(i) *A generalized hybrid mapping with a fixed point is quasi-nonexpansive;*
(ii) *a firmly nonexpansive mapping is* $(2-\lambda, 1-\lambda)$*-generalized hybrid.*

Proof. We show (i). Since $T : C \longrightarrow E$ is a generalized hybrid mapping, there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{3.3}$$

for all $x, y \in C$. Let $u \in F(T)$. Then we have that for any $y \in C$,

$$\alpha\|u - Ty\|^2 + (1-\alpha)\|u - Ty\|^2 \leq \beta\|u - y\|^2 + (1-\beta)\|u - y\|^2 \tag{3.4}$$

and hence $\|u - Ty\|^2 \leq \|u - y\|^2$. This implies that T is quasi-nonexpansive. We next show (ii). Let T be a firmly nonexpansive mapping of C into E. Then we have that for $x, y \in C$ and $j \in J(Tx - Ty)$,

$$\|Tx - Ty\|^2 \leq \langle x - y, j\rangle.$$

From Theorem 2.1 we have that

$$\begin{aligned}
\|Tx-Ty\|^2 &\leq \langle x - y, j\rangle \\
&\iff 0 \leq 2\langle x - Tx - (y - Ty), j\rangle \\
&\Longrightarrow 0 \leq \|x - y\|^2 - \|Tx - Ty\|^2 \\
&\iff \|Tx - Ty\|^2 \leq \|x - y\|^2 \\
&\iff \|Tx - Ty\| \leq \|x - y\|.
\end{aligned}$$

So, for $\lambda \in [0,1]$ we have

$$\lambda\|Tx - Ty\|^2 \leq \lambda\|x - y\|^2. \tag{3.5}$$

Futhermore, we have that for $x, y \in C$ and $j \in J(Tx - Ty)$,

$$\begin{aligned}
\|Tx-Ty\|^2 &\leq \langle x - y, j\rangle \\
&\iff 0 \leq 2\langle x - Tx - (y - Ty), j\rangle \\
&\iff 0 \leq 2\langle x - Tx, j\rangle + 2\langle Ty - y, j\rangle \\
&\Longrightarrow 0 \leq \|x - Ty\|^2 - \|Tx - Ty\|^2 + \|Tx - y\|^2 - \|Tx - Ty\|^2 \\
&\iff 0 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - 2\|Tx - Ty\|^2 \\
&\iff 2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2.
\end{aligned}$$

Thus, for $\lambda \in [0,1]$ we have

$$2(1-\lambda)\|Tx - Ty\|^2 \leq (1-\lambda)\|x - Ty\|^2 + (1-\lambda)\|y - Tx\|^2. \tag{3.6}$$

Therefore, we have from (3.5) and (3.6) that

$$(2-\lambda)\|Tx - Ty\|^2 \leq (1-\lambda)\|x - Ty\|^2 + (1-\lambda)\|y - Tx\|^2 + \lambda\|x - y\|^2$$

and hence

$$(2-\lambda)\|Tx - Ty\|^2 + (\lambda - 1)\|x - Ty\|^2 \leq (1-\lambda)\|y - Tx\|^2 + \lambda\|x - y\|^2.$$

This implies that T is a $(2-\lambda, 1-\lambda)$-generalized hybrid mapping. $\square$

Using Takahashi and Jeong's result [22], Hsu, Takahashi and Yao [9] also proved the following fixed point theorem for generalized hybrid mappings in a Banach space.

Theorem 3.3. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a generalized hybrid mapping. Then the following are equivalent:*

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Using Theorem 3.3, they also proved the following fixed point theorems in a Banach space.

Theorem 3.4. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \longrightarrow C$ be a nonexpansive mapping, i.e.,*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Theorem 3.5. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \longrightarrow C$ be a nonspreading mapping, i.e.,*

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Theorem 3.6. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \longrightarrow C$ be a hybrid mapping, i.e.,*

$$3\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 + \|x - y\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

4. SOME PROPERTIES OF GENERALIZED HYBRID MAPPINGS

Let E be a Banach space. Let C be a nonempty subset of E. Let $T : C \longrightarrow C$ be a mapping. Then, $p \in C$ is called an asymptotic fixed point of T [18] if there exists $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\lim_{n \longrightarrow \infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T. A mapping $I - T$ of C into E is said to be demiclosed on C if $\hat{F}(T) = F(T)$.

Theorem 4.1. *Let E be a Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let $\alpha, \beta \in \mathbb{R}$ and let T be an (α, β)-generalized hybrid mapping of C into itself such that $\alpha \geq 1$ and $\beta \geq 0$. Then $\hat{F}(T) = F(T)$, i.e., $I - T$ is demiclosed.*

Proof. Let $T : C \to C$ be an (α, β)-generalized hybrid mapping, i.e., there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{4.1}$$

for all $x, y \in C$. The inclusion $F(T) \subset \hat{F}(T)$ is obvious. Thus we show $\hat{F}(T) \subset F(T)$. Let $u \in \hat{F}(T)$ be given. Then we have a sequence $\{x_n\}$ of C such that

$x_n \rightharpoonup u$ and $\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0$. Since $T : C \longrightarrow C$ is a generalized hybrid mapping, we obtain that

$$\alpha\|Tx_n - Tu\|^2 + (1-\alpha)\|x_n - Tu\|^2 \leq \beta\|Tx_n - u\|^2 + (1-\beta)\|x_n - u\|^2. \tag{4.2}$$

From $\alpha \geq 1$, $\beta \geq 0$ and (4.2), we have

$$\begin{aligned}\alpha\|Tx_n - Tu\|^2 \leq \beta(\|Tx_n - x_n\| + \|x_n - u\|)^2 + (1-\beta)\|x_n - u\|^2 \\ + (\alpha-1)(\|x_n - Tx_n\| + \|Tx_n - Tu\|)^2.\end{aligned}$$

So, we have that

$$\begin{aligned}(\alpha - (\alpha-1))\|Tx_n - Tu\|^2 \leq (\beta + (1-\beta))\|x_n - u\|^2 + (\beta+\alpha-1)\|x_n - Tx_n\|^2 \\ + 2(\beta+\alpha-1)(\|x_n - u\| + \|Tx_n - Tu\|)\|Tx_n - x_n\|\end{aligned}$$

and hence

$$\begin{aligned}\|Tx_n - Tu\|^2 \leq \|x_n - u\|^2 + (\beta+\alpha-1)\|x_n - Tx_n\|^2 \\ + 2(\beta+\alpha-1)(\|x_n - u\| + \|Tx_n - Tu\|)\|Tx_n - x_n\|.\end{aligned} \tag{4.3}$$

From $x_n \rightharpoonup u$, we obtain that $\{x_n\}$ is bounded. From $\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0$ we also have that $\{Tx_n\}$ is bounded. So, we can take a positive constant M such that

$$\sup\{\|x_n - u\| + \|Tx_n - Tu\| : n \in \mathbb{N}\} \leq M. \tag{4.4}$$

Suppose $Tu \neq u$. Then we have from Opial's condition, (4.3) and (4.4) that

$$\begin{aligned}\liminf_{n\longrightarrow\infty} \|x_n - u\|^2 &< \liminf_{n\longrightarrow\infty} \|x_n - Tu\|^2 \\ &= \liminf_{n\longrightarrow\infty} \|x_n - Tx_n + Tx_n - Tu\|^2 \\ &= \liminf_{n\longrightarrow\infty} \|Tx_n - Tu\|^2 \\ &\leq \liminf_{n\longrightarrow\infty}(\|x_n - u\|^2 + (\beta+\alpha-1)\|x_n - Tx_n\|^2 \\ &\quad + 2(\beta+\alpha-1)M\|Tx_n - x_n\|) \\ &= \liminf_{n\longrightarrow\infty} \|x_n - u\|^2.\end{aligned}$$

This is a contradiction. So, we have $Tu = u$ and hence $\hat{F}(T) \subset F(T)$. This completes the proof. $\square$

Remark. We do not know that the demiclosedness property for a generalized hybrid mapping holds or not in a uniformly convex Banach space.

Using Theorem 4.1, we can prove the following theorems in a Banach space.

Theorem 4.2. *Let E be a Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let $T : C \longrightarrow C$ be a nonexpansive mapping, i.e.,*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Then, $I - T$ is demiclosed on C.

Proof. In Theorem 4.1, a $(1, 0)$-generalized hybrid mapping of C into itself is nonexpansive. Further, $\alpha = 1 \geq 1$ and $\beta = 0 \geq 0$. By Theorem 4.1, $I - T$ is demiclosed on C. $\square$

Theorem 4.3. *Let E be a Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let $T : C \longrightarrow C$ be a nonspreading mapping, i.e.,*

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Then, $I - T$ is demiclosed on C.

Proof. In Theorem 4.1, a (2, 1)-generalized hybrid mapping of C into itself is nonspreading. Further, $\alpha = 2 > 1$ and $\beta = 1 > 0$. By Theorem 4.1, $I - T$ is demiclosed on C. □

Theorem 4.4. *Let E be a Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let $T : C \longrightarrow C$ be a hybrid mapping, i.e.,*

$$3\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 + \|x - y\|^2, \quad \forall x, y \in C.$$

Then, $I - T$ is demiclosed on C.

Proof. In Theorem 4.1, a $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mapping of C into itself is hybrid. Further, $\alpha = \frac{3}{2} > 1$ and $\beta = \frac{1}{2} > 0$. By Theorem 4.1, $I - T$ is demiclosed on C. □

Next, we have the following property of the fixed point set of a generalized hybrid mapping in a Banach space.

Theorem 4.5. *Let E be a strictly convex Banach space, let C be a nonempty closed convex subset of E and let T be a generalized hybrid mapping of C into itself. Then $F(T)$ is closed and convex.*

Proof. Let $T : C \to C$ be a generalized hybrid mapping, i.e., there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{4.5}$$

for all $x, y \in C$. If $F(T)$ is empty, then $F(T)$ is closed and convex. If $F(T)$ is nonempty, then we have from Theorem 3.2 that T is quasi-nonexpansive. From Itoh and Takahashi [11], we have that $F(T)$ is closed and convex. □

Let E be a Banach space and let C be a nonempty subset of E. A mapping $T : C \longrightarrow C$ is called asymptotically regular if for any $x \in C$,

$$T^{n+1}x - T^n x \longrightarrow 0.$$

Theorem 4.6. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let T be a generalized hybrid mapping of C into itself with $F(T) \neq \emptyset$ and let γ be a real number with $0 < \gamma < 1$. Define a mapping $S : C \longrightarrow C$ by*

$$S = \gamma I + (1 - \gamma)T.$$

Then, for any $x \in C$, $S^{n+1}x - S^n x$ converges strongly to 0, i.e., S is asymptotically regular.

Proof. Let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Then, from Theorem 3.2 we have that T is quasi-nonexpansive. Using that T is quasi-nonexpansive, we have that for any $u \in F(T)$, $x \in C$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\|S^{n+1}x - u\| &= \|SS^n x - u\| \\
&= \|\gamma S^n x + (1 - \gamma)TS^n x - u\| \\
&= \|\gamma(S^n x - u) + (1 - \gamma)(TS^n x - u)\| \\
&\leq \gamma\|S^n x - u\| + (1 - \gamma)\|TS^n x - u\|
\end{aligned}$$

$$\begin{aligned} &\leq \gamma\|S^n x - u\| + (1-\gamma)\|S^n x - u\| \\ &= \|S^n x - u\|. \end{aligned}$$

So, we have that $\lim_{n\longrightarrow\infty}\|S^n x - u\|$ exists. Then, $\{S^n x\}$ is bounded. Since T is quasi-nonexpansive, $\{TS^n x\}$ is also bounded. Let

$$r = \max\{\sup_{n\in\mathbb{N}}\|S^n x - u\|, \sup_{n\in\mathbb{N}}\|TS^n x - u\|\}.$$

Then, from Theorem 2.3, there exists a strictly increasing, continuous and convex function $g : [0,\infty) \longrightarrow [0,\infty)$ such that $g(0) = 0$ and

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$. So, we have that for any $u \in F(T)$, $x \in C$ and $n \in \mathbb{N}$,

$$\begin{aligned} \|S^{n+1} - u\|^2 &= \|SS^n x - u\|^2 \\ &= \|\gamma S^n x + (1-\gamma)TS^n x - u\|^2 \\ &\leq \gamma\|S^n x - u\|^2 + (1-\gamma)\|TS^n x - u\|^2 - \gamma(1-\gamma)g(\|S^n x - TS^n x\|) \\ &\leq \gamma\|S^n x - u\|^2 + (1-\gamma)\|S^n x - u\|^2 - \gamma(1-\gamma)g(\|S^n x - TS^n x\|) \\ &= \|S^n x - u\|^2 - \gamma(1-\gamma)g(\|S^n x - TS^n x\|) \\ &\leq \|S^n x - u\|^2 \end{aligned}$$

and hence

$$\gamma(1-\gamma)g(\|S^n x - TS^n x\|) \leq \|S^n x - u\|^2 - \|S^{n+1}x - u\|^2.$$

Since $\lim_{n\longrightarrow\infty}\|S^n x - u\|^2$ exists and $0 < \gamma < 1$, we have

$$\lim_{n\longrightarrow\infty} g(\|S^n x - TS^n x\|) = 0.$$

From the properties of g, we have $\lim_{n\longrightarrow\infty}\|S^n x - TS^n x\| = 0$. From

$$\|S^{n+1}x - TS^n x\| = \|\gamma S^n x + (1-\gamma)TS^n x - TS^n x\| = \gamma\|S^n x - TS^n x\|,$$

we have that

$$\begin{aligned} \|S^{n+1}x - S^n x\| &= \|S^{n+1}x - TS^n x + TS^n x - S^n x\| \\ &\leq \|S^{n+1}x - TS^n x\| + \|TS^n x - S^n x\| \\ &= \gamma\|S^n x - TS^n x\| + \|TS^n x - S^n x\| \longrightarrow 0. \end{aligned}$$

This completes the proof. □

5. WEAK CONVERGENCE THEOREMS

In this section, we first prove a weak convergence theorem of Mann's type [16] for generalized hybrid mappings in a Banach space satisfying Opial's condition.

Theorem 5.1. *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty closed convex subset of E. Let $\alpha, \beta \in \mathbb{R}$ and let T be an (α,β)-generalized hybrid mapping of C into itself such that $\alpha \geq 1$ and $\beta \geq 0$. Let $\{\gamma_n\}$ be a sequence of real numbers with $0 < a \leq \gamma_n \leq b < 1$ and define a sequence $\{x_n\}$ of C as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \gamma_n x_n + (1-\gamma_n)Tx_n, \quad \forall n \in \mathbb{N}.$$

If $F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to some element $z \in F(T)$.

Proof. Let $T : C \to C$ be an (α, β)-generalized hybrid mapping., i.e.,

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. Since $F(T) \neq \emptyset$, we know from Theorem 3.2 that T is quasi-nonexpansive. Using the fact that T is quasi-nonexpansive, we have that for any $u \in F(T)$, $x \in C$ and $n \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+1} - u\| &= \|\gamma_n x_n + (1-\gamma_n)Tx_n - u\| \\ &= \|\gamma_n(x_n - u) + (1-\gamma_n)(Tx_n - u)\| \\ &\le \gamma_n\|x_n - u\| + (1-\gamma_n)\|Tx_n - u\| \\ &\le \gamma_n\|x_n - u\| + (1-\gamma_n)\|x_n - u\| \\ &= \|x_n - u\|.\end{aligned}$$

So, we have that $\lim_{n\longrightarrow\infty} \|x_n - u\|$ exists. Then, $\{x_n\}$ is bounded. Since T is quasi-nonexpansive, $\{Tx_n\}$ is also bounded. Let

$$r = \max\{\sup_{n\in\mathbb{N}} \|x_n - u\|, \sup_{n\in\mathbb{N}} \|Tx_n - u\|\}.$$

Then, from Theorem 2.3, there exists a strictly increasing, continuous and convex function $g : [0,\infty) \longrightarrow [0,\infty)$ such that $g(0) = 0$ and

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \le \lambda \le 1$, where $B_r = \{z \in E : \|z\| \le r\}$. So, we have that for any $u \in F(T)$, $x \in C$ and $n \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|\gamma_n x_n + (1-\gamma_n)Tx_n - u\|^2 \\ &= \|\gamma_n(x_n - u) + (1-\gamma_n)(Tx_n - u)\|^2 \\ &\le \gamma_n\|x_n - u\|^2 + (1-\gamma_n)\|Tx_n - u\|^2 - \gamma_n(1-\gamma_n)g(\|x_n - Tx_n\|) \\ &\le \gamma_n\|x_n - u\|^2 + (1-\gamma_n)\|x_n - u\|^2 - \gamma_n(1-\gamma_n)g(\|x_n - Tx_n\|) \\ &= \|x_n - u\|^2 - \gamma_n(1-\gamma_n)g(\|x_n - Tx_n\|) \\ &\le \|x_n - u\|^2\end{aligned}$$

and hence

$$\gamma_n(1-\gamma_n)g(\|x_n - Tx_n\|) \le \|x_n - u\|^2 - \|x_{n+1} - u\|^2.$$

Since $\lim_{n\longrightarrow\infty} \|x_n - u\|^2$ exists, we have from $0 < a \le \gamma_n \le b < 1$ that

$$\lim_{n\longrightarrow\infty} g(\|x_n - Tx_n\|) = 0.$$

From the properties of g, we have

$$\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0. \tag{5.1}$$

Since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $u \in C$. Using Theorem 4.1 and (5.1), we have $Tu = u$. Let us show that the entire sequence $\{x_n\}$ converges weakly to some point of $F(T)$. To show it, let us take two subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$. Suppose $u \neq v$. From $u, v \in F(T)$, we know that $\lim_{n\longrightarrow\infty} \|x_n - u\|$ and $\lim_{n\longrightarrow\infty} \|x_n - v\|$ exist. Since E satisfies Opial's condition, we have that

$$\begin{aligned}\lim_{n\longrightarrow\infty} \|x_n - u\| &= \lim_{i\longrightarrow\infty} \|x_{n_i} - u\| \\ &< \lim_{i\longrightarrow\infty} \|x_{n_i} - v\|\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n \longrightarrow \infty} \|x_n - v\| \\
&= \lim_{j \longrightarrow \infty} \|x_{n_j} - v\| \\
&< \lim_{j \longrightarrow \infty} \|x_{n_j} - u\| \\
&= \lim_{n \longrightarrow \infty} \|x_n - u\|.
\end{aligned}
$$

This is a contradiction. So, we must have $u = v$. This implies that $\{x_n\}$ converges weakly to a point of $F(T)$. □

Remark. We do not know that such a weak convergence theorem for a generalized hybrid mapping holds or not in a uniformly convex Banach space which has a Fréchet differentiable norm.

Using Theorem 5.1, we obtain the following result.

Theorem 5.2. *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty closed convex subset of E. Let T be a generalized hybrid mapping of C into itself with $F(T) \neq \emptyset$ and let γ be a real number with $0 < \gamma < 1$. Define a mapping $S : C \longrightarrow C$ by*

$$S = \gamma I + (1 - \gamma)T.$$

Then, for any $x \in C$, $S^n x$ converges weakly to an element $z \in F(T)$.

Proof. Putting $\gamma_n = \gamma$ for all $n \in \mathbb{N}$ and $S = \gamma I + (1 - \gamma)T$, we have that for any $x \in C$,

$$x_2 = Sx_1 = Sx, x_3 = S^2 x_1 = S^2 x, \ldots$$

in Theorem 5.1. So, we have from Theorem 5.1 that $S^n x$ converges weakly to an element $z \in F(T)$. This completes the proof. □

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