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THE HAHN-BANACH THEOREM AND THE SEPARATION THEOREM IN A PARTIALLY ORDERED VECTOR SPACE

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ABSTRACT. In this paper, using the Bourbaki-Kneser fixed point theorem, we give a new proof of the Hahn-Banach theorem in a partially ordered vector space and the separation theorem in the Cartesian product of a vector space and a partially ordered vector space.

KEYWORDS : Fixed point theorem; Hahn-Banach theorem; Separation theorem; Partially ordered vector space.

1. INTRODUCTION

The Hahn-Banach theorem is one of the most fundamental theorems in the functional analysis theory and the separation theorem is one of the most fundamental theorems in the optimization theory. These theorems are known well in the case where the range space is the real number system. The following is the Hahn-Banach theorem: *Let p be a sublinear mapping from a vector space X to the real number system R, Y a vector subspace of X and q a linear mapping from Y to R such that $q \leq p$ on Y. Then q can be extended to a linear mapping g defined on the whole space X to R such that $g \leq p$.*

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Moreover, the following is the separation theorem:

Let X be a normed space, X^* a its dual space, A, B subsets of X such that A is closed convex and B is compact convex subset with $A \cap B = \emptyset$, then there exists an $f \in X^* \setminus \{0\}$ such that $\inf\{f(y) \mid y \in B\} \geq \sup\{f(x) \mid x \in A\}$.

It is known that the Hahn-Banach theorem establishes in the case where the range space is a Dedekind complete Riesz space; see [4, 17, 19] and the separation theorem establishes in the Cartesian product space of a vector space and a Dedekind complete ordered vector space; see [6, 7, 15, 16].

The Hahn-Banach theorem is proved often using the Zorn lemma. For the proof of the Hahn-Banach theorem, there exist several approaches. For instance, Hirano, Komiya, and Takahashi [8] showed the Hahn-Banach theorem by using the Markov-Kakutani fixed point theorem [10] in the case where the range space is the real number system.

In this paper, in Section 3, using the Bourbaki-Kneser fixed point theorem, we give a new proof of the Hahn-Banach theorem and the Mazur-Orlicz theorem in the case where the range space is a Dedekind complete partially ordered vector space (Theorem 3.1 and Theorem 3.2). In Section 4, we give a new proof of the separation theorem in the Cartesian product of a vector space and a Dedekind complete partially ordered vector space (Theorem 4.1); see [6, 7, 15, 16]. The Bourbaki-Kneser fixed point theorem is proved without using the Zorn lemma; see [11]. Therefore the theorems above are proved without using the Zorn lemma.

2. PRELIMINARIES

Let R be the set of real numbers, N the set of natural numbers, I an indexed set, $(E, \leq)$ a partially ordered set and F a subset of E. The set F is called a *chain* if any two elements are comparable, that is, $x \leq y$ or $y \leq x$ for any x, $y \in F$. An element $x \in E$ is called a *lower bound* of F if $x \leq y$ for any $y \in F$. An element $x \in E$ is called the *minimum* of F if x is a lower bound of F and $x \in F$. If there exists a lower bound of F, then F is said to be *bounded from below*. An element $x \in E$ is called an *upper bound* of F if $y \leq x$ for any $y \in F$. An element $x \in E$ is called the *maximum* of F if x is an upper bound and $x \in F$. If there exists an upper bound of F, then F is said to be *bounded from above*. If the set of all lower bounds of F has the maximum, then the maximum is called an *infimum* of F and denoted by $\inf F$. If the set of all upper bounds of F has the minimum, then the minimum is called a *supremum* of F and denoted by $\sup F$. A partially ordered set E is said to be *complete* if every nonempty chain of E has an infimum; E is said to be *chain complete* if every nonempty chain of E which is bounded from below has an infimum; E is said to be *Dedekind complete* if every nonempty subset of E which is bounded from below has an infimum. A mapping f from E to E is called *decreasing* if $f(x) \leq x$ for every $x \in E$. For the further information of a partially ordered set; see [1, 4, 5, 14, 17].

In a complete partially ordered set, the following theorem is obtained; see [3, 11, 12].

Theorem 2.1 (Bourbaki-Kneser). *Let E be a complete partially ordered set. Let f be a decreasing mapping from E to E. Then f has a fixed point.*

Recently, T. C. Lim [13] proved a common fixed point theorem for the family of decreasing commutative mapping, which is a generalization of Theorem 2.1.

A partially ordered set E is called a partially ordered vector space if E is a vector space and $x+z \le y+z$ and $\alpha x \le \alpha y$ hold whenever $x, y, z \in E$, $x \le y$, and α is a nonnegative real number. If a partially ordered vector space E is a lattice, that is, any two elements have a supremum and an infimum, then E is called a *Riesz space*.

Let X be a vector space and E a partially ordered vector space. A mapping f from X to E is said to be *concave* if

$$f(tx+(1-t)y) \ge tf(x)+(1-t)f(y)$$

for any $x, y \in X$ and $t \in [0,1]$. A mapping f from X to E is called *sublinear* if the following conditions are satisfied.
(S1) For any $x, y \in X$, $p(x+y) \le p(x)+p(y)$.
(S2) For any $x \in X$ and $\alpha \ge 0$ in R, $p(\alpha x) = \alpha p(x)$.

3. THE HAHN-BANACH THEOREM

Lemma 3.1. *Let p be a sublinear mapping from a vector space X to a Dedekind complete partially ordered vector space E, K a nonempty convex subset of X and q a concave mapping from K to E such that $q \le p$ on K. For any $x \in X$, let*

$$f(x) = \inf\{p(x+ty) - tq(y) \mid t \in [0,\infty) \text{ and } y \in K\}.$$

Then f is sublinear such that $f \le p$. Moreover if g is a linear mapping from X to E, then $g \le f$ is equivalent to $g \le p$ on X and $q \le g$ on K.

Proof. For any $x \in X$,

$$\{p(x+ty) - tq(y) \mid t \in [0,\infty) \text{ and } y \in K\}$$

is bounded from below. Indeed, since

$$p(x+ty) - tq(y) \ge p(ty) - p(-x) - tq(y) \ge -p(-x),$$

it is established. Since E is Dedekind complete, f is well-defined and we have $f(x) \ge -p(-x)$. By definition of f, we have $f(x) \le p(x)$ and $f(\alpha x) = \alpha f(x)$ for any $\alpha \ge 0$. Thus (S2) is established. Let $x_1, x_2 \in X$. For any $y_1, y_2 \in K$ and $s, t > 0$, we have

$$\begin{aligned} p(x_1+sy_1) - sq(y_1) + p(x_2+ty_2) - tq(y_2) \\ \ge p(x_1+x_2+(s+t)w) - (s+t)q(w) \\ \ge f(x_1+x_2), \end{aligned}$$

where $w = \frac{1}{s+t}(sy_1+ty_2) \in K$. For $p(x_1+sy_1) - sq(y_1)$, take infimum with respect to $s > 0$ and $y_1 \in K$, we have

$$f(x_1) + p(x_2+ty_2) - tq(y_2) \ge f(x_1+x_2)$$

and for $p(x_2+ty_2) - tq(y_2)$, also take infimum with respect to $t > 0$ and $y_2 \in K$, we have

$$f(x_1)+f(x_2) \ge f(x_1+x_2).$$

Thus (S1) is established. Suppose that g is a linear mapping from X to E. If $g \le f$, then we have $g \le p$. Moreover for any $y \in K$, since

$$-g(y) = g(-y) \le f(-y) \le p(-y+y) - q(y) = -q(y),$$

we have $g \ge q$ on K. To prove the converse, suppose that $g \le p$ on X and $q \le g$ on K. For any $x \in X$, $y \in K$ and $t > 0$, we have

$$g(x) = g(x+ty) - tg(y) \le p(x+ty) - tq(y).$$

This implies that $g \le f$. □

The above lemma is proved in case where the range space is a Dedekind complete Riesz space, see [17, Lemma 1.5.1].

By Theorem 2.1 and Lemma 3.1, we can give a following lemma.

Lemma 3.2. *Let f be a sublinear mapping from a vector space X to a Dedekind complete partially ordered vector space E. Then there exists a linear mapping g from X to E such that $g \le f$.*

Proof. Let E^X be the set of mappings of X into E. Define $f \le g$ for $f, g \in E^X$ by $f(x) \le g(x)$ for all $x \in X$. Then $(E^X, \le)$ is a partially ordered vector space. Put $f^*(x) = -f(-x)$ for any $x \in X$. Let

$$Y = \{h \in E^X \mid h \text{ is sublinear, } f^* \le h \le f\}.$$

Then Y is an ordered set. Since E is Dedekind complete, E^X is also so. Consider an arbitrary chain $Z \subset Y$. Since E^X is Dedekind complete and Z is bounded from below, there exists a $g = \inf Z$ in E^X. Then g is sublinear. Since Y is bounded from below, it holds that $g \in Y$. Thus Y is complete. Let $K = \{y\}$. Then h is also a concave mapping from K to E. We define a decreasing operator S by

$$Sh(x) = \inf\{h(x+ty) - th(y) \mid t \in [0, \infty),\, y \in K\}$$

for any $h \in Y$. By Lemma 3.1, Sh is sublinear and S is a mapping from Y to Y. Thus by Theorem 2.1, we have a fixed point $g \in Y$. Then for any $x \in X$, we have $g(x) \le g(x+y) - g(y)$ and

$$g(x) + g(y) \le g(x+y) \le g(x) + g(y).$$

Since

$$0 = g(0) = g(-\alpha x + \alpha x) = g(-\alpha x) + \alpha g(x)$$

for any $\alpha > 0$ and $x \in X$, we have $g(-\alpha x) = -\alpha g(x)$. Thus $g(\alpha x) = \alpha g(x)$ for any $\alpha \in R$ and $x \in X$. Therefore, g is linear. □

By Lemma 3.2 and Lemma 3.1, we can prove the Hahn-Banach theorem and the Mazur-Orlicz theorem in case where the range space is a Dedekind complete partially ordered vector space.

Theorem 3.1. *Let p be a sublinear mapping from a vector space X to a Dedekind complete ordered vector space E, Y a vector subspace of X and q a linear mapping from Y to E such that $q \le p$ on Y. Then q can be extended to a linear mapping g defined on the whole space X to E such that $g \le p$.*

Proof. By Lemma 3.1, there exists a sublinear mapping f such that $f \le p$. By Lemma 3.2, there exists a linear mapping g such that $g \le f$. Then putting $K = Y$ in Lemma 3.1, we have $g \le p$ on X and $q \le g$ on Y. Since q is linear, for any $y \in Y$, we have

$$g(-y) \le f(-y) \le p(-y+y) - q(y) = -q(y) = q(-y).$$

Then we have $g \le q$ on Y. Thus $q = g$ on Y. Therefore, the assertion holds. □

We obtain the Mazur-Orlicz theorem in a Dedekind complete partially ordered vector space.

Theorem 3.2. *Let p be a sublinear mapping from a vector space X to a Dedekind complete partially ordered vector space E. Let $\{x_j \mid j \in I\}$ be a family of elements of X and $\{y_j \mid j \in I\}$ a family of elements of E. Then the following* (1) and (2) *are equivalent.*
(1) *There exists a linear mapping f from X to E such that $f(x) \leq p(x)$ for any $x \in X$ and $y_j \leq f(x_j)$ for any $j \in I$.*
(2)*For any $n \in N$, $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ and $j_1, j_2, \ldots, j_n \in I$, we have*

$$\sum_{i=1}^{n} \alpha_i y_{j_i} \leq p\left(\sum_{i=1}^{n} \alpha_i x_{j_i}\right).$$

Proof. The assertion from (1) to (2) is clear. For any $x \in X$, by (2), we have

$$-p(-x) \leq p\left(x + \sum_{i=1}^{n} \alpha_i x_{j_i}\right) - \sum_{i=1}^{n} \alpha_i y_{j_i}.$$

Put

$$p_0(x) = \inf\left\{ p\left(x + \sum_{i=1}^{n} \alpha_i x_{j_i}\right) - \sum_{i=1}^{n} \alpha_i y_{j_i} \;\middle|\; n \in N, \alpha_i \geq 0 \text{ and } j_i \in I \right\}.$$

Since E is Dedekind complete, p_0 is well-defined and p_0 is sublinear. Thus by Lemma 3.2, there exists a linear mapping f from X to E such that $f(x) \leq p_0(x)$ for any $x \in X$. Since $p_0(-x_j) \leq -y_j$, we have

$$y_j \leq -p_0(-x_j) \leq f(x_j).$$

Since $p_0(x) \leq p(x)$, we have $f(x) \leq p(x)$. Thus the assertion holds. □

4. THE SEPARATION THEOREM

Let X be a vector space, E a Dedekind complete partially ordered vector space and $X \times E$ the Cartesian product of X and E. Let A be a nonempty subset of X and $L(A)$ denotes the affine manifold spanned by A. We denotes the *algebraical relative interior* of A, that is,

$$Int(A) = \left\{ x \in X \;\middle|\; \begin{array}{l} \text{For any } x' \in L(A) \text{ there exists } \varepsilon > 0 \text{ such that} \\ x + \lambda(x' - x) \in A \text{ for any } \lambda \in [0,\, \varepsilon) \end{array} \right\}.$$

If $L(A) = X$, then we write $I(A)$ instead of $Int(A)$. Let f be a linear mapping from X to E, g a linear mapping from E to E and u_0 a point in E. Then

$$H = \{(x,\, y) \in X \times E \mid f(x) + g(y) = u_0\}$$

is empty or an affine manifold in $X \times E$. Let A, B be nonempty subsets of $X \times E$. A nonempty subset $A \subset X \times E$ is said to be *cone* (with the vertex in $x_0 \in X \times E$) if $\lambda > 0$ implies $\lambda(A - x_0) \subset (A - x_0)$. It is said that an affine manifold H separates A and B if

$$H_- \supset A \text{ and } H_+ \supset B$$

where we set

$$H_- = \{(x,\, y) \in X \times E \mid f(x) + g(y) \leq u_0\}$$

and

$$H_+ = \{(x,\, y) \in X \times E \mid f(x) + g(y) \geq u_0\}.$$

The operator P_X defined by $P_X(x, y) = x$ for any $(x, y) \in X \times E$ is called the *projection* of $X \times E$ onto X. Then P_X is a linear mapping from $X \times E$ to X. We define

$$P_X(A) = \{x \in X \mid \text{ there exists } y \in E \text{ such that } (x, y) \in A\}.$$

Then we have

$$P_X(A + B) = P_X(A) + P_X(A)$$

for $A \neq \emptyset$ and $B \neq \emptyset$. The subset

$$C(A) = \{\lambda z \in X \times E \mid \lambda \geq 0,\ z \in A\}$$

is called the *cone spanned by* A. If A is convex, then $C(A)$ is convex. By Lemma 3.2, we obtain the separation theorem in the Cartesian product of a vector space and a Dedekind complete partially ordered vector space.

Theorem 4.1. *Let A and B be subsets of $X \times E$ such that $C(A - B)$ is convex, and $P_X(A - B)$ satisfies the following* (i) *and* (ii) *:*
(i) $0 \in I(P_X(A - B))$,
(ii) *if $(x, y_1) \in A$ and $(x, y_2) \in B$, then $y_1 \geq y_2$ holds.*
Then there exists a linear mapping f from X to E and a $y_0 \in E$ such that the affine manifold

$$H = \{(x, y) \in X \times E \mid f(x) - y = y_0\}$$

separates A and B.

Proof. By assumption (i) and the definition of $I(P_X(A - B))$, for any $x \in X$ there exists $\varepsilon > 0$ and for any $\lambda \in [0, \varepsilon)$, there exists $y \in E$ such that $(\lambda x, y) \in A - B$. Then there exist $x_1, x_2 \in X$ and $y_1, y_2 \in E$ such that

$$(\lambda x, y) = (x_1 - x_2, y_1 - y_2) = (x_1, y_1) - (x_2, y_2) \in A - B.$$

Define

$$E_x = \{y \in E \mid (x, y) \in C(A - B)\}, \text{ for any } x \in X.$$

Since $\lambda^{-1}(y_1 - y_2) \in E_x$ for any $\lambda \in (0, \varepsilon)$, we have $E_x \neq \emptyset$. Let $y \in E_0$ and $y \neq 0$, then there exists $\lambda > 0$, $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$ such that

$$(0, y) = \lambda\{(x_1, y_1) - (x_2, y_2)\}$$

and $x_1 = x_2$. By assumption (ii), we have $y = \lambda(y_1 - y_2) \geq 0$. We define $E_+ = \{y \in E \mid y \geq 0\}$. Then we have $y \in E_+$. Since $C(A - B)$ is convex cone, we have $E_x + E_{x'} \subset E_{x+x'}$ for any $x, x' \in X$. We prove that for every $x \in X$ the subset E_x possesses a lower bound in E. Since E_x is nonempty, for any $x \in X$, there exists $y' \in E$ with $-y' \in E_{-x}$. Then we have

$$y - y' \in E_x + E_{-x} \subset E_0 \subset E_+$$

for any $y \in E_x$. This implies $y' \leq y$ for any $y \in E_x$. Since E is Dedkind complete, operator $p(x) = \inf\{y \mid y \in E_x\}$ is well defined. Then $p(x)$ is sublinear. By Lemma 3.2, there exists a linear mapping f from X to E such that $f(x) \leq p(x)$ for all $x \in X$. Then for any $(x_1, y_1) \in A$, $(x_2, y_2) \in B$, take $x = x_1 - x_2$, we have

$$f(x_1 - x_2) \leq p(x_1 - x_2) \leq y_1 - y_2.$$

Therefore,

$$f(x_1) - y_1 \leq f(x_2) - y_2.$$

Since E is Dedekind complete, there exists a $y_0 \in E$ such that

$$f(x_1) - y_1 \leq y_0 \leq f(x_2) - y_2$$

for any $(x_1, y_1) \in A$, $(x_2, y_2) \in B$. Thus the affine manifold H separates A and B. □

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