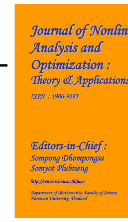
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 2, No. 1, (2011), 97-110
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



THE ALTERNATELY FIBONACCI COMPLEMENTARY DUALITY IN QUADRATIC OPTIMIZATION PROBLEM

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ABSTRACT. In this paper, we consider a pair of primal and dual quadratic optimization problems, and we compare optimal values and optimal points of both problems. The optimal values and optimal points of both problems have a triple Fibonacci property as follows. (i) The value of maximum and minimum are the same (duality). (ii) The maximum point and the minimum point are two-step alternate Fibonacci sequences (2-step alternately Fibonacci). (iii) Both the optimal points constitute alternately two consecutive positive numbers and two consecutive negative numbers of Fibonacci sequence (alternately Fibonacci complement). This triplet is called the *alternately Fibonacci complementary duality*. Moreover, we show a two-step alternate DA VINCI Code by using optimal points of their quadratic optimization problems, and we propose a method the *alternately Fibonacci section* to find optimal points for their problems.

KEYWORDS : Quadratic optimization problem; Fibonacci sequence; Alternately Fibonacci complementary duality.

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Article history : Received 14 December 2010. Accepted 8 February 2011.

1. INTRODUCTION

A recent move “The DA VINCI Code" (2006) shows the following finite sequence:

$$1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 1 \quad 3 \quad 2 \quad 1. \tag{1.1}$$

The code utilizes the Fibonacci sequence as a mysterious code [5], which is the first eight numbers

$$F_1,\ F_2,\ F_3,\ F_4,\ F_5,\ F_6,\ F_7,\ F_8$$

in the Fibonacci sequence (Table 1).

Definition 1.1. The *Fibonacci sequence* $\{F_n\}$ is defined as the solution to the second-order linear difference equation,

$$F_{n+2} - F_{n+1} - F_n = 0, \quad F_1 = 1,\ F_0 = 0. \tag{1.2}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987

Table 1 Fibonacci sequence $\{F_n\}$

The many relationships between optimization theory and the code were studied in [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Iwamoto, Kira and Ueno [14] proposed the *Fibonacci complementary duality* for two pairs of primal and dual optimization problems. The Fibonacci complementary duality proved that there are some beautiful relations during optimal points of both optimization problems, and whose optimal values and optimal points are characterized by the Fibonacci sequence. Our results in this paper become paired their results in [14].

This paper considers a pair of primal and dual quadratic optimization problems, and we compare optimal values and optimal points of both problems. We show that both optimal solutions are characterized by an alternate Fibonacci sequence in the following sense. (i) The value of maximum and minimum are the same (duality). (ii) The maximum point and the minimum point are two-step alternate Fibonacci sequences (2-step alternately Fibonacci). “*Alternate*" means that a positive number and a negative number appear alternately for a sequence. (iii) Both the optimal points constitute alternately two consecutive positive numbers and two consecutive negative numbers of Fibonacci sequence (alternately Fibonacci complement). This triplet is called the *alternately Fibonacci complementary duality*. Moreover, this paper considers a pair of primal and dual optimizaiton problems of four variables together with their respective reversed problems. We show a two-step alternate DA VINCI Code by using optimal points of their problems, and we propose a method the *alternately Fibonacci section* to find optimal points for the problems.

This paper is organized as follows. In section 2, we establish the alternately Fibonacci complementary duality about two pairs of primal and dual quadratic optimization problems. Section 3 proves the two-step alternate DA VINCI Code by using optimal points of four variables for the problems. In section 4, we propose the alternately Fibonacci section to find optimal points for quadratic optimization problems.

2. THE ALTERNATELY FIBONACCI COMPLEMENTARY DUALITY

In this section, we consider a pair of primal and dual quadratic optimization problems

$$(\mathrm{P}_n) \qquad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=0}^{n-1} \left[(x_k + x_{k+1})^2 + x_{k+1}^2\right] \\ \text{subject to} & \text{(i) } x \in R^n \\ & \text{(ii) } x_0 = c, \end{array}$$

where $c \in R$, $x = (x_1, x_2, \ldots, x_n)$, and

$$(\mathrm{D}_n) \qquad \begin{array}{ll} \text{Maximize} & 2c\mu_0 - \mu_0^2 - \displaystyle\sum_{k=0}^{n-2} \left[(\mu_k + \mu_{k+1})^2 + \mu_{k+1}^2\right] - \mu_{n-1}^2 \\ \text{subject to} & \text{(i) } \mu \in R^n \end{array}$$

where $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1})$.

Theorem 2.1. *For the problems* (P_n) *and* (D_n), *let* $x = (x_0, x_1, \ldots, x_n)$ *be feasible of the primal problem* (P_n) *and* $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1})$ *be feasible of the dual problem* (D_n). *Then,* $\min(\mathrm{P}_n) \geq \max(\mathrm{D}_n)$.

Proof Let $x = (x_0, x_1, \ldots, x_n)$ be feasible of the primal problem (P_n), and $I(x)$ be the evaluated value

$$I(x) := \sum_{k=0}^{n-1} \left[(x_k + x_{k+1})^2 + x_{k+1}^2\right]. \tag{2.1}$$

Let

$$u_k = x_k + x_{k+1} \qquad 0 \leq k \leq n-1. \tag{2.2}$$

Then we have for any $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1}) \in R^n$,

$$I(x) = \sum_{k=0}^{n-1} \left[u_k^2 + x_{k+1}^2 - 2\mu_k \left(x_{k+1} + x_k - u_k\right)\right]. \tag{2.3}$$

Since

$$\begin{aligned} I(x) &= 2x_0\mu_0 + \sum_{k=1}^{n-1}[x_k + 2(\mu_{k-1}+\mu_k)x_k] + x_n^2 + 2\mu_{n-1}x_n \\ &+ \sum_{k=0}^{n-1}\left(u_k^2 - 2\mu_k u_k\right), \end{aligned}$$

we have

$$\begin{aligned} I(x) &= 2x_0\mu_0 - \mu_0^2 - \sum_{k=0}^{n-2}\left[(\mu_k+\mu_{k+1})^2 + \mu_{k+1}^2\right] - \mu_{n-1}^2 \\ &\quad + \sum_{k=1}^{n-1}(x_k+\mu_{k-1}+\mu_k)^2 + (x_n+\mu_{n-1})^2 + \sum_{k=0}^{n-1}(u_k-\mu_k)^2 \\ &\geq 2x_0\mu_0 - \mu_0^2 - \sum_{k=0}^{n-2}\left[(\mu_k+\mu_{k+1})^2 + \mu_{k+1}^2\right] - \mu_{n-1}^2 \end{aligned}$$

for any $x = (x_0, x_1, \ldots, x_n) \in R^{n+1}$, $u = (u_0, \ldots, u_{n-1}) \in R^n$ satisfying (2.2) and any $\mu = (\mu_0, \ldots, \mu_{n-1}) \in R^n$. Let us take

$$J(\mu) := 2c\mu_0 - \mu_0^2 - \sum_{k=0}^{n-2}\left[(\mu_k+\mu_{k+1})^2 + \mu_{k+1}^2\right] - \mu_{n-1}^2. \tag{2.4}$$

Then we have an inequality

$$I(x) \geq J(\mu) \tag{2.5}$$

for any feasible $x = (x_0, x_1, \ldots, x_n) \in R^{n+1}$ of the primal problem (P_n) and any feasible $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1}) \in R^n$ of the dual problem (D_n).

Lemma 2.2. *(Lucas formula) Let $\{F_k\}$ be the Fibonacci sequence. For any $n \geq 1$, we have*

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}. \tag{2.6}$$

Theorem 2.3. *The primal problem (P_n) has the minimum value* $m = \dfrac{F_{2n}}{F_{2n+1}}c^2$ *at the point*

$$\begin{aligned} \hat{x} &= (x_0,\ \hat{x}_1,\ \hat{x}_2,\ \hat{x}_3,\ \ldots,\ \hat{x}_k,\ \ldots,\ \hat{x}_{n-1},\ \hat{x}_n) \\ &= \frac{c}{F_{2n+1}}\Big(F_{2n+1}, -F_{2n-1}, F_{2n-3}, \ldots, \\ &\qquad \ldots, (-1)^k F_{2n-2k+1}, \ldots, (-1)^{n-1}F_3, (-1)^n F_1\Big). \end{aligned}$$

Proof Since the objective function $I(x)$ (see also (2.1)) for (P_n) is convex and differentiable for any $x = (x_1, x_2, \ldots, x_n)$, the minimum point for (P_n) satisfies the first order optimality condition,

$$\frac{\partial I}{\partial x_k} = 0 \qquad 1 \le k \le n. \tag{2.7}$$

These conditions (2.7) are equivalent to the following n equations:

$$(\mathrm{AF})_{\mathrm{P}} \qquad (-1)^k \frac{x_k + x_{k+1}}{F_{2n-2k}} = (-1)^{k+1} \frac{x_{k+1}}{F_{2n-2k-1}} \qquad 0 \le k \le n-1. \tag{2.8}$$

The condition $(\mathrm{AF})_{\mathrm{P}}$ is called the *alternately Fibonacci condition* for (P_n). From the condition $(\mathrm{AF})_{\mathrm{P}}$,

$$x_{k+1} = -\frac{F_{2n-2k-1}}{F_{2n-2k+1}} x_k \qquad 0 \le k \le n-1. \tag{2.9}$$

Thus we have

$$\begin{aligned} \hat{x} &= (x_0,\ \hat{x}_1,\ \hat{x}_2,\ \hat{x}_3,\ \ldots,\ \hat{x}_k,\ \ldots,\ \hat{x}_{n-1},\ \hat{x}_n) \\ &= \frac{c}{F_{2n+1}} \Big(F_{2n+1}, -F_{2n-1}, F_{2n-3}, \ldots, \\ &\qquad \ldots, (-1)^k F_{2n-2k+1}, \ldots, (-1)^{n-1} F_3, (-1)^n F_1 \Big). \end{aligned} \tag{2.10}$$

Next, we prove the minimum value $m = \frac{F_{2n}}{F_{2n+1}} c^2$. From (2.1) and (2.10),

$$\begin{aligned} F_{2n+1}^2 \frac{I(\hat{x})}{c^2} &= \left[(F_{2n+1} - F_{2n-1})^2 + (-F_{2n-1})^2 \right] \\ &+ \left[(-F_{2n-1} + F_{2n-3})^2 + F_{2n-3}^2 \right] + \\ &\qquad \cdots + \left[\left\{ (-1)^{n-1} F_3 + (-1)^n F_1 \right\}^2 + \left\{ (-1)^n F_1 \right\}^2 \right] \\ &= \sum_{k=0}^{n-1} \left(F_{2n-2k}^2 + F_{2n-2k-1}^2 \right) \\ &= \sum_{k=1}^{2n} F_k^2 \\ &= F_{2n} F_{2n+1} \quad \text{(by Lucas formula)}. \end{aligned}$$

Consequently, we get

$$m = I(\hat{x}) = \frac{F_{2n}}{F_{2n+1}} c^2. \tag{2.11}$$

Theorem 2.4. *The dual problem* (D_n) *has the maximum value* $M = \dfrac{F_{2n}}{F_{2n+1}} c^2$ *at the point*

$$\begin{aligned} \mu^* &= (\mu_0^*,\ \mu_1^*,\ \mu_2^*,\ \mu_3^*,\ \ldots,\ \mu_k^*,\ \ldots,\ \mu_{n-1}^*) \\ &= \frac{c}{F_{2n+1}} \Big(F_{2n}\,, -F_{2n-2}\,, F_{2n-4}\,, \ldots\,, \\ &\qquad\qquad , \ldots\,, (-1)^k F_{2n-2k}\,, \ldots\,, (-1)^{n-1} F_2 \Big). \end{aligned}$$

Proof Since the objective function $J(\mu)$ (see also (2.4)) for (D_n) is concave and differentiable for any $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1})$, the maximum point for (D_n) satisfies the first order optimality condition,

$$\frac{\partial J}{\partial \mu_k} = 0 \qquad 0 \le k \le n-1. \tag{2.12}$$

These conditions (2.12) are equivalent to the following n equations:

$$(\mathrm{AF})_{\mathrm{D}} \quad \frac{c-\mu_0}{F_{2n-1}} = \frac{\mu_0+\mu_1}{F_{2n-1}}, \quad (-1)^k \frac{\mu_k+\mu_{k+1}}{F_{2n-2k-1}} = (-1)^{k+1} \frac{\mu_{k+1}}{F_{2n-2k-2}} \tag{2.13}$$

for $0 \le k \le n-2$. The condition $(\mathrm{AF})_{\mathrm{D}}$ is called the *alternately Fibonacci condition* for (D_n). From the condition $(\mathrm{AF})_{\mathrm{D}}$,

$$\mu_0 = \frac{F_{2n-2}+F_{2n-1}}{F_{2n-2}+F_{2n-1}+F_{2n-1}} c, \quad \mu_{k+1} = -\frac{F_{2n-2k-2}}{F_{2n-2k}} \mu_k \qquad 0 \le k \le n-2. \tag{2.14}$$

Thus we have

$$\begin{aligned} \mu^* &= (\mu_0^*,\ \mu_1^*,\ \mu_2^*,\ \mu_3^*,\ \ldots,\ \mu_k^*,\ \ldots,\ \mu_{n-1}^*) \\ &= \frac{c}{F_{2n+1}} \Big(F_{2n}\,, -F_{2n-2}\,, F_{2n-4}\,, \ldots\,, \\ &\qquad\qquad , \ldots\,, (-1)^k F_{2n-2k}\,, \ldots\,, (-1)^{n-1} F_2 \Big). \end{aligned} \tag{2.15}$$

Next, we prove the maximum value $M = \dfrac{F_{2n}}{F_{2n+1}} c^2$. From (2.4) and (2.15),

$$\begin{aligned} F_{2n+1}^2 \frac{J(\mu^*)}{c^2} &= 2F_{2n}F_{2n+1} - F_{2n}^2 - \left[(F_{2n} - F_{2n-2})^2 + (-F_{2n-2})^2 \right] - \\ &\qquad \cdots - \left[(-F_{2n-2} + F_{2n-4})^2 + F_{2n-4}^2 \right] \\ &\qquad - \left[(F_4 - F_2)^2 + (-F_2)^2 \right] - (-F_2)^2 \\ &= 2F_{2n}F_{2n+1} - \sum_{k=1}^{2n} F_k^2 \\ &= 2F_{2n}F_{2n+1} - F_{2n}F_{2n+1} \quad \text{(by Lucas formula)} \\ &= F_{2n}F_{2n+1}. \end{aligned}$$

Consequently, we get

$$M \;=\; J(\mu^*) \;=\; \frac{F_{2n}}{F_{2n+1}}c^2. \tag{2.16}$$

There are the following triplet relations between the minimum point $\hat{x}$ of the primal problem (P_n) and the maximum point μ^* of the dual problem (D_n).

(i) (duality) The value of maximum and minimum are the same:

$$m = M \;=\; \frac{F_{2n}}{F_{2n+1}}\, c^2.$$

It is a quadratic function of c, whose coefficient is ratio of adjacent Fibonacci number. This is the first alternately Fibonacci complementary duality.

(ii) (2-step alternately Fibonacci) Both the minimum point $\hat{x}$ and the maximum point μ^* are two-step alternate Fibonacci sequence:

$$\begin{aligned}\hat{x} \;&=\; (x_0,\ \hat{x}_1,\ \hat{x}_2,\ \hat{x}_3,\ \ldots,\ \hat{x}_k,\ \ldots,\ \hat{x}_{n-1},\ \hat{x}_n)\\ &=\; \frac{c}{F_{2n+1}}\Big(\, F_{2n+1}\,, -F_{2n-1}\,, F_{2n-3}\,, \ldots\,, \ldots\,, (-1)^k F_{2n-2k+1}\,, \ldots\,, (-1)^n F_1 \Big)\end{aligned}$$

and

$$\begin{aligned}\mu^* \;&=\; (\mu_0^*,\ \mu_1^*,\ \mu_2^*,\ \mu_3^*,\ \ldots,\ \mu_k^*,\ \ldots,\ \mu_{n-1}^*)\\ &=\; \frac{c}{F_{2n+1}}\Big(\, F_{2n}\,, -F_{2n-2}\,, F_{2n-4}\,, \ldots\,, (-1)^k F_{2n-2k}\,, \ldots\,, (-1)^{n-1} F_2 \Big).\end{aligned}$$

This is the second.

(iii) (alternately Fibonacci complement) Both the optimal points constitute alternately the (1-step) alternate two-run Fibonacci sequence:

$$\begin{aligned}&(x_0\,, \mu_0^*\,, \hat{x}_1\,, \mu_1^*\,, \ldots\,, \hat{x}_k\,, \mu_k^*\,, \ldots\,, \mu_{n-1}^*\,, \hat{x}_n)\\ =\;& \frac{c}{F_{2n+1}}\Big(\, F_{2n+1}\,, F_{2n}\,, -F_{2n-1}\,, -F_{2n-2}\,, \ldots\,, (-1)^k F_{2n-2k+1}\,, (-1)^k F_{2n-2k}\,,\\ &\qquad \ldots\,, (-1)^{n-1} F_2\,, (-1)^n F_1 \Big).\end{aligned}$$

This is the third.

This triplet is called the *alternately Fibonacci complementary duality*.

Theorem 2.5. *If* (P_n) *has an optimal solution, then there is a feasible solution of* (D_n) *and the two objectives have the same values. Moreover,*

$$\begin{aligned}\hat{x} &= (x_0,\ \hat{x}_1,\ \hat{x}_2,\ \hat{x}_3,\ \ldots,\ \hat{x}_k,\ \ldots,\ \hat{x}_{n-1},\ \hat{x}_n)\\ &= \frac{c}{F_{2n+1}}\Big(F_{2n+1}\,, -F_{2n-1}\,, F_{2n-3}\,, \ldots\,, \ldots\,, (-1)^k F_{2n-2k+1}\,, \ldots\,, (-1)^n F_1\Big)\end{aligned}$$

is the optimal solution of (P_n)*, and*

$$\begin{aligned}\mu^* &= (\mu_0^*,\ \mu_1^*,\ \mu_2^*,\ \mu_3^*,\ \ldots,\ \mu_k^*,\ \ldots,\ \mu_{n-1}^*)\\ &= \frac{c}{F_{2n+1}}\Big(F_{2n}\,, -F_{2n-2}\,, F_{2n-4}\,, \ldots\,, (-1)^k F_{2n-2k}\,, \ldots\,, (-1)^{n-1} F_2\Big)\end{aligned}$$

is the optimal solution of (D_n)*. Hence, the alternately Fibonacci complementary duality holds between* (P_n) *and* (D_n)*.*

Proof It is obvious to prove this theorem from the proofs of theorem 2.1, 2.3, and 2.4.

3. THE ALTERNATE DA VINCI CODE

We introduced the DA VINCI Code (see also (1.1)) in the introduction, and we showed that the code utilizes the Fibonacci sequence. In this section, we consider a pair of primal problem (P_4) and its dual (D_4) together with their respective reversed problems. We prove the two-step alternate DA VINCI Code by using optimal points of their problems. Two-step alternate DA VINCI Code is defined as the following sequence:

$$1\quad -1\quad -2\quad 3\quad 5\quad -8\quad -13\quad 21. \tag{3.1}$$

Let us now consider the following primal quadratic optimization problem (P_4):

$$(\mathrm{P}_4)\qquad \begin{array}{ll}\text{minimize} & \displaystyle\sum_{k=0}^{3}\left[(x_k + x_{k+1})^2 + x_{k+1}^2\right]\\ \text{subject to} & \text{(i)}\ -\infty < x_k < \infty \qquad k = 1, 2, 3, 4\\ & \text{(ii)}\ x_0 = c.\end{array}$$

By theorem 2.3, the primal problem (P_4) has the minimum value

$$m_4 \;=\; \frac{F_8}{F_9}c^2 \tag{3.2}$$

at the point

$$\hat{x} \;=\; (x_0\,, \hat{x}_1\,, \hat{x}_2\,, \hat{x}_3\,, \hat{x}_4) \;=\; \frac{c}{F_9}\,(F_9\,, -F_7\,,\ F_5\,, -F_3\,,\ F_1)\,. \tag{3.3}$$

From theorem 2.1, a dual problem (D_4) for the primal problem (P_4) is given by

$$(\mathrm{D}_4) \quad \begin{array}{ll} \text{Maximize} & 2c\mu_0 - \mu_0^2 - \sum_{k=0}^{2} \left[(\mu_k + \mu_{k+1})^2 + \mu_{k+1}^2 \right] - \mu_3^2 \\ \text{subject to} & \text{(i)} \; -\infty < \mu_k < \infty \quad k = 0, 1, 2, 3. \end{array}$$

By theorem 2.4, the dual problem (D_4) has the maximum value

$$M_4 = \frac{F_8}{F_9} c^2 \tag{3.4}$$

at the point

$$\mu^* = (\mu_0^*, \mu_1^*, \mu_2^*, \mu_3^*) = \frac{c}{F_9} (F_8, -F_6, F_4, -F_2). \tag{3.5}$$

Let us now appreciate a triplet alternately Fibonacci complementary duality about both the optimal solutions for the primal problem (P_4) and the dual problem (D_4).

(i) (duality) The value of maximum and minimum are the same:

$$m_4 = M_4 = \frac{F_8}{F_9} c^2. \tag{3.6}$$

This is the first alternately Fibonacci complementary duality.

(ii) (2-step alternately Fibonacci) Both the minimum point

$$\hat{x} = \frac{c}{F_9} (F_9, -F_7, F_5, -F_3, F_1) \tag{3.7}$$

and the maximum point

$$\mu^* = \frac{c}{F_9} (F_8, -F_6, F_4, -F_2) \tag{3.8}$$

are 2-step alternate Fibonacci sequences, as was shown. This is the second.

(iii) (alternately Fibonacci complement) Both the optimal points constitute alternately the (1-step) alternate two-run Fibonacci sequence:

$$\begin{aligned} & (x_0, \; \mu_0^*, \; \hat{x}_1, \; \mu_1^*, \; \hat{x}_2, \; \mu_2^*, \; \hat{x}_3, \; \mu_3^*, \; \hat{x}_4) \\ = \; & \frac{c}{F_9} \Big(F_9, \; F_8, \; -F_7, \; -F_6, \; F_5, \; F_4, \; -F_3, \; -F_2, \; F_1 \Big). \end{aligned} \tag{3.9}$$

This is the third.

Hence, the alternately Fibonacci complementary duality holds between the primal problem (P_4) and the dual problem (D_4).

Next we consider a reversed problem (RP_4) for the problem (P_4) as follows:

$$(\mathrm{RP}_4) \quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{4} \left[x_k^2 + (x_k + x_{k+1})^2\right] \\ \text{subject to} & \text{(i)} \; -\infty < x_k < \infty \quad k = 1, 2, 3, 4 \\ & \text{(ii)} \; x_5 = c, \end{array}$$

that is, a variable $(x_0, x_1, x_2, x_3, x_4)$ for (P_4) was replaced by a variable $(x_5, x_4, x_3, x_2, x_1)$. Moreover, its dual problem is the following problem:

$$(\mathrm{RD}_4) \quad \begin{array}{ll} \text{Maximize} & -\mu_1^2 - \displaystyle\sum_{k=1}^{3} \left[\mu_k^2 + (\mu_k + \mu_{k+1})^2\right] - \mu_4^2 + 2c\mu_4 \\ \text{subject to} & \text{(i)} \; -\infty < \mu_k < \infty \quad k = 1, 2, 3, 4. \end{array}$$

From (3.2) and (3.3), the problem (RP_4) has the minimal value

$$m_4' = \frac{21}{34} c^2 \tag{3.10}$$

at the point

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, x_5) = \frac{c}{34}(1, -2, 5, -13, 34). \tag{3.11}$$

From (3.4) and (3.5), its dual problem (RD_4) has the maximum value

$$M_4' = \frac{21}{34} c^2 \tag{3.12}$$

at the point

$$\mu^\star = (\mu_1^\star, \mu_2^\star, \mu_3^\star, \mu_4^\star) = \frac{c}{34}(-1, 3, -8, 21). \tag{3.13}$$

Both the reversed optimization problems (RP_4) and (RD_4) have the alternately Fibonacci complementary duality.

(i) (duality) The value of maximum and minimum are the same:

$$m_4' = M_4' = \frac{21}{34} c^2.$$

(ii) (2-step alternately Fibonacci) Both the minimum point

$$\tilde{x} = \frac{c}{34}(1, -2, 5, -13, 34) \tag{3.14}$$

and the maximum point

$$\mu^\star = \frac{c}{34}(-1, 3, -8, 21) \tag{3.15}$$

are 2-step alternate Fibonacci sequences, as was shown.

(iii) (alternately Fibonacci complement) Both the optimal points constitute alternately the (1-step) alternate two-run Fibonacci sequence:

$$\begin{aligned}&(\tilde{x}_1\,,\ \mu_1^\star\,,\ \tilde{x}_2\,,\ \mu_2^\star\,,\ \tilde{x}_3\,,\ \mu_3^\star\,,\ \tilde{x}_4\,,\ \mu_4^\star\,,\ x_5) \\ =\ &\frac{c}{34}\Big(1\,,-1\,,-2\,,\ 3\,,\ 5\,,-8\,,-13\,,\ 21\,,\ 34\Big). \end{aligned} \tag{3.16}$$

For (3.16), we take a constant $c = 34$, then the sequence $(\tilde{x}_1\,,\ \mu_1^\star\,,\ \tilde{x}_2\,,\ \mu_2^\star\,,\ \tilde{x}_3\,,\ \mu_3^\star\,,\ \tilde{x}_4\,,\ \mu_4^\star)$ constitutes a two-step alternate DA VINCI Code.

4. THE ALTERNATELY FIBONACCI SECTION

In this section, for the problem (P_4) and (D_4), we propose the *alternately Fibonacci sections*, which are a method to find optimal points for the quadratic optimization problems. The alternately Fibonacci sections are given by the alternately Fibonacci conditions $(\mathrm{AF})_\mathrm{P}$ and $(\mathrm{AF})_\mathrm{D}$.

First, let us now propose the alternately Fibonacci section for the primal problem (P_4). From (2.8) in the proof of theorem 2.3, the alternately Fibonacci condition $(\mathrm{AF})_\mathrm{P}$ for the problem (P_4) is given by the following four equations:

$$(\mathrm{AF})_\mathrm{P} \qquad \frac{c+x_1}{F_8} = \frac{x_1}{-F_7}, \quad \frac{x_1+x_2}{-F_6} = \frac{x_2}{F_5}, \quad \frac{x_2+x_3}{F_4} = \frac{x_3}{-F_3}, \quad \frac{x_3+x_4}{-F_2} = \frac{x_4}{F_1}. \tag{4.1}$$

The condition $(\mathrm{AF})_\mathrm{P}$ means that the following eight quantities:

$$c+x_1, \quad x_1, \quad x_1+x_2, \quad x_2, \quad x_2+x_3, \quad x_3, \quad x_3+x_4, \quad x_4 \tag{4.2}$$

are allocated on the basis of the alternate two-run Fibonacci sequence $F_8 : -F_7 : -F_6 : F_5 : F_4 : -F_3 : -F_2 : F_1$. We take an interval $[0, c]$, where $c > 0$. The first equation of $(\mathrm{AF})_\mathrm{P}$ means that an optimal point $-\hat{x}_1$ is an internally dividing point of the interval $[0, c]$ depending on a ratio $F_7 : F_8$. That is,

$$-\hat{x}_1 \ = \ \frac{F_7}{F_7+F_8}c \ = \ \frac{F_7}{F_9}c.$$

Next the second equation of $(\mathrm{AF})_\mathrm{P}$ means that an optimal point $\hat{x}_2$ is an internally dividing point of the interval $[0, -\hat{x}_1]$ depending on a ratio $F_5 : F_6$. That is,

$$\hat{x}_2 \ = \ \frac{F_5}{F_5+F_6}\,(-\hat{x}_1) \ = \ \frac{F_5}{F_7}\,(-\hat{x}_1)\,.$$

The third equation of $(\mathrm{AF})_{\mathrm{P}}$ means that an optimal point $-\hat{x}_3$ is an internally dividing point of the interval $[0, \hat{x}_2]$ depending on a ratio $F_3 : F_4$. That is,

$$-\hat{x}_3 \;=\; \frac{F_3}{F_3+F_4}\hat{x}_2 \;=\; \frac{F_3}{F_5}\hat{x}_2.$$

At the last, the fourth equation of $(\mathrm{AF})_{\mathrm{P}}$ means that an optimal point $\hat{x}_4$ is an internally dividing point of the interval $[0, -\hat{x}_3]$ depending on a ratio $F_1 : F_2$. That is,

$$\hat{x}_4 \;=\; \frac{F_1}{F_1+F_2}\,(-\hat{x}_3) \;=\; \frac{F_1}{F_3}\,(-\hat{x}_3)\,.$$

Thus we can find the optimal point $\hat{x} = (x_0, \hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4)$ as follows,

$$\begin{aligned}
\hat{x}_1 &= -\frac{F_7}{F_9}c,\\
\hat{x}_2 &= \frac{F_5}{F_7}\cdot\frac{F_7}{F_9}c \;=\; \frac{F_5}{F_9}c,\\
\hat{x}_3 &= -\frac{F_3}{F_5}\cdot\frac{F_5}{F_7}\cdot\frac{F_7}{F_9}c \;=\; -\frac{F_3}{F_9}c,\\
\hat{x}_4 &= \frac{F_1}{F_3}\cdot\frac{F_3}{F_5}\cdot\frac{F_5}{F_7}\cdot\frac{F_7}{F_9}c, \;=\; \frac{F_1}{F_9}c.
\end{aligned}$$

Consequently, we get

$$\hat{x} \;=\; (x_0, \hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) \;=\; \frac{c}{F_9}\,(F_9\,, -F_7\,, F_5\,, -F_3\,, F_1)\,.$$

Let us now propose the alternately Fibonacci section for the dual problem (D_4). From (2.13) in the proof of theorem 2.4, the alternately Fibonacci condition $(\mathrm{AF})_{\mathrm{D}}$ for the problem (D_4) is given by the following four equations:

$$(\mathrm{AF})_{\mathrm{D}} \qquad \frac{c-\mu_0}{F_7} = \frac{\mu_0+\mu_1}{F_7} = \frac{\mu_1}{-F_6},\;\; \frac{\mu_1+\mu_2}{-F_5} = \frac{\mu_2}{F_4},\;\; \frac{\mu_2+\mu_3}{F_3} = \frac{\mu_3}{-F_2}. \tag{4.3}$$

The condition $(\mathrm{AF})_{\mathrm{D}}$ means that the following seven quantities:

$$\mu_0, \quad \mu_0+\mu_1, \quad \mu_1, \quad \mu_1+\mu_2, \quad \mu_2, \quad \mu_2+\mu_3, \quad \mu_3. \tag{4.4}$$

are allocated on the basis of the alternate two-run Fibonacci sequence $F_8 : F_7 : -F_6 : -F_5 : F_4 : F_3 : -F_2$. We take an interval $[0, c]$, where $c > 0$. The first equation of $(\mathrm{AF})_{\mathrm{D}}$ means that an optimal point μ_0^* is an internally dividing point of the interval $[0, c]$ depending on a ratio $F_8 : F_7$. That is,

$$\mu_0^* \;=\; \frac{F_8}{F_8+F_7}c \;=\; \frac{F_8}{F_9}c.$$

Next the second equation of $(\mathrm{AF})_{\mathrm{D}}$ means that an optimal point $-\mu_1^*$ is an internally dividing point of the interval $[0, \mu_0^*]$ depending on a ratio $F_6 : F_7$. That is,

$$-\mu_1^* \;=\; \frac{F_6}{F_6+F_7}\mu_0^* \;=\; \frac{F_6}{F_8}\mu_0^*.$$

The third equation of $(\mathrm{AF})_{\mathrm{D}}$ means that an optimal point μ_2^* is an internally dividing point of the interval $[0, -\mu_1^*]$ depending on a ratio $F_4 : F_5$. That is,

$$\mu_2^* \;=\; \frac{F_4}{F_4+F_5}\left(-\mu_1^*\right) \;=\; \frac{F_4}{F_6}\left(-\mu_1^*\right).$$

At the last, the fourth equation of $(\mathrm{AF})_{\mathrm{D}}$ means that an optimal point $-\mu_3^*$ is an internally dividing point of the interval $[0, \mu_2^*]$ depending on a ratio $F_2 : F_3$. That is,

$$-\mu_3^* \;=\; \frac{F_2}{F_2+F_3}\mu_2^* \;=\; \frac{F_2}{F_4}\mu_2^*.$$

Thus we can find the optimal point $\mu^* = (\mu_0^*, \mu_1^*, \mu_2^*, \mu_3^*)$ as follows,

$$\begin{aligned}
\mu_0^* &= \frac{F_8}{F_9}c, \\
\mu_1^* &= -\frac{F_6}{F_8}\cdot\frac{F_8}{F_9}c \;=\; -\frac{F_6}{F_9}c, \\
\mu_2^* &= \frac{F_4}{F_6}\cdot\frac{F_6}{F_8}\cdot\frac{F_8}{F_9}c \;=\; \frac{F_4}{F_9}c, \\
\mu_3^* &= -\frac{F_2}{F_4}\cdot\frac{F_4}{F_6}\cdot\frac{F_6}{F_8}\cdot\frac{F_8}{F_9}c \;=\; -\frac{F_2}{F_9}c.
\end{aligned}$$

Consequently, we get

$$\mu^* \;=\; (\mu_0^*, \mu_1^*, \mu_2^*, \mu_3^*) \;=\; \frac{c}{F_9}\left(F_8\,, -F_6\,, F_4\,, -F_2\right).$$

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