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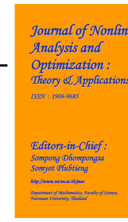

FIXED POINT AND MEAN CONVERGENCE THEOREMS FOR A FAMILY OF λ-HYBRID MAPPINGS

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ABSTRACT. The aim of this paper is to prove fixed point and mean convergence theorems for a sequence of λ-hybrid mappings in Hilbert spaces.

KEYWORDS : λ-hybrid mapping; Fixed point; Mean convergence theorem.

1. INTRODUCTION

In this paper, we show fixed point and mean convergence theorems for a sequence of λ-hybrid mappings in Hilbert spaces. Particularly, we focus on pointwise convergent sequences of such mappings.

According to [2] and §2, every nonexpansive mapping [5, 6, 10] is a 1-hybrid mapping and every nonspreading mapping introduced by Kohsaka and Takahashi [7] is a 0-hybrid mapping. Thus our results may be regarded as generalizations of results of [1] and [8]. Akatsuka, Aoyama, and Takahashi [1] showed a mean convergence theorem for a pointwise convergent sequence of nonexpansive mappings; Kurokawa and Takahashi [8] proved some mean convergence theorems for nonspreading mappings in Hilbert spaces.

Moreover, since the convex combination of the identity mapping and a strictly pseudononspreading mapping introduced by Osilike and Isiogugu [9] is λ-hybrid for some real number λ, our mean convergence theorem is a generalization of [9]. Osilike and Isiogugu [9] showed some mean convergence theorems for strictly pseudononspreading mappings in Hilbert spaces.

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2. PRELIMINARIES

Throughout the present paper, H denotes a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, C a nonempty closed convex subset of H, I the identity mapping on H, and $\mathbb{N}$ the set of positive integers. Strong convergence of a sequence $\{x_n\}$ in H to x is denoted by $x_n \longrightarrow x$ and weak convergence by $x_n \rightharpoonup x$. The metric projection of H onto C is denoted by P_C, that is, for each $x \in H$, $P_C x$ is the unique point in C such that $\|P_C x - x\| = \min\{\|y - x\| : y \in C\}$. It is known that P_C is nonexpansive and

$$\langle y - P_C x, x - P_C x \rangle \leq 0 \tag{2.1}$$

for all $x \in H$ and $y \in C$; see [10].

Let D be a nonempty subset of H. The set of fixed points of a mapping $T\colon D \longrightarrow H$ is denoted by $F(T)$. A mapping $T\colon D \longrightarrow H$ is said to be quasi-nonexpansive if $F(T)$ is nonempty and $\|Tx - z\| \leq \|x - z\|$ for all $x \in D$ and $z \in F(T)$. Let λ be a real number. A mapping $T\colon D \longrightarrow H$ is said to be λ-hybrid [2] if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - 2\lambda \langle x - Tx, y - Ty \rangle$$

or equivalently

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1 - \lambda) \langle x - Tx, y - Ty \rangle$$

for all $x, y \in D$. Let κ be a real number with $\kappa \in [0, 1)$. A mapping $T\colon D \longrightarrow H$ is said to be κ-strictly pseudononspreading [9] if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle + \kappa \|x - Tx - (y - Ty)\|^2$$

for all $x, y \in D$. It is known that

- T is λ-hybrid for every $\lambda \in [0, 1]$ if T is a firmly nonexpansive mapping [3, 4, 5, 6];
- T is 1-hybrid if and only if T is nonexpansive;
- T is 0-hybrid if and only if T is nonspreading in the sense of [7];
- T is $1/2$-hybrid if and only if T is hybrid in the sense of [11];
- $F(T)$ is closed and convex if $T\colon C \longrightarrow H$ is a quasi-nonexpansive mapping;
- T is quasi-nonexpansive if T is a λ-hybrid mapping with a fixed point.

The following lemma plays an important role in the present paper.

Lemma 2.1. *Let H be a Hilbert space, D a nonempty subset of H, γ and κ real numbers, and $T\colon D \longrightarrow H$ a mapping such that*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\gamma \langle x - Tx, y - Ty \rangle + \kappa \|x - Tx - (y - Ty)\|^2 \tag{2.2}$$

for all $x, y \in D$. Let $T_\alpha\colon D \longrightarrow H$ be a mapping defined by $T_\alpha = \alpha I + (1 - \alpha)T$, where α is a real number with $\alpha < 1$. Then

$$\begin{aligned} \|T_\alpha x - T_\alpha y\|^2 &+ \frac{\alpha - \kappa}{1 - \alpha} \|x - T_\alpha x - (y - T_\alpha y)\|^2 \\ &\leq \|x - y\|^2 + \frac{2\gamma}{1 - \alpha} \langle x - T_\alpha x, y - T_\alpha y \rangle \end{aligned} \tag{2.3}$$

for all $x, y \in D$. Moreover, if $\kappa \leq \alpha$, then T_α is $(1 - \alpha - \gamma)/(1 - \alpha)$-hybrid.

Proof. Let $x, y \in D$ be fixed. Since $1 - \alpha > 0$ and $I - T = (I - T_\alpha)/(1 - \alpha)$, it follows from (2.2) that

$$\begin{aligned} &(1 - \alpha)\|Tx - Ty\|^2 \\ &\quad \leq (1 - \alpha)\big(\|x - y\|^2 + 2\gamma \langle x - Tx, y - Ty \rangle + \kappa \|x - Tx - (y - Ty)\|^2\big) \end{aligned}$$

$$= (1-\alpha)\|x-y\|^2 + \frac{2\gamma}{1-\alpha}\langle x - T_\alpha x, y - T_\alpha y\rangle + \frac{\kappa}{1-\alpha}\|x - T_\alpha x - (y - T_\alpha y)\|^2$$

and hence

$$\begin{aligned}
&\|T_\alpha x - T_\alpha y\|^2 \\
&= \alpha\|x-y\|^2 + (1-\alpha)\|Tx - Ty\|^2 - \alpha(1-\alpha)\|x - Tx - (y - Ty)\|^2 \\
&= \alpha\|x-y\|^2 + (1-\alpha)\|Tx - Ty\|^2 - \frac{\alpha}{1-\alpha}\|x - T_\alpha x - (y - T_\alpha y)\|^2 \\
&\le \|x-y\|^2 + \frac{2\gamma}{1-\alpha}\langle x - T_\alpha x, y - T_\alpha y\rangle - \frac{\alpha-\kappa}{1-\alpha}\|x - T_\alpha x - (y - T_\alpha y)\|^2.
\end{aligned}$$

Thus (2.3) holds. Now we suppose that $\kappa \le \alpha$. Then $(\alpha-\kappa)/(1-\alpha) \ge 0$, and (2.3) yields that

$$\|T_\alpha x - T_\alpha y\|^2 \le \|x-y\|^2 + 2\left(1 - \frac{1-\alpha-\gamma}{1-\alpha}\right)\langle x - T_\alpha x, y - T_\alpha y\rangle.$$

Therefore, T_α is $(1-\alpha-\gamma)/(1-\alpha)$-hybrid. □

Lemma 2.1 implies the following lemma.

Lemma 2.2. *Let H be a Hilbert space, D a nonempty subset of H, λ a real number, and $T\colon D \longrightarrow H$ a λ-hybrid mapping. Let $T_\alpha\colon D \longrightarrow H$ be a mapping defined by $T_\alpha = \alpha I + (1-\alpha)T$, where α a real number with $0 \le \alpha < 1$. Then T_α is $(\lambda-\alpha)/(1-\alpha)$-hybrid.*

Proof. Assuming that $\gamma = 1-\lambda$ and $\kappa = 0$ in Lemma 2.1, we obtain the conclusion. □

Using Lemma 2.2, we can show the following corollary.

Corollary 2.3. *Let H be a Hilbert space and D a nonempty convex subset of H. Suppose that every nonspreading self-mapping on D has a fixed point. If $\lambda \in [0,1)$, then every λ-hybrid mapping $T\colon D \longrightarrow D$ has a fixed point.*

Proof. Let $\lambda \in [0,1)$ and let $T\colon D \longrightarrow D$ be a λ-hybrid mapping. Then it follows from Lemma 2.2 that $T_\lambda = \lambda I + (1-\lambda)T$ is a nonspreading mapping of D into itself. Hence, by assumption, we know that $F(T_\lambda)$ is nonempty. On the other hand, it obviously holds that $F(T_\lambda) = F(T)$. Thus $F(T)$ is nonempty. □

Remark 2.4. It is known that every nonspreading self-mapping on C has a fixed point if C is a nonempty bounded closed convex subset of H; see [7, Theorem 4.1].

Lemma 2.1 also implies the following lemma, which was essentially proven in [9].

Lemma 2.5. *Let H be a Hilbert space, D a nonempty subset of H, κ and β real numbers with $0 \le \kappa \le \beta < 1$, $T\colon D \longrightarrow H$ a κ-strictly pseudononspreading mapping, and $T_\beta\colon D \longrightarrow H$ the mapping defined by $T_\beta = \beta I + (1-\beta)T$. Then T_β is $-\beta/(1-\beta)$-hybrid.*

Proof. Assuming that $\alpha = \beta$ and $\gamma = 1$ in Lemma 2.1, we obtain the conclusion. □

We need the following lemmas in order to prove our results in the remainder sections.

Lemma 2.6. *Let $\{x_n\}$ and $\{y_n\}$ be sequences in a Hilbert space H and $\{\eta_n\}$ a sequence of real numbers. Suppose that $\{x_n\}$ is bounded and both $\{y_n\}$ and $\{\eta_n\}$ are convergent. Then*

$$\frac{1}{n}\sum_{k=1}^{n}\eta_k\langle x_{k+1}-x_k, y_k\rangle \longrightarrow 0$$

as $n \longrightarrow \infty$.

Proof. Let y and η be the limits of $\{y_n\}$ and $\{\eta_n\}$, respectively. Since $\{x_n\}$ is bounded, it follows that $\langle x_{n+1}-x_n, y_n-y\rangle \longrightarrow 0$ and hence

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\langle x_{k+1}-x_k, y_k\rangle &= \frac{1}{n}\sum_{k=1}^{n}\langle x_{k+1}-x_k, y\rangle + \frac{1}{n}\sum_{k=1}^{n}\langle x_{k+1}-x_k, y_k-y\rangle \\ &= \frac{1}{n}\langle x_{n+1}-x_1, y\rangle + \frac{1}{n}\sum_{k=1}^{n}\langle x_{k+1}-x_k, y_k-y\rangle \longrightarrow 0\end{aligned}$$

as $n \longrightarrow \infty$. Therefore, since $\{\langle x_{n+1}-x_n, y_n\rangle\}$ is bounded, it follows that

$$\frac{1}{n}\sum_{k=1}^{n}\eta_k\langle x_{k+1}-x_k, y_k\rangle = \frac{1}{n}\sum_{k=1}^{n}\eta\langle x_{k+1}-x_k, y_k\rangle+\frac{1}{n}\sum_{k=1}^{n}(\eta_k-\eta)\langle x_{k+1}-x_k, y_k\rangle \longrightarrow 0$$

as $n \longrightarrow \infty$. □

The following lemma was essentially shown in [1, Lemma 3.1], where $\{\xi_n\}$ was assumed to be convergent to 0. For the sake of completeness, we give the proof.

Lemma 2.7. *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $T\colon C \longrightarrow H$ a mapping. Let $\{x_n\}$ be a sequence in C, $\{\xi_n\}$ a sequence of real numbers, $\{z_n\}$ a sequence in C defined by $z_n = (1/n)\sum_{k=1}^{n} x_k$ for $n \in \mathbb{N}$, and z a weak cluster point of $\{z_n\}$. Suppose that*

$$\xi_n \le \|x_n - z\|^2 - \|x_{n+1} - Tz\|^2$$

for every $n \in \mathbb{N}$ and $(1/n)\sum_{k=1}^{n}\xi_k \longrightarrow 0$ as $n \longrightarrow \infty$. Then z is a fixed point of T.

Proof. By assumption, it is clear that

$$\begin{aligned}\xi_k &\le \|x_k - z\|^2 - \|x_{k+1} - Tz\|^2 \\ &= \|x_k - Tz + Tz - z\|^2 - \|x_{k+1} - Tz\|^2 \\ &= \|x_k - Tz\|^2 - \|x_{k+1} - Tz\|^2 + 2\langle x_k - Tz, Tz - z\rangle + \|Tz - z\|^2\end{aligned}$$

for every $k \in \mathbb{N}$. Summing these inequalities from $k = 1$ to n and dividing by n, we have

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\xi_k &\le \frac{1}{n}\left(\|x_1 - Tz\|^2 - \|x_{n+1} - Tz\|^2\right) \\ &\qquad + 2\left\langle \frac{1}{n}\sum_{k=1}^{n} x_k - Tz, Tz - z\right\rangle + \|Tz - z\|^2 \\ &\le \frac{1}{n}\|x_1 - Tz\|^2 + 2\langle z_n - Tz, Tz - z\rangle + \|Tz - z\|^2\end{aligned}$$

for every $n \in \mathbb{N}$. Since z is a weak cluster point of $\{z_n\}$, there is a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z$. Replacing n by n_i in the above inequality, we obtain

$$\frac{1}{n_i}\sum_{k=1}^{n_i}\xi_k \le \frac{1}{n_i}\|x_1 - Tz\|^2 + 2\langle z_{n_i} - Tz, Tz - z\rangle + \|Tz - z\|^2.$$

Since $(1/n_i)\sum_{k=1}^{n_i}\xi_k \longrightarrow 0$ and $z_{n_i} \rightharpoonup z$, we conclude that

$$0 \leq 2\langle z - Tz, Tz - z\rangle + \|Tz - z\|^2 = -\|Tz - z\|^2$$

and hence $Tz = z$. □

Lemma 2.8 (Takahashi and Toyoda [12]). *Let F be a nonempty closed convex subset of a Hilbert space H, P the metric projection of H onto F, and $\{x_n\}$ a sequence in H such that $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $u \in F$ and $n \in \mathbb{N}$. Then $\{Px_n\}$ converges strongly to some point in F.*

3. FIXED POINT THEOREMS

In this section, we study existence of fixed points of λ-hybrid mappings.

The following theorem is a generalization of [1, Theorem 3.2] and [2, Theorem 4.1].

Theorem 3.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $\{\lambda_n\}$ a sequence of real numbers such that $\lambda_n \longrightarrow \lambda$, and $T_n : C \longrightarrow C$ a λ_n-hybrid mapping for $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = T_n x_n, \text{ and } z_n = \frac{1}{n}\sum_{k=1}^{n} x_k$$

for $n \in \mathbb{N}$. Suppose that $\{T_n\}$ is pointwise convergent and T denotes the pointwise limit of $\{T_n\}$, that is, $Tx = \lim_{n\longrightarrow\infty} T_n x$ for $x \in C$. Then the following hold:

- (i) *The mapping T is λ-hybrid and $\bigcap_{n=1}^{\infty} F(T_n) \subset F(T)$;*
- (ii) *if $\{x_n\}$ is bounded, then T has a fixed point and every weak cluster point of $\{z_n\}$ is a fixed point of T.*

Proof. We first prove (1). Let $x, y \in C$ be fixed. Since each T_n is λ_n-hybrid, it follows that

$$\|T_n x - T_n y\|^2 \leq \|x - y\|^2 + 2(1 - \lambda_n)\langle x - T_n x, y - T_n y\rangle$$

for every $n \in \mathbb{N}$. Taking the limit $n \longrightarrow \infty$, we have

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Tx, y - Ty\rangle.$$

Thus T is λ-hybrid. Furthermore, let $u \in \bigcap_{n=1}^{\infty} F(T_n)$. Since T_n is pointwise convergent, $Tu = \lim_{n\longrightarrow\infty} T_n u = u$ and hence $u \in F(T)$.

We next prove (2). Assume that $\{x_n\}$ is bounded. Then $\{z_n\}$ is also bounded and thus there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z \in C$. It is enough to show that z is a fixed point of T. Since T_n is λ_n-hybrid and $x_{n+1} = T_n x_n$, we have

$$\begin{aligned}
\|x_{n+1} - Tz\|^2 &= \|x_{n+1} - T_n z + T_n z - Tz\|^2 \\
&= \|x_{n+1} - T_n z\|^2 + \|T_n z - Tz\|^2 + 2\langle x_{n+1} - T_n z, T_n z - Tz\rangle \\
&\leq \|x_n - z\|^2 + 2(1 - \lambda_n)\langle x_n - x_{n+1}, z - T_n z\rangle \\
&\quad + \|T_n z - Tz\|(\|T_n z - Tz\| + 2\|x_{n+1} - T_n z\|).
\end{aligned}$$

Therefore, we conclude that

$$\mu_n + \epsilon_n \leq \|x_n - z\|^2 - \|x_{n+1} - Tz\|^2$$

for every $n \in \mathbb{N}$, where $\mu_n = 2(1 - \lambda_n)\langle x_{n+1} - x_n, z - T_n z\rangle$ and

$$\epsilon_n = -\|T_n z - Tz\|(\|T_n z - Tz\| + 2\|x_{n+1} - T_n z\|).$$

Since $\{x_n\}$ is bounded and both $\{\lambda_n\}$ and $\{T_n z\}$ are convergent, Lemma 2.6 shows that $(1/n)\sum_{k=1}^{n}\mu_k \longrightarrow 0$, and hence $(1/n)\sum_{k=1}^{n}(\mu_k+\epsilon_k) \longrightarrow 0$. Thus Lemma 2.7 implies that z is a fixed point of T. □

A direct consequence of Theorem 3.1 is as follows:

Corollary 3.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H, λ a real number, $T\colon C \longrightarrow C$ a λ-hybrid mapping, and $\{\alpha_n\}$ a sequence in $[0,1)$ such that $\alpha_n \longrightarrow 0$. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \text{ and } z_n = \frac{1}{n}\sum_{k=1}^{n} x_k$$

for $n \in \mathbb{N}$. Suppose that $\{x_n\}$ is bounded. Then T has a fixed point and every weak cluster point of $\{z_n\}$ is a fixed point of T.

Proof. Put $T_n = \alpha_n I + (1-\alpha_n)T$ for $n \in \mathbb{N}$. Then Lemma 2.2 shows that $T_n\colon C \longrightarrow C$ is $(\lambda-\alpha_n)/(1-\alpha_n)$-hybrid. It is clear that $(\lambda-\alpha_n)/(1-\alpha_n) \longrightarrow \lambda$ and T is the pointwise limit of $\{T_n\}$. Therefore, Theorem 3.1 implies the conclusion. □

In particular, assuming that $\alpha_n = 0$ for each $n \in \mathbb{N}$ in Corollary 3.1, we obtain the following:

Corollary 3.2. *([2, Theorem 4.1]). Let H, C, λ, and T be the same as in Corollary 3.1. Let x be a point in C and $\{z_n\}$ a sequence in C defined by*

$$z_n = \frac{1}{n}\sum_{k=1}^{n} T^{k-1}x$$

for $n \in \mathbb{N}$, where T^0 is the identity mapping on C. Suppose that $\{T^n x\}$ is bounded. Then T has a fixed point and every weak cluster point of $\{z_n\}$ is a fixed point of T.

4. MEAN CONVERGENCE THEOREMS

In this section, we prove some mean convergence theorems for a family of λ-hybrid mappings.

We first prove the following lemma, which is a variant of [2, Lemma 5.1].

Lemma 4.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $T_n\colon C \longrightarrow C$ a quasi-nonexpansive mapping for $n \in \mathbb{N}$. Suppose that $\{T_n\}$ has a common fixed point. Let F be the set of common fixed points of $\{T_n\}$ and P the metric projection of H onto F. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = T_n x_n, \text{ and } z_n = \frac{1}{n}\sum_{k=1}^{n} x_n$$

for $n \in \mathbb{N}$. Then the following hold:

- (i) *The sequence $\{x_n\}$ is bounded and $\{Px_n\}$ converges strongly;*
- (ii) *if each weak cluster point of $\{z_n\}$ belongs to F, then $\{z_n\}$ converges weakly to the strong limit of $\{Px_n\}$.*

Proof. We first prove (1). Since T_n is quasi-nonexpansive,

$$\|x_{n+1} - u\| = \|T_n x_n - u\| \le \|x_n - u\|$$

for all $u \in F$ and $n \in \mathbb{N}$. Thus $\{x_n\}$ is bounded and Lemma 2.8 implies that $\{Px_n\}$ converges strongly.

We next prove (2). Since $\{z_n\}$ is bounded by (1), there exists a weak cluster point z of $\{z_n\}$. Let $\{z_{n_i}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z$ and w the strong limit of $\{Px_n\}$. It is enough to show that $z = w$. Since P is the metric projection of H onto F and $z \in F$, it follows from (2.1) that

$$\langle z - Px_k, x_k - Px_k \rangle \leq 0$$

for every $k \in \mathbb{N}$. Since each T_k is quasi-nonexpansive and $Px_k \in F$, it follows from the definition of P that

$$\|x_{k+1} - Px_{k+1}\| \leq \|x_{k+1} - Px_k\| = \|T_k x_k - Px_k\| \leq \|x_k - Px_k\|$$

for every $k \in \mathbb{N}$. Therefore

$$\begin{aligned} \langle z - w, x_k - Px_k \rangle &= \langle z - Px_k, x_k - Px_k \rangle + \langle Px_k - w, x_k - Px_k \rangle \\ &\leq \langle Px_k - w, x_k - Px_k \rangle \\ &\leq \|Px_k - w\| \|x_k - Px_k\| \\ &\leq \|Px_k - w\| \|x_1 - Px_1\| \end{aligned}$$

for every $k \in \mathbb{N}$. Summing these inequalities from $k = 1$ to n_i and dividing by n_i, we have

$$\left\langle z - w, z_{n_i} - \frac{1}{n_i} \sum_{k=1}^{n_i} Px_k \right\rangle \leq \frac{1}{n_i} \sum_{k=1}^{n_i} \|Px_k - w\| \|x_1 - Px_1\|.$$

Since $z_{n_i} \rightharpoonup z$ as $i \longrightarrow \infty$ and $Px_n \longrightarrow w$ as $n \longrightarrow \infty$, we obtain $\langle z - w, z - w \rangle \leq 0$ and hence $z = w$. This completes the proof. □

Using Theorem 3.1 and Lemma 4.1, we obtain the following:

Theorem 4.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $\{\lambda_n\}$ a sequence of real numbers such that $\lambda_n \longrightarrow \lambda$, and $T_n \colon C \longrightarrow C$ a λ_n-hybrid mapping for $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = T_n x_n, \text{ and } z_n = \frac{1}{n} \sum_{k=1}^{n} x_k$$

for $n \in \mathbb{N}$. Suppose that $\{T_n\}$ is pointwise convergent, T denotes the pointwise limit of $\{T_n\}$, and $F(T) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{z_n\}$ converges weakly to the strong limit of $\{Px_n\}$, where P is the metric projection of H onto $F(T)$.

Proof. Since T_n is λ_n-hybrid and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, each T_n is quasi-nonexpansive. Thus it follows from Lemma 4.1 that $\{x_n\}$ is bounded. Hence Theorem 3.1 shows that every weak cluster point of $\{z_n\}$ belongs to $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Therefore, Lemma 4.1 implies the conclusion. □

The following corollary is a direct consequence of Theorem 4.1.

Corollary 4.2. *Let H be a Hilbert space, C a nonempty closed convex subset of H, λ a real number, $T \colon C \longrightarrow C$ a λ-hybrid mapping with a fixed point, and $\{\alpha_n\}$ a sequence in $[0, 1)$ such that $\alpha_n \longrightarrow 0$. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \text{ and } z_n = \frac{1}{n} \sum_{k=1}^{n} x_k$$

for $n \in \mathbb{N}$. Then $\{z_n\}$ converges weakly to the strong limit of $\{Px_n\}$, where P is the metric projection of H onto $F(T)$.

Proof. Put $T_n = \alpha_n I + (1-\alpha_n)T$ for $n \in \mathbb{N}$. Then Lemma 2.2 shows that each $T_n : C \longrightarrow C$ is $(\lambda-\alpha_n)/(1-\alpha_n)$-hybrid. It is clear that $(\lambda-\alpha_n)/(1-\alpha_n) \longrightarrow \lambda$ and T is the pointwise limit of $\{T_n\}$. It is also clear that $F(T_n) = F(T)$ for every $n \in \mathbb{N}$ and hence $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Therefore, Theorem 4.1 implies the conclusion. □

Using Corollary 4.2, we immediately obtain the following weak convergence theorem for a strictly pseudononspreading mapping, which is a generalization of [8, Theorem 3.1].

Corollary 4.3. *(Osilike and Isiogugu [9, Theorem 3.1]) Let H, C, $\{\alpha_n\}$, and P be the same as in Corollary 4.2. Let κ and β be real numbers with $0 \le \kappa \le \beta < 1$ and $T : C \longrightarrow C$ a κ-strictly pseudononspreading mapping with a fixed point. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\big(\beta x_n + (1-\beta)Tx_n\big), \text{ and } z_n = \frac{1}{n}\sum_{k=1}^{n} x_k$$

for $n \in \mathbb{N}$. Then $\{z_n\}$ converges weakly to the strong limit of $\{Px_n\}$.

Proof. Set $T_\beta = \beta I + (1-\beta)T$. Then it follows from Lemma 2.5 that $-\beta/(1-\beta)$-hybrid. Obviously, $F(T) = F(T_\beta)$. Thus Corollary 4.2 implies the conclusion. □

Assuming that $\alpha_n = 0$ for each $n \in \mathbb{N}$ in Corollary 4.2, we obtain the following:

Corollary 4.4. *([2, Theorem 5.2]) Let H, C, λ, T, and P be the same as in Corollary 4.2. Let x be a point in C and $\{z_n\}$ a sequence in C defined by*

$$z_n = \frac{1}{n}\sum_{k=1}^{n} T^{k-1}x$$

for $n \in \mathbb{N}$, where T^0 is the identity mapping on C. Then $\{z_n\}$ converges weakly to the strong limit of $\{PT^n x\}$.

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