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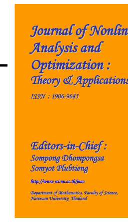

CONVERGENCE THEOREM IN CAT(O) SPACE

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ABSTRACT. Some new iterative process for multivalued mappings in CAT(0) spaces are introduced. Strong and $\triangle$-convergence theorems for such iterative process are established.

KEYWORDS : Fixed point; Strong convergence; $\triangle$-convergence; Quasi nonexpansive multivalued mapping; Condition (E).

1. INTRODUCTION

Fixed point theory in CAT(0) spaces was first studied by W. A. Kirk (see [11, 12].) He showed that every nonexpansive single valued mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single valued and multivalued mappings in CAT(0) spaces has been developed and a number of papers have appeared (see for example, [1, 3, 6, 15]). In [6], Dhompongsa and Panyanak obtained $\triangle$-convergence theorems for the Mann and Ishikawa iterations for nonexpansive single valued mappings in CAT(0) spaces. Very recently, Khan and Abbas [10] introduced a new iterative process for nonexpansive single valued mappings and proved convergence theorems for such iterative process in CAT(0) spaces. On the other hand some authors introduced and studied Mann and Ishikawa iteration for multivalued mappings in Hilbert spaces as well as in Banach spaces (see [16, 17, 18, 19].) The purpose of this paper is to introduce some iterative process for quasi nonexpansive multivalued mappings and prove $\triangle$-convergence and strong convergence theorems for such iterative process in CAT(0) spaces.

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2. PRELIMINARIES

Let (X,d) be a metric space. A geodesic path joining $x \in X$ and $y \in X$ is a map c from a closed interval $[0,r] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(r) = y$ and $d(c(t),c(s)) = |t-s|$ for all $s,t \in [0,r]$. In particular, the mapping c is an isometry and $d(x,y) = r$. The image of c is called a geodesic segment joining x and y which when unique is denoted by $[x,y]$. For any $x,y \in X$, we denote the point $z \in [x,y]$ such that $d(x,z) = \alpha d(x,y)$ by $z = (1-\alpha)x \oplus \alpha y$, where $0 \leq \alpha \leq 1$. The space (X,d) is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y. A subset D of X is called convex if D includes every geodesic segment joining any two points of D.

A geodesic triangle $\triangle(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of points (the edges of $\triangle$). A comparison triangle for $\triangle(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1,x_2,x_3) := \triangle(\overline{x_1},\overline{x_2},\overline{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i},\overline{x_j}) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$.

A geodesic metric space X is called a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom:

Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be its comparison triangle in $\mathbb{R}^2$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x,y \in \triangle$ and all comparison points $\overline{x},\overline{y} \in \overline{\triangle}$, $d(x,y) \leq d_{R^2}(\overline{x},\overline{y})$.

The following properties of a CAT(0) space are useful (see [2]):

- (i) A CAT(0) space X is uniquely geodesic;
- (ii) For any $x \in X$ and any closed convex subset $D \subset X$ there is a unique closest point to $x \in D$.

A notion of $\triangle$-convergence in CAT(0) spaces based on the fact that in Hilbert spaces a bounded sequence is weakly convergent to its unique asymptotic center has been studied in [13]. Let $\{x_n\}$ be a bounded sequence in X and D be a nonempty bounded subset of X. We associate this sequence with the number

$$r = r(D,\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in D\},$$

where

$$r(x,\{x_n\}) = \limsup_{n \longrightarrow \infty} d(x_n,x),$$

and the set

$$A = A(D,\{x_n\}) = \{x \in D : r(x,\{x_n\}) = r\}.$$

The number r is known as the *asymptotic radius* of $\{x_n\}$ relative to D. Similarly, the set A is called the *asymptotic center* of $\{x_n\}$ relative to D.

In a CAT(0) space, the asymptotic center $A = A(D,\{x_n\})$ of (x_n) consists of exactly one point whenever D is closed and convex. A sequence $\{x_n\}$ in a CAT(0) space X is said to be $\triangle$-convergent to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. Notice that given $\{x_n\} \subset X$ such that $\{x_n\}$ is $\triangle$-convergent to x and given $y \in X$ with $x \neq y$,

$$\lim sup_{n \longrightarrow \infty} d(x,x_n) < \lim sup_{n \longrightarrow \infty} d(y,x_n).$$

Thus every CAT(0) space X satisfies the Opial property.

Lemma 2.1. *([13]) Every bounded sequence in a complete CAT(0) space has a $\triangle$-convergent subsequence.*

Lemma 2.2. *([4]) If D is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in D, then the asymptotic center of $\{x_n\}$ is in D.*

Lemma 2.3. *([6]) If $\{x_n\}$ is a bounded sequence in complete CAT(0) space X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.4. *([6]) Let (X,d) be a CAT(0) space. For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = td(x, y) \qquad \text{and} \qquad d(y, z) = (1 - t)d(x, y).$$

We use the notation $(1 - t)x \oplus ty$ for the unique point z.

Let (X, d) be a geodesic metric space. We denote by $CB(X)$ the collection of all nonempty closed bounded subsets of X, we also write $K(X)$ to denote the collection of all nonempty compact subsets of X. Let H be the Hausdorff metric with respect to d, that is,

$$H(A, B) := \max\{\sup_{x\in A} dist(x, B), \sup_{y\in B} dist(y, A)\},$$

for all $A, B \in CB(X)$ where $dist(x, B) = \inf_{y\in B} d(x, y)$.
Let $T : X \longrightarrow 2^X$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$. The set of fixed points of T is denoted by $F(T)$.

Definition 2.5. A multivalued mapping $T : X \longrightarrow CB(X)$ is called

(i) nonexpansive if

$$H(Tx, Ty) \leq d(x, y), \quad x, y \in X.$$

(ii) quasi nonexpansive if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq d(x, p)$ for all $x \in X$ and all $p \in F(T)$.

In [7], Garcia-Falset et al. introduced condition (E) for single valued mappings. The current authors in [1] stated this condition for multivalued mappings as follows:

Definition 2.6. A multivalued mapping $T : X \longrightarrow CB(X)$ is said to satisfy condition (E_μ) provided that

$$dist(x, Ty) \leq \mu\, dist(x, Tx) + d(x, y), \quad x, y \in X.$$

We say that T satisfies condition (E) whenever T satisfies (E_μ) for some $\mu \geq 1$.

Lemma 2.7. *Let $T : X \longrightarrow CB(X)$ be a multivalued nonexpansive mapping, then T satisfies the condition (E_1).*

The following lemmas can be found in [6].

Lemma 2.8. *Let X be a CAT(0) space. Then for all $x, y, z \in X$ and all $t \in [0, 1]$ we have*

(i) $d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z)$,
(ii) $d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2$.

3. MAIN RESULTS

In this section we use the following iteration process.

(A): Let X be a CAT(0) space, D be a nonempty convex subset of X and $T : D \longrightarrow CB(D)$ be a given mapping. Then, for $x_1 \in D$, and $a_n, b_n \in [0,1]$, we consider the following iterative process:

$$y_n = (1-b_n)x_n \oplus b_n z_n, \quad n \geq 1,$$
$$x_{n+1} = (1-a_n)u_n \oplus a_n w_n, \quad n \geq 1,$$

where $z_n, u_n \in Tx_n$ and $w_n \in Ty_n$.

Theorem 3.1. *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $T : D \longrightarrow CB(D)$ be a quasi nonexpansive multivalued mapping such that $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in F(T)$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_n, b_n \in [a,b] \subset (0,1)$. Assume that*

$$\lim_{n \longrightarrow \infty} dist(x_n, T(x_n)) = 0 \implies \liminf_{n \longrightarrow \infty} dist(x_n, F(T)) = 0.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Let $p \in F(T)$. Then, using (A) and quasi nonexpansiveness of T we have

$$\begin{aligned} d(y_n, p) &= d((1-b_n)x_n \oplus b_n z_n, p) \\ &\leq (1-b_n)d(x_n,p) + b_n d(z_n,p) \\ &= (1-b_n)d(x_n,p) + b_n dist(z_n, Tp) \\ &\leq (1-b_n)d(x_n,p) + b_n H(Tx_n, Tp) \\ &\leq (1-b_n)d(x_n,p) + b_n d(x_n,p) = d(x_n,p). \end{aligned}$$

We also have

$$\begin{aligned} d(x_{n+1}, p) &= d((1-a_n)u_n \oplus a_n w_n, p) \\ &\leq (1-a_n)d(u_n,p) + a_n d(w_n,p) \\ &= (1-a_n)dist(u_n, Tp) + a_n dist(w_n, Tp) \\ &\leq (1-a_n)H(Tx_n, Tp) + a_n H(Ty_n, Tp) \\ &\leq (1-a_n)d(x_n,p) + a_n d(y_n,p) \leq d(x_n,p). \end{aligned}$$

Thus, the sequence $\{d(x_n,p)\}$ is decreasing and bounded below. It now follows that $\lim_{n \longrightarrow \infty} d(x_n,p)$ exists for any $p \in F(T)$. From Lemma 2.8, we get

$$\begin{aligned} d(y_n,p)^2 &= d((1-b_n)x_n \oplus b_n z_n, p)^2 \\ &\leq (1-b_n)d(x_n,p)^2 + b_n d(z_n,p)^2 - b_n(1-b_n)d(x_n,z_n)^2 \\ &= (1-b_n)d(x_n,p)^2 + b_n dist(z_n,Tp)^2 - b_n(1-b_n)d(x_n,z_n)^2 \\ &\leq (1-b_n)d(x_n,p)^2 + b_n H(Tx_n,Tp)^2 - b_n(1-b_n)d(x_n,z_n)^2 \\ &\leq (1-b_n)d(x_n,p)^2 + b_n d(x_n,p)^2 - b_n(1-b_n)d(x_n,z_n)^2 \\ &= d(x_n,p)^2 - b_n(1-b_n)d(x_n,z_n)^2. \end{aligned}$$

By another application of Lemma 2.8 we obtain

$$\begin{aligned} d(x_{n+1},p)^2 &= d((1-a_n)u_n \oplus a_n w_n, p)^2 \\ &\leq (1-a_n)d(u_n,p)^2 + a_n d(w_n,p)^2 - a_n(1-a_n)d(u_n,w_n)^2 \\ &\leq (1-a_n)dist(u_n,Tp)^2 + a_n dist(w_n,Tp)^2 \\ &\leq (1-a_n)H(Tx_n,Tp)^2 + a_n H(Ty_n,Tp)^2 \\ &\leq (1-a_n)d(x_n,p)^2 + a_n d(y_n,p)^2 \end{aligned}$$

$$\leq \quad (1-a_n)d(x_n,p)^2 + a_n d(x_n,p)^2 - a_n b_n(1-b_n)d(x_n,z_n)^2,$$

so that

$$a^2(1-b)d(x_n,z_n)^2 \leq a_n b_n(1-b_n)d(x_n,z_n) \leq d(x_n,p)^2 - d(x_{n+1},p)^2.$$

This implies that

$$\sum_{n=1}^{\infty} a^2(1-b)d(x_n,z_n)^2 \leq d(x_1,p)^2 < \infty,$$

and hence $\lim_{n\longrightarrow\infty} d(x_n,z_n) = 0$. Thus $dist(x_n,Tx_n) \leq d(x_n,z_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Note that by our assumption $\lim_{n\longrightarrow\infty} dist(x_n,F(T)) = 0$. Therefore, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence p_k in $F(T)$ such that for all $k \in \mathbb{N}$

$$d(x_{n_k},p_k) < \frac{1}{2^k}.$$

Since the sequence $\{d(x_n,p)\}$ is decreasing, we get

$$d(x_{n_{k+1}},p_k) \leq d(x_{n_k},p_k) < \frac{1}{2^k}.$$

Hence

$$d(p_{k+1},p_k) \leq d(x_{n_{k+1}},p_{k+1}) + d(x_{n_{k+1}},p_k) < \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}.$$

Consequently, we conclude that $\{p_k\}$ is a Cauchy sequence in D and hence converges to $q \in D$. Since

$$dist(p_k,T(q)) \leq H(T(p_k),T(q)) \leq d(p_k,q)$$

and $p_k \longrightarrow q$ as $k \longrightarrow \infty$, it follows that $dist(q,T(q)) = 0$ and hence $q \in F(T)$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\longrightarrow\infty} d(x_n,q)$ exists, it follows that $\{x_n\}$ converges strongly to q. □

Theorem 3.2. *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Suppose $T : D \longrightarrow K(D)$ satisfies the condition (E). If $\{x_n\}$ is a sequence in D such that $\lim_{n\longrightarrow\infty} dist(Tx_n,x_n) = 0$ and $\triangle-\lim_n x_n = v$. Then $v \in D$ and $v \in Tv$.*

Proof. Let $\triangle - \lim_n x_n = v$. We note that by Lemma 2.2, $v \in D$. For each $n \geq 1$, we choose $z_n \in Tv$ such that$d(x_n,z_n) = dist(x_n,Tv)$.

By the compactness of Tv there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\lim_{k\longrightarrow\infty} z_{n_k} = w \in Tv$. Since T satisfies the condition (E) we have

$$dist(x_{n_k},Tv) \leq \mu\, dist(x_{n_k},T(x_{n_k})) + d(x_{n_k},v).$$

for some $\mu \geq 1$. Note that

$$d(x_{n_k},w) \leq d(x_{n_k},z_{n_k}) + d(z_{n_k},w) \leq \mu\, dist(x_{n_k},T(x_{n_k})) + d(x_{n_k},v) + d(z_{n_k},w).$$

Thus

$$\limsup_{k\longrightarrow\infty} d(x_{n_k},w) \leq \limsup_{k\longrightarrow\infty} d(x_{n_k},v).$$

From the Opial property of CAT(0) space X, we have $v = w \in Tv$. □

Now, we are ready to prove a $\triangle$-convergence theorem.

Theorem 3.3. *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $T : D \longrightarrow K(D)$ be a quasi nonexpansive multivalued mapping satisfying condition (E) and such that $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in F(T)$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_n, b_n \in [a,b] \subset (0,1)$. Then $\{x_n\}$ is $\triangle$-convergent to a fixed point of T.*

Proof. As in the proof of Theorem 3.1 we have $\lim_{n\longrightarrow\infty} dist(Tx_n, x_n) = 0$. Now we let $W_w(x_n) := \cup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $W_w(x_n) \subset F(T)$. Let $u \in W_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.1 and 2.2 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle - \lim_n v_n = v \in D$. Since $\lim_{n\longrightarrow\infty} dist(Tv_n, v_n) = 0$, by Theorem 3.2 we have $v \in F(T)$, and $\lim_{n\longrightarrow\infty} d(x_n, v)$ exists. Hence $u = v \in F(T)$ by Lemma 2.3. This shows that $W_w(x_n) \subset F(T)$. Next we show that $W_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in W_w(x_n) \subset F(T)$ and $d(x_n, v)$ converges, by Lemma 2.3 we have $x = u$. □

By using Theorem 3.3 along with Lemma 2.7 we obtain the following corollary.

Corollary 3.1. *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $T : D \longrightarrow K(D)$ be a multivalued nonexpansive mapping such that $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in F(T)$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_n, b_n \in [a, b] \subset (0, 1)$. Then $\{x_n\}$ is $\triangle$-convergent to a fixed point of T.*

Theorem 3.4. *Let D be a nonempty compact convex subset of a complete CAT(0) space X. Let $T : D \longrightarrow CB(D)$ be a quasi nonexpansive multivalued mapping satisfying condition (E) and such that $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in F(T)$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_n, b_n \in [a, b] \subset (0, 1)$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. As in the proof of Theorem 3.1 we have $\lim_{n\longrightarrow\infty} dist(Tx_n, x_n) = 0$. Since D is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim x_{n_k} = w$ for some $w \in D$. Since T satisfies the condition (E), for some $\mu \geq 1$ we have

$$\begin{aligned} dist(w, Tw) &\leq d(w, x_{n_k}) + dist(x_{n_k}, Tw) \\ &\leq \mu\, dist(x_{n_k}, T(x_{n_k})) + 2d(w, x_{n_k}) \longrightarrow 0 \qquad as \quad k \longrightarrow \infty. \end{aligned}$$

This implies that $w \in F(T)$. Since $\{x_{n_k}\}$ converges strongly to a point w and $\lim_{n\longrightarrow\infty} d(x_n, w)$ exists (as the proof of Theorem 3.1 shows), it follows that $\{x_n\}$ converges strongly to w. □

We now define the following iteration process.

(B): Let $T : D \longrightarrow P(D)$ be a given mapping and

$$P_T(x) = \{y \in Tx :\| x - y \|= dist(x, Tx)\}.$$

For fixed $x_1 \in D$, and $a_n, b_n \in [0, 1]$, we consider the iterative process defined by:

$$y_n = (1 - b_n)x_n \oplus b_n z_n, \qquad n \geq 1,$$

$$x_{n+1} = (1 - a_n)u_n \oplus a_n w_n, \qquad n \geq 1,$$

where $z_n, u_n \in P_T x_n$ and $w_n \in P_T y_n$.

Theorem 3.5. *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $T : D \longrightarrow CB(D)$ be a multivalued mapping with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the iterative process defined by (B), and $a_n, b_n \in [a, b] \subset (0, 1)$. Assume that*

$$\lim_{n\longrightarrow\infty} dist(x_n, T(x_n)) = 0 \implies \liminf_{n\longrightarrow\infty} dist(x_n, F(T)) = 0.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Let $p \in F(T)$. Then $p \in P_T p = \{p\}$. Hence using (B) and Lemma 2.8 we have

$$\begin{aligned} d(y_n, p) &= d((1-b_n)x_n \oplus b_n z_n, p) \\ &\leq (1-b_n)d(x_n,p) + b_n d(z_n,p) \\ &= (1-b_n)d(x_n,p) + b_n dist(z_n, P_T p) \\ &\leq (1-b_n)d(x_n,p) + b_n H(P_T x_n, P_T p) \\ &\leq (1-b_n)d(x_n,p) + b_n d(x_n,p) = d(x_n,p), \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d((1-a_n)u_n \oplus a_n w_n, p) \\ &\leq (1-a_n)d(u_n,p) + a_n d(w_n,p) \\ &= (1-a_n)dist(u_n, P_T p) + a_n dist(w_n, P_T p) \\ &\leq (1-a_n)H(P_T x_n, P_T p) + a_n H(P_T y_n, P_T p) \\ &\leq (1-a_n)d(x_n,p) + a_n d(y_n,p) \leq d(x_n,p). \end{aligned}$$

Hence, the sequence $\{d(x_n,p)\}$ is decreasing and bounded below. It now follows that $\lim_{n\longrightarrow\infty} d(x_n,p)$ exists for any $p \in F(T)$. From Lemma 2.8, we get

$$\begin{aligned} d(y_n,p)^2 &= d((1-b_n)x_n \oplus b_n z_n, p)^2 \\ &\leq (1-b_n)d(x_n,p)^2 + b_n d(z_n,p)^2 - b_n(1-b_n)d(x_n,z_n)^2 \\ &= (1-b_n)d(x_n,p)^2 + b_n dist(z_n, P_T p)^2 - b_n(1-b_n)d(x_n,z_n)^2 \\ &\leq (1-b_n)d(x_n,p)^2 + b_n H(P_T x_n, P_T p)^2 - b_n(1-b_n)d(x_n,z_n)^2 \\ &\leq (1-b_n)d(x_n,p)^2 + b_n d(x_n,p)^2 - b_n(1-b_n)d(x_n,z_n)^2 \\ &= d(x_n,p)^2 - b_n(1-b_n)d(x_n,z_n)^2. \end{aligned}$$

By applying Lemma 2.8 we infer that

$$\begin{aligned} d(x_{n+1},p)^2 &= d((1-a_n)u_n \oplus a_n w_n, p)^2 \\ &\leq (1-a_n)d(u_n,p)^2 + a_n d(w_n,p)^2 - a_n(1-a_n)d(u_n,w_n)^2 \\ &\leq (1-a_n)dist(u_n, P_T p)^2 + a_n dist(w_n, P_T p)^2 \\ &\leq (1-a_n)H(P_T x_n, P_T p)^2 + a_n H(P_T y_n, P_T p)^2 \\ &\leq (1-a_n)d(x_n,p)^2 + a_n d(y_n,p)^2 \\ &\leq (1-a_n)d(x_n,p)^2 + a_n d(x_n,p)^2 - a_n b_n(1-b_n)d(x_n,z_n)^2. \end{aligned}$$

As in the proof of Theorem 3.1, $\lim_{n\longrightarrow\infty} dist(x_n, F(T)) = 0$. Therefore, we can choose a subsequence $\{x_{n_k}\}$ and a sequence p_k in $F(T)$ such that for all $k \in \mathbb{N}$

$$d(x_{n_k}, p_k) < \frac{1}{2^k}.$$

Again, $\{p_k\}$ is a Cauchy sequence in D and hence converges to some $q \in D$. Since

$$dist(p_k, Tq) \leq dist(p_k, P_T q) \leq H(P_T p_k, P_T q) \leq d(p_k, q)$$

and $p_k \longrightarrow q$ as $k \longrightarrow \infty$, it follows that $dist(q, Tq) = 0$ and hence $q \in F(T)$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\longrightarrow\infty} d(x_n, q)$ exists, we conclude that $\{x_n\}$ converges strongly to q.

□

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