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EXISTENCE AND UNIQUENESS OF NASH EQUILIBRIUM IN A SUPPLY CHAIN NEWSVENDOR GAME

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ABSTRACT. This paper characterizes properties of Nash equilibriums for a supply chain newsvendor game with resource constraints. The newsvendor game has been modelled as a Stackelberg game in $\mathbb{R}^N$ and also been reformulated as a fixed-point problem for proving the existence and uniqueness.

KEYWORDS : Fixed-point; Contraction mapping; Lipschitz condition; Lagrangian; Kuhn-Tucker condition; Lagrange-Burmann expansion.

1. INTRODUCTION

The classic newsvendor (or newsboy or single-period) model is a mathematical model in operations management and applied economics. It used to determine optimal inventory levels that reap the most profits under selling price p and uncertain demand D with cumulative distribution function F_D. Please refer to the review to this topic over the past 40 years [8]. Determining the fulfillment amount is elaborate, as order too much will lead to waste and order too little will results to business lost. Since the inventory at hand is fixed, every demand above this quantity will be lost, that is, the expected sales $S(q) = E\min(q, D)$. A standard newsvendor problem solves the profit maximization problem

$$\max_q \pi = \max_q E[p\min(q, D) - cq]$$

where $c < p$ denotes the procurement cost for the good and E is the expectation operator with respect to the random variable D. Rewrite the function $\min(a, b) =$

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$b-(b-a)^+$ for $(a)^+ = \max(a,0)$. The celebrated form of solution for the optimal quantity to the classic newsvendor problem read as

$$q^* = \arg\max_q \pi = F_D^{-1}(\frac{p-c}{p}) \tag{1.1}$$

where F_D^{-1} denotes the inverse function of the well-defined cumulative distribution function F of D. The ratio $\frac{p-c}{p}$ sometimes is referred to a critical ratio. This paper considers a newsvendor selling N products in a supply chain while procuring them from a supplier, the problem is often modelled as a non-cooperative game. The expected profit functions for the retailer can be written as

$$\pi_r(q,w) = \sum_{i=1}^{N} p_i E\min(q_i, D_i) - w_i q_i, \tag{1.2}$$

while the one for the supplier is

$$\pi_s(q,w) = \sum_{i=1}^{N} (w_i - c_i) q_i. \tag{1.3}$$

The retailer determine an optimal order quantity $q = (q_1, \ldots, q_N)$ within a strategic space $Q \subset \mathbb{R}^N$ while the supplier determine an optimal wholesale price $w = (w_1, \ldots, w_N)$ within a strategic space $W \subset \mathbb{R}^N$ for maximizing their own profits. c represents the unit production cost. Through the rest of the paper, we use bold face letter for vector representation, e.g., $c = (c_1, \ldots, c_N)$. The non-cooperative game between two vertical players in a supply chain is

$$\begin{bmatrix} \max_{q\in Q} \pi_r(q,w) \\ \max_{w\in W} \pi_s(q,w) \end{bmatrix} \tag{1.4}$$

In certain situations, the order quantities and wholesale prices need to be confined to particular conditions. For example the constraint conditions can be generalized to, for example, a minimum service level $q > R_i$ [3], or resource limitation $\sum_{i\in\mathbb{C}_j} q_i < C_j$ [1, 6] for multiple product newsvendor problems. In this paper, we consider the resource constraint,

$$\sum_{i=1}^{N} q_i \leq \bar{C}. \tag{1.5}$$

2. PRELIMINARIES

For ease of derivation, the expected sales $S(q_i) = E[\min(q_i, D_i)]$ rewrite to $E[D_i - (D_i - q_i)^+]$. By using integral by part, we can further write

$$\begin{aligned} S(q_i) &= \int_0^\infty x f_i(x)dx - \int_{q_i}^\infty (x-q_i) f_i(x)dx \\ &= q_i F_i(q_i) - \int_0^{q_i} F_i(x)dx + q_i(1 - F_i(q_i)) \\ &= q_i - \int_0^{q_i} F_i(x)dx \end{aligned}$$

The problem (1.4) with constraint (1.5) can be solved by the Lagrangian,

$$L_r(q,w,\lambda) = \pi_r(q,w) + \lambda\left(\sum_{i=1}^{N} q_i - \bar{q}\right). \tag{2.1}$$

The best response function for the retailer has been solved by the Karush-Kuhn-Tucker condition, that is,

$$q_i(w_i, \lambda) = F_i^{-1}\left(\frac{p_i - w_i - \lambda}{p_i}\right) \tag{2.2}$$

$$\sum_{i=1}^{N} q_i - \bar{C} \leq 0 \tag{2.3}$$

$$\lambda\left(\sum_{i=1}^{N} q_i - \bar{C}\right) = 0 \tag{2.4}$$

Since the supplier possesses the freedom to pricing its goods, the supply chain game is deemed a Stackelberg game with supplier as the leader and the retailer as the follower. Substituting the best follower response function q, the supplier profit (1.3) becomes

$$\pi_s(q, w, \lambda) = \sum_{i=1}^{N}(w_i - c_i)F_i^{-1}\left(\frac{p_i - w_i - \lambda}{p_i}\right), \tag{2.5}$$

and the corresponding first order condition

$$\frac{\partial \pi_s}{\partial w_i} = F_i^{-1}\left(\frac{p_i - w_i - \lambda}{p_i}\right) - (w_i - c_i)\nabla F_i^{-1}\left(\frac{p_i - w_i - \lambda}{p_i}\right) = 0 \tag{2.6}$$

Define $w_i' = \frac{p_i - w_i - \lambda}{p_i}$ and $g_i(w_i') = (p_i - c_i - \lambda - p_i w_i')\nabla F_i^{-1}(w_i')$. Equ. (2.6) becomes a fixed point problem

$$w_i' = F(g_i(w_i')) = F_i \circ g_i(w_i'), \tag{2.7}$$

where the operator $\circ$ represents the function composition.

3. MAIN RESULTS

Before proving the nature of equilibriums, we need some definitions. A function $f : X \times Y \to \mathbb{R}$ has *increasing difference* in $(x, y) \in X \times Y \subset \mathbb{R} \times \mathbb{R}$ if for all $x' \geq x$ and $y' \geq y$, f satisfies

$$f(x', y') - f(x, y') \geq f(x', y) - f(x, y).$$

The definition of increasing difference implies that the incremental gain or payoff to choosing a higher x is greater when y is larger. A game is a *supermodular game* if the strategy spaces are compact and the best response function $u(s_i, s_{-i})$ is continuous and has increasing difference in (s_i, s_{-i}).

Lemma 3.1. [11] *Given a twice differentiable function f, it has increasing difference in (x, y) if and only if* $\frac{\partial^2 f(x,y)}{\partial x \partial y} \geq 0$.

Lemma 3.2. [11] *A supermodular game admits at least one pure strategy Nash equilibrium.*

The existence of Nash equilibriums have been proved in many reports by either the convexity or supermodularity of payoff functions[7, 4, 9, 10].

It is also essential to apply fixed point theorems for proving the existence of Nash equilibriums, for example, Brouwer, Kakutani and Tarski's fixed point theorems [2].

Theorem 3.1. *The game (1.4) admits at least one pure strategy Nash equilibrium, if F is non-decreasing.*

Proof. Consider the game (1.4) with the fixed point problem (2.7). The composition mapping $F \circ g$ is continuous and it maps a convex and compact set W into a closed convex subset of W. By the Kakutani fixed point theorem, there exists a point $w^0 \in W$ such that $w^0 \in F \circ g(w)$. □

In order to prove uniqueness, we the following definitions. A mapping $f : W \to W$ is a contraction if there exists a Lipschitz constant $0 < k < 1$, such that

$$||f(x) - f(y)|| \leq k||x - y||, \forall x, y \in W.$$

This Lipschitz condition amounts to the condition $||f'(w)|| < 1$ for all $w \in W$. It is well-known that a contraction has a unique fixed point.

Define the coefficient of relative risk aversion $\epsilon(w)$ as $\frac{-\nabla^2 F(w)}{\nabla F(w)}$ and it is a measure of risk attitude which is relative to the strength of preference [5].

Theorem 3.2. *If the accumulative distribution function F for the uncertain demand is well-defined and*

$$\nabla^2 F_i < \frac{g_i}{1 + p_i g_i}, \forall w_i \in W, \tag{3.1}$$

the game (1.4) has a unique solution.

Proof. Taking derivative to the composition (2.7),

$$\begin{aligned}
\|\nabla(F_i \circ g_i)(w_i)\| &= ||(\nabla g_i \cdot (\nabla F_i) \circ g_i)(w_i)|| \\
&= ||((-p_i \nabla F_i^{-1} + g_i \nabla^2 F_i^{-1}) \cdot (\nabla F_i) \circ g_i)(w_i)|| \\
&\leq ||(-p_i \nabla F_i^{-1} + g_i \nabla^2 F_i^{-1})|| \cdot ||((\nabla F_i) \circ g_i)(w)||
\end{aligned}$$

By the Lagrange inversion theorem and Lagrange-Bürmann formula, the derivative of an inversion reads as

$$\nabla \left(F^{-1}\right)(w) = \frac{1}{\nabla F(F^{-1}(w))}.$$

Let $F^{-1}(w) = a$. Given $\nabla^2 F < 0$, therefore,

$$\begin{aligned}
&||\frac{-p_i}{\nabla F_i(a)} + \frac{g_i}{\nabla^2 F_i(a)}|| \leq 1, \text{and} \\
&||\nabla F_i||||g_i(w)|| \leq 1
\end{aligned}$$

It is necessary that

$$|| - p_i g_i + \frac{g_i}{\nabla^2 F_i(a)}|| \leq 1, \ \forall a \in W.$$

If the condition (3.1) holds, $\|\nabla(F_i \circ g_i)(w_i)\| < 1$. Therefore the game has a unique solution. □

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