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DOOB'S DECOMPOSITION OF FUZZY SUBMARTINGALES VIA ORDERED NEAR VECTOR SPACES[◇]

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ABSTRACT. We use ideas from measure-free martingale theory and Rådström's completion of a near vector space to derive a Doob decomposition of submartingales in ordered near vector spaces, which is a generalization of result noted by Daures, Ni and Zhang, and an analogue of the Doob decomposition of submartingales in the fuzzy setting, as noted by Shen and Wang.

1. INTRODUCTION

The aim of this paper is to complete the work of [11] and extend those results to the fuzzy setting. In [11] we focussed on the Doob's decomposition of submartingales and used the notion of near vector spaces to overcome the problems faced.

In this paper, we again concern ourselves with Doob's decomposition of submartingales. This decomposition was extended from the classical setting of real valued martingales to set-valued martingales by Daures, Ni and Zhang (see [12, 13]) and also by Shen and Wang (see [16]).

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When modelling events which involve inherent uncertainty due to incomplete information, one method is to use fuzzy sets. This leads to the concept of fuzzy martingales. In the fuzzy setting we immediately encounter the same type of problem the one faces in the set-valued setting. That is that neither the spaces of submartingales nor the range spaces of the submartingales are vector spaces. Since they are in fact near vector spaces we can apply the near vector ideas developed in [10, 14].

Our aim is to build on the work done in [10] and [11] and once again use ideas from measure-free martingale theory (see [1, 2, 3, 7, 15, 17]), together with Rådström's completion of a near vector space, to give an elementary proof for a version of Doob's decomposition of fuzzy submartingales.

After introducing the necessary preliminaries and notation, we consider Doob's decomposition of a submartingale in an ordered vector space. From this, and with the aid of Rådström's completion of a near vector space (see [10]), we obtain a Doob decomposition of a submartingale in an ordered near vector space. We then specialize the ordered near vector space to the appropriate fuzzy set-valued space of submartingales that are integrable. As special cases, we obtain the Daures, Ni and Zhang result by using the fact that martingales which are integrably bounded are integrable (see [12]). We also derive an analogue of the Doob decomposition of fuzzy submartingales, as noted by Shen and Wang.

2. PRELIMINARIES

Let (Ω, Σ, μ) be a finite measure space. If Σ_0 is a sub σ-algebra of Σ, denote by $L^0(\Omega, \Sigma_0, \mu)$ the set of Σ_0-measurable functions $f : \Omega \longrightarrow \mathbb{R}$. If $f \in L^1(\Omega, \Sigma_0, \mu)$ is a random variable, we denote by $\mathbb{E}[f|\Sigma_0]$ the conditional expectation of f with respect to Σ_0. If (Σ_i) an increasing sequence of sub σ-algebras of Σ, then (f_i, Σ_i) is a martingale (submartingale) provided that

$$f_i \in L^0(\Omega, \Sigma_i, \mu) \text{ and } f_i = \, (\leq) \, \mathbb{E}[f_{i+1}|\Sigma_i]$$

for all $i \in \mathbb{N}$. The following well-known result relates submartingales to martingales:

Theorem 2.1. (Doob's Decomposition) *If (Σ_i) an increasing sequence of sub σ-algebras of Σ, and (f_i, Σ_i) is a submartingale, then (f_i, Σ_i) has a unique decomposition*

$$f_i(\omega) = M_i(\omega) + A_i(\omega) \text{ a.e.}$$

where (M_i, Σ_i) is a set-valued martingale and (A_i) is a predictable (i.e., A_i is Σ_{i-1}-measurable for all $i \geq 2$), increasing sequence such that

(a) $A_1(\omega) = 0$ *a.e.,*
(b) $A_j(\omega) = \sum_{i=1}^{j-1} \Big(\mathbb{E}[f_{i+1}|\Sigma_i](\omega) - f_i(\omega)\Big)$ *a.e. for* $j \geq 2$,
(c) $M_j(\omega) = f_j(\omega) - A_j(\omega)$ *a.e. for all* $j \in \mathbb{N}$.

Daures, Ni and Zhang proved an analogue of Doob's decomposition for set-valued submartingales (see [3, 13]). Before we state our main result, as can be found in [12], we first recall some terminology from [6, 12].

Our main focus is on the application of near vector spaces to fuzzy submartingales so we present an overview of the important notions associated with fuzzy sets and fuzzy random variables. For a more comprehensive treatment of fuzzy sets and the application of fuzzy sets in functional analysis the reader is referred to [12].

Definition 2.1. Let X be a set and I the unit interval $[0,1]$. A *fuzzy set* on X (*fuzzy-subset of* X) is a map from X into I. That is, if A is a fuzzy subset of X then $A \in I^X$, where I^X denotes the collection of all maps from X into I.

I^X is naturally equipped with an order structure induced by I. If $A, B \in I^X$ then we say that A is a *fuzzy subset* of B if $A(x) \leq B(x)$ for all $x \in X$.
For a given fuzzy set we associate collections of crisp subsets of X with it.

If $A \in I^X$ and $\alpha \in I$ we define,

$$A^\alpha = \{x \in X : A(x) > \alpha\};$$

$$A_\alpha = \{x \in X : A(x) \geq \alpha\}.$$

These crisp sets are referred to as α-*levels* (or *cuts*), strong and weak respectively. We call A^0 the *support* of A and denote it by $\operatorname{supp}(A)$. We have the following useful relationship between fuzzy sets and their corresponding α-cuts.

Lemma 2.2. *Let A and B be fuzzy sets on a set X. Then for all $\alpha \in (0,1]$:*

(i) $A = B \iff A_\alpha = B_\alpha$ *for all* $\alpha \in (0,1]$*, and*
(ii) $A \leq B \iff A_\alpha \subseteq B_\alpha$.

The following theorems enable us to decompose a given fuzzy set into a supremum of a collection of crisp sets.

Theorem 2.2. *For a $A \in I^X$ and $x \in X$ we have*

$$A(x) = \sup_{\alpha \in (0,1]} \{\alpha \chi_{A_\alpha}(x)\}.$$

If $A \in \mathcal{P}(X)$ and $\alpha \in I$ we define

$$\alpha\chi_A(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

So

$$\alpha\chi_{\{x\}} = \begin{cases} \alpha & \text{on } x \\ 0 & \text{elsewhere} \end{cases}$$

We call $\alpha\chi_{\{x\}}$ a *fuzzy point* with support at x and *value* α. We will denote the set of fuzzy points in I^X by $\tilde{X}$. A fuzzy point is clearly a generalization of a point in ordinary set theory.
If A, B are crisp subsets of a vector space X and t a scalar we have
$t \cdot A = \{ta : a \in A\}$ and $A + B = \{a + b : a \in A, b \in B\}$. We define addition and scalar multiplication of fuzzy sets in the natural way which is a direct consequence of the image of a fuzzy mapping.

Definition 2.3. Let X be a vector space. For $A, B \in I^X, t \in K$ and $x \in X$

(a) [Addition] $(A + B)(x) = \sup_{x_1 + x_2 = x}\{A(x_1) \wedge A(x_2)\}$.
(b) [Scalar multiplication] $t \cdot A(x) = A(\frac{x}{t})$ for $t \neq 0$. If $t = 0$:

$$t \cdot A(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \sup A & \text{if } x = 0. \end{cases}$$

Denote the power set of X by $\mathcal{P}(X)$. It is well-known that the set $\mathcal{P}_0(X) = \mathcal{P}(X)\backslash\emptyset$ does not, in general, form a vector space with respect to the above defined operations.

A crisp subset A of a vector space X is said to be convex of for any $a, b \in A$ and any $k \in [0,1]$ we have that $ka + (1-k)b \in A$. We have an analagous notion in the fuzzy setting.

Definition 2.4. Let A be a fuzzy subset of a vector space X. Then A is *(fuzzy)convex* if $A(kx + (1-k)y) \geq A(x) \wedge A(y)$ whenever $x, y \in X$ and $0 \leq k \leq 1$.

It is once again a simple matter to confirm that if a fuzzy set A is convex then for each $\alpha \in [0,1)$, A_α is convex in the classical sense.
In [11] we considered the certain collections of sets.

$$\mathrm{f}(X) := \{A \in \mathcal{P}_0(X) : A \text{ is closed}\}.$$

For all $A, C \in \mathrm{f}(X)$, define

$$A \oplus C = \overline{A + C},$$

where the closure is taken with respect to the norm on X. Then $\mathrm{f}(X)$ is closed under $\oplus$.
In [10] we introduced the following notation:

$$\begin{aligned} \mathrm{cf}(X) : &= \{A \in \mathrm{f}(X) : A \text{ is convex}\}, \\ \mathrm{bf}(X) : &= \{A \in \mathrm{f}(X) : A \text{ is bounded}\}, \\ \mathrm{cbf}(X) : &= \{A \in \mathrm{bf}(X) : A \text{ is convex}\}. \end{aligned}$$

We define $\mathrm{F}(X)$ as the collection of fuzzy sets $A : X \to I$ such that

(a) A is uppersemicontinuous,
(b) supp(A) is compact,
(c) $\{x \in X : A(x) = 1\} \neq \emptyset$.

Lemma 2.5. *Let $A \subseteq \mathrm{F_c}(X)$ if and only if $A_\alpha \in \mathrm{cf}(X)$for all $\alpha \in I$.*

If $A \in \mathcal{P}_0(X)$ and $x \in X$, the *distance* between x and A is defined by

$$d(x, A) = \inf\{\|x - y\|_X : y \in A\}.$$

Define d_H for all $A, B \in \mathrm{cf}(X)$ by

$$d_H(A, B) = \sup_{a \in A} d(a, B) \vee \sup_{b \in B} d(b, A).$$

Then d_H is a metric on $\mathrm{cf}(X)$, which is called the *Hausdorff metric*, and $(\mathrm{cf}(X), d_H)$ is a complete metric space (cf. [12]). In the special case where $B = \{0\}$, let

$$\|A\|_H = d_H(A, \{0\});$$

in general $\|\cdot\|_H$ is not a norm. Furthermore, $\mathrm{cbf}(X)$ is a closed subspace of $\mathrm{bf}(X)$ (cf. [12]).
We generalize the Hausdorff metric to the fuzzy setting in the natural way by defining d_∞ as

$$d_\infty(F_1, F_2) = d_H(\text{supp}(F_1), \text{supp}(F_2)),$$

for fuzzy random variables F_1 and F_2.
In [11] we defined the following operation the $\ominus$ operation on crisp sets A and B in the following way: $A \ominus B := \{x \in X : x + B \subseteq A\}$. We can naturally extend this to fuzzy sets in the following way.

Definition 2.6. Let $A, B \in I^X$. We define $A \ominus B$ as

$$A \ominus B = \sup_{\alpha \in (0,1]} \{\alpha \chi_{A_\alpha \ominus B_\alpha}(x)\}.$$

The following definition provides us with the fundamental notions of random variables in the fuzzy setting.

Definition 2.7. (a) A *fuzzy set-valued random variable* (f.r.v.) or a *fuzzy random set* is a function $F : \Omega \to \mathrm{F}(X)$ such that $F_\alpha(\omega) = \{x \in X : F(\omega)(x) \geq \alpha\}$ is a set-valued random variable for all $\alpha \in (0, 1]$. We denote by $\mathbf{M}[\Sigma, \mathrm{F}(X)]$ the collection of all Σ-measurable fuzzy random variables. We denote by $\mathbf{M}[\Sigma, \mathrm{F_c}(X)]$ the collection of measurable and integrable functions $f : \Sigma \to \mathrm{F_c}(X)$ respectively.

(b) The *expectation* of a fuzzy random variable F, denoted $E(F)$, is defined by

$$E(F) = \int_\Omega \text{supp}(F) d\mu.$$

(c) Let Σ_0 be a sub σ-algebra of Σ. Then the *conditional expectation* of F relative to Σ_0 is defined as $\mathcal{E}[F|\Sigma_0] = \mathcal{E}[\text{supp}(F)|\Sigma_0]$.

(d) A *selection* of $F \in \mathbf{M}[\Sigma, \mathrm{F}(X)]$ is a function $f \in L^1(\mu, \tilde{\mathbb{R}})$ such that $f(\omega) \leq F(\omega)$ for all $\omega \in \Omega$ a.e. We denote the set of selections of F by S^1_F and we say that F is *integrable* if $S^1_F \neq \emptyset$. We denote by $\mathcal{L}[\Sigma, \mathrm{F}(X)]$ the collection $\{F \in \mathbf{M}[\Sigma, \mathrm{F}(X)] : S^1_F \neq \emptyset\}$ and $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ denotes the set $\{F \in \mathcal{L}[\Sigma, \mathrm{F}(X)] : F(\omega) \in \mathrm{F_c}(X), \forall \omega \in \Omega\}$.

Hiai and Umegaki proved in [6] that, if $F \in \mathcal{L}[\Sigma, \mathrm{f}(X)]$, then there exists a unique $G \in \mathbf{M}[\Sigma_0, \mathrm{f}(X)]$ such that

$$S^1_G(\Sigma_0) = \overline{\{\mathbb{E}[f|\Sigma_0] : f \in S^1_F\}},$$

where the closure is taken in $L^1(\Sigma, \mu, X)$, and $\mathbb{E}[f|\Sigma_0]$ denotes the conditional expectation of $f \colon \Omega \longrightarrow X$ with respect to Σ_0. As is customary, we denote G by $\mathcal{E}[F|\Sigma_0]$ and call $\mathcal{E}[F|\Sigma_0]$ the *conditional expectation* of $F \colon \Omega \longrightarrow \mathrm{f}(X)$ relative to Σ_0 (cf. [6, 12]). It is well-known that the following properties hold:

Theorem 2.3. (see [6, 12]) *Let Σ_0 be a sub σ-algebra of Σ. If $F \in \mathcal{L}[\Sigma, \mathrm{F_c}(X)]$, then the conditional expectation $\mathcal{E}[F|\Sigma_0] \in \mathcal{L}[\Sigma_0, \mathrm{F_c}(X)]$ of F with respect to Σ_0 has the following properties:*

(E1) *If $F_1, F_2 \in \mathcal{L}[\Sigma, \mathrm{F_c}(X)]$, then $\mathcal{E}[F_1 + F_2|\Sigma_0] = \mathcal{E}[F_1|\Sigma_0] \oplus \mathcal{E}[F_2|\Sigma_0]$.*
(E2) *If $F \in \mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ and $\lambda \in \mathbb{R}_+$, then $\mathcal{E}[\lambda F|\Sigma_0] = \lambda \mathcal{E}[F|\Sigma_0]$.*
(E3) *If $F_1, F_2 \in \mathcal{L}[\Sigma, \mathrm{F_c}(X)]$, then $F_1 \leq F_2$ implies $\mathcal{E}[F_1|\Sigma_0] \leq \mathcal{E}[F_2|\Sigma_0]$.*
(E4) *If $F \in \mathcal{L}[\Sigma_0, \mathrm{F_c}(X)]$, then $\mathcal{E}[F|\Sigma_0] = F$.*
(E5) *If $\Sigma_0 \subseteq \Sigma_2 \subseteq \Sigma$ and $F \in \mathcal{L}[\Sigma_0, \mathrm{F_c}(X)]$, then $\mathcal{E}\left[\,\mathcal{E}[F|\Sigma_2]\,|\,\Sigma_0\right] = \mathcal{E}[F|\Sigma_0]$.*

If $F \in \mathbf{M}[\Sigma, \mathrm{F}(X)]$, then F is called *integrably bounded* provided that there exists $\rho \in L^1(\mu)$ such that $\|x\|_X \leq \rho(\omega)$ for all $x \in F(\omega)$ and for all $\omega \in \Omega$. In this case, $F(\omega) \in \mathrm{F_c}(X)$ a.e. and $\|F(\omega)\|_H = \sup\{\|x\|_X : x \in \text{supp}(F)(\omega)\} \leq \rho(\omega)$ for all $\omega \in \Omega$.
Let $\mathcal{L}^1[\Sigma, \mathrm{F}(X)]$ denote the set of all equivalence classes of a.e. equal $F \in \mathbf{M}[\Sigma, \mathrm{F}(X)]$ which are integrably bounded. If $\Delta \colon \mathcal{L}^1[\Sigma, \mathrm{F}(X)] \times \mathcal{L}^1[\Omega, \mathrm{F}(X)] \longrightarrow \mathbb{R}_+$ is defined by

$$\Delta(F_1, F_2) = \int_\Omega d_H\big(\text{supp}(F_1(\omega)), \text{supp}(F_2(\omega))\big) d\mu,$$

then $(\mathcal{L}^1[\Sigma, \mathrm{F}(X)], \Delta)$ is a complete metric space. Define addition $+$, scalar multiplication $\cdot$ and an order relation pointwise on $\mathcal{L}^1[\Sigma, \mathrm{F}(X)]$.
Let

$$\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)] = \{F \in \mathcal{L}^1[\Sigma, \mathrm{F}(X)] : F(\omega) \in \mathrm{F_c}(X) \text{ a.e.}\}.$$

Note that $\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)] \subseteq \mathcal{L}^1[\Sigma, \mathrm{F}(X)]$ and for Σ_0 a sub σ-algebra of Σ we have $\mathcal{E}[F|\Sigma_0] \in \mathcal{L}^1[\Sigma_0, \mathrm{F}(X)]$ for all $F \in \mathcal{L}^1[\Sigma, \mathrm{F}(X)]$.
We generalize the following from [12]:

Definition 2.8. Let $(F_i) \subseteq \mathcal{L}[\Sigma, \mathrm{F}(X)]$ and (Σ_i) an increasing sequence of sub σ-fields of Σ. Then $(F_i, \Sigma_i)_{i\in\mathbb{N}}$ is called a *martingale* (respectively, *submartingale*) in $\mathcal{L}[\Sigma, \mathrm{F}(X)]$ provided that $F_i \in \mathbf{M}[\Sigma_i, \mathrm{F}(X)]$ and $F_i(\omega) = (\leq)\ \mathcal{E}[F_{i+1}|\Sigma_i](\omega)$ a.e. for all $i \in \mathbb{N}$.

Let $X^* = \{x^* : X \longrightarrow \mathbb{R} : x^* \text{ is linear and continuous}\}$. For every bounded subset C of X and each $x^* \in X^*$, let

$$s(x^*, C) := \sup\{x^*(x) : x \in \text{supp}(C)\}.$$

We are now in a position to state a fuzzy version of Doob decomposition theorem of Daures, Ni and Zhang, as can be found in [12]:

Theorem 2.4. *Let (F_i, Σ_i) be a set-valued submartingale in $\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)]$; i.e., (F_i, Σ_i) be a submartingale in $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ and $(F_i) \subseteq \mathcal{L}^1[\Sigma, \mathrm{F_c}(X)]$. If there exists $B \in \Sigma$ with $\mu(B) = 0$ such that for any $\omega \notin B$ and all $i \in \mathbb{N}$*

(i) $s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)]) - s(\,\cdot\,, F_{i-1}(\omega))$ *and*
(ii) $s(\,\cdot\,, F_i(\omega)) - s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)])$

are convex functions on X^*, then (F_i, Σ_i) can be decomposed as

$$F_i(\omega) = M_i(\omega) + A_i(\omega) \text{ for all } \omega \notin B$$

where (M_i, Σ_i) is a fuzzy martingale and (A_i) is a fuzzy set-valued predictable increasing sequence such that for all $\omega \notin B$

(a) $A_1(\omega) = 0$,
(b) $A_j(\omega) = \overline{\left(\sum_{i=1}^{j-1} \mathcal{E}[F_{i+1}|\Sigma_i](\omega) \ominus F_i(\omega)\right)}$ *for all* $j \geq 2$,
(c) $M_1(\omega) = F_1(\omega)$, *and*
(d) $M_j(\omega) = \overline{\left(\sum_{i=2}^{j} [F_i(\omega) \ominus \mathcal{E}[F_i|\Sigma_{i-1}](\omega)\right) + F_1(\omega)}$ *for all* $j \geq 2$.

The proof of Theorem original crisp version of 2.4, as given in [12], exploits the properties of the the functions $s(\cdot, C)$ where $C \in \mathrm{F_c}(X)$.
To achieve our aim, we derive a Doob decomposition theorem for submartingales in ordered near vector spaces as considered in [10]. As a consequence and as an intermediate step, we obtain a Doob decomposition for fuzzy submartingales which are integrable in §4. The latter then yields the Daures, Ni and Zhang result (see [3, 13]) as a special case, as integrably bounded functions are integrable (see [12, p.31]). It also yields an analogue of the Doob decomposition of fuzzy submartingales, as noted by Shen and Wang (see [16]).

3. DOOB'S DECOMPOSITION IN AN ORDERED NEAR VECTOR SPACE

It was noted in [10] that if X is a Banach space, then $(\mathrm{cf}(X), +, \cdot)$ is a near vector space. For the convenience of the reader, we recall the definition from [10].

Let S be a nonempty set. As in [10], S is said to be a *near vector space*, provided that addition $+\colon S\times S\longrightarrow S$ is defined such that $(S,+)$ is a cancellative commutative semigroup; i.e., for all $x,y,z\in S$:

$$x+z=y+z\Rightarrow x=y,\quad x+y=y+x,\quad (x+y)+z=x+(y+z),$$

and multiplication $\cdot:\mathbb{R}_+\times S\longrightarrow S$ by positive scalars is defined such that for all $x,y\in S$ and $\lambda,\delta\in\mathbb{R}_+$:

$$\lambda x+\lambda y=\lambda(x+y),\ (\lambda+\delta)x=\lambda x+\delta x,\ (\lambda\delta)x=\lambda(\delta x),\ 1x=x.$$

Let S be a (near) vector space. If $(S,\leq)$ is a partially ordered set such that $\leq$ is compatible with addition and multiplication by positive scalars; i.e., for all $x,y\in S$ and $\alpha\in\mathbb{R}_+$,

$$x\leq y\ \Rightarrow\ [x+z\leq y+z \text{ and } \alpha x\leq\alpha y],$$

then S is called an *ordered (near) vector space*.
It was noted in [10] that if X is a Banach space, then $(\mathrm{cf}(X),\subseteq,+,\cdot)$ is an ordered near vector space.
Rådström proved the following result in [14, Theorem 1]:

Theorem 3.1. *If S is a near vector space, then there exist a vector space $R(S)$ and a map $j\colon S\longrightarrow R(S)$ such that*

(a) *j is injective,*
(b) *$j(\alpha x+\beta y)=\alpha j(x)+\beta j(y)$ for all $\alpha,\beta\in\mathbb{R}_+$ and $x,y\in S$,*
(c) *$R(S)=j(S)-j(S):=\{j(x)-j(y):x,y\in S\}$.*

An outline of the proof of the previous theorem can be found in [10].
Let S be an ordered near vector space. Define an order $\leq$ on $R(S)$ by

$$[x,y]\leq[x_1,y_1]\Longleftrightarrow x+y_1\leq y+x_1,$$

Then $R(S)$ is an ordered vector space and $j\colon S\longrightarrow R(S)$ is an order embedding (see also [10]); i.e.,

$$s\leq t\Leftrightarrow j(s)\leq j(t).$$

Let S be an ordered near vector space which also satisfies

(Z) there exists $0\in S$ such that $x+0=x$ for all $x\in S$ and $\lambda 0=0$ for all $\lambda\in\mathbb{R}_+$.

Then S is said to be an *ordered near vector space with a zero*.
If X is a separable Banach space, then

(a) $(\mathbf{M}[\Sigma,\mathrm{F_c}(X)],+,\cdot,\leq)$,
(b) $(\mathcal{L}[\Sigma,\mathrm{F_c}(X)],+,\cdot,\leq)$, and
(c) $(\mathcal{L}^1[\Sigma,\mathrm{F_c}(X)],+,\cdot,\leq)$

are ordered near vector space with with $\chi_{\{0\}}$ as zero where χ_A is the characteristic function of A. In fact, $(\mathcal{L}[\Sigma,\mathrm{F_c}(X)],+,\cdot,\leq)$ is a sub ordered near vector space of $(\mathbf{M}[\Sigma,\mathrm{F_c}(X)],+,\cdot,\leq)$ and $(\mathcal{L}^1[\Sigma,\mathrm{F_c}(X)],+,\cdot,\leq)$ is a sub ordered near vector space of $(\mathcal{L}[\Sigma,\mathrm{F_c}(X)],+,\cdot,\leq)$.
It is clear that if S is an ordered near vector space S with a zero, then there exists a subtraction operation on $R(S)$, but this does not guarantee the existence of a subtraction operation on S under which S is closed.
To overcome this problem, we consider the following:

Definition 3.1. Let S be an ordered near vector space with a zero and define $\sqsubseteq$ by

$$y\sqsubseteq x\Longleftrightarrow\exists\, z\in S\,[0\leq z \text{ and } y+z=x].$$

Then, by the cancellation law, z is unique in Definition 3.1 and we define

$$z := x - y.$$

Also,

$$x \sqsubseteq y \Rightarrow x \leq y \text{ for all } x, y \in S$$

and it follows that $\sqsubseteq$ is a partial ordering on S. We call $\sqsubseteq$ the *ordering associated with* $\leq$.
Also note that, for all $x \in S$,

$$0 \sqsubseteq x \Longleftrightarrow 0 \leq x;$$

i.e.,

$$S_+ := \{x \in S : 0 \leq x\} = \{x \in S : 0 \sqsubseteq x\}.$$

It is readily verified that $(S, \sqsubseteq)$ is an ordered near vector space with 0 as zero. Furthermore, if we consider the Rådström completion $R(S)$ of $(S, +, \cdot, \sqsubseteq)$, then

$$\begin{aligned} y \sqsubseteq x &\Longleftrightarrow \exists\, z \in S \ (0 \leq z \text{ and } [z, 0] = [x, y]) \\ &\Longleftrightarrow \exists\, x - y \in S \ (0 \leq x - y \text{ and } [x - y, 0] = [x, y]). \end{aligned}$$

Our strategy is now as follows. We first consider Doob's decomposition of a submartingale in an ordered vector space. Then we use this ordered vector space result to obtain a Doob decomposition of a submartingale in an ordered near vector space. We specialize the ordered near vector space to the appropriate fuzzy set-valued space of submartingales that are integrable and obtain the Daures, Ni and Zhang result as a special case from the latter for integrably bounded martingales.

We now define martingales in terms of projections rather than sub σ-algebras. By considering martingales in this way, we can apply the theory of martingales to near vector spaces.

Definition 3.2. Let S be any nonempty set, (T_i) a commuting sequence (i.e., $T_iT_j = T_jT_i = T_i$ for all $i \leq j$) of projections on S and $(f_i) \subseteq S$. Then

(a) (f_i, T_i) is a *martingale* in S, provided that $f_i = T_i f_j$ for all $i \leq j$.

If, in addition, $(S, \leq)$ is a partially ordered set and each T_i is order preserving; i.e., $u \leq v \Rightarrow T_i u \leq T_i v$ for all $u, v \in S$, then

(b) (f_i, T_i) is called a *submartingale* in S, provided that $f_i \in \mathcal{R}(T_i)$ for all i (where $\mathcal{R}(T_i)$ is the range of T_i) and $f_i \leq (\geq)\, T_i f_j$ for all $i \leq j$.

As was noted in [9], it follows from Theorem 2.3 that if (Σ_i) is a filtration and if we set

$$T_i(F) = \mathcal{E}[F|\Sigma_i] \text{ for all } F \in \mathcal{L}[\Sigma, \mathrm{F_c}(X)]\Big(F \in \mathcal{L}^1[\Sigma, \mathrm{F_c}(X)]\Big) \text{ and } i \in \mathbb{N},$$

then (T_i) is a commuting sequence of order preserving projections on the ordered near vector space $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ $(\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)])$ and the range $\mathcal{R}(T_i)$ of T_i is $\mathcal{L}[\Sigma_i, \mathrm{F_c}(X)]$ $(\mathcal{L}^1[\Sigma_i, \mathrm{F_c}(X)])$ for each i. Furthermore, if $(F_i) \subseteq \mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ $(\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)])$ and (Σ_i) is an increasing sequence of sub σ-fields of Σ, then (F_i, T_i) is a *martingale* (respectively, *submartingale*) in the ordered near vector space $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ $(\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)])$ in the sense of Definition 3.2.
The following result, which is based on a vector lattice version in [7], is the first step in achieving our aim of proving the Daures, Ni and Zang result in an elementary way:

Theorem 3.2. *Let E be an ordered vector space, $(f_i) \subseteq E$ and (T_i) a commuting sequence of positive linear projections on E. If (f_i, T_i) is a submartingale,*

(i) $A_1 = 0$,
(ii) $A_j = \sum_{i=1}^{j-1} \big(T_i f_{i+1} - f_i\big)$ *for all* $j \geq 2$ *and*
(iii) $M_j = f_j - A_j$ *for all* $j \in \mathbb{N}$,

then the decomposition $f_i = M_i + A_i, i \in \mathbb{N}$, *is the unique decomposition of* (f_i, T_i) *with* (M_j, T_j) *a martingale in* E, $(A_j) \subseteq E$ *a positive and increasing sequence and* $A_{j+1} \in \mathcal{R}(T_j)$ *for all* $j \in \mathbb{N}$.

Let S be an ordered near vector space. A map $T\colon S \longrightarrow S$ is called $\mathbb{R}_+$-*linear* provided that $T(\alpha x + \beta y) = \alpha Tx + \beta Ty$ for all $x, y \in S$ and $\alpha, \beta \in \mathbb{R}_+$. It was shown in [10] that if S is an ordered near vector space and $T\colon S \longrightarrow S$ is an order preserving $\mathbb{R}_+$-linear map, then $\hat{T}$, defined by $\hat{T}[x, y] = [Tx, Ty]$ for all $[x, y] \in R(S)$, is an order preserving linear map from $R(S)$ to $R(S)$.
Let (f_i, T_i) be a submartingale in an ordered near vector space S, where (T_i) is a commuting sequence of order preserving $\mathbb{R}_+$-linear idempotents on S. Then $([f_i, 0], \hat{T}_i)$ is a submartingale in $R(S)$ and $(\hat{T}_i)$ is a commuting sequence of order preserving linear projections on $R(S)$.
We need the following notion:

Definition 3.3. Let S be an ordered near vector space with a zero, $(f_i) \subseteq S$, (T_i) a commuting sequence of order preserving $\mathbb{R}_+$-linear idempotents on S and (f_i, T_i) We call (f_i, T_i) a $\sqsubseteq$-*submartingale* in S if $f_i \in \mathcal{R}(T_i)$ for all i and $f_j \sqsubseteq T_j(f_i)$ for all $j \leq i$.

Theorem 3.3. *Let* S *be an ordered near vector space with a zero,* $(f_i) \subseteq S$ *and* (T_i) *a commuting sequence of increasing* $\mathbb{R}_+$*-linear projections on* S. *If* (f_i, T_i) *is a* $\sqsubseteq$*-submartingale,*

(a) $A_1 = 0$,
(b) $A_j = \sum_{i=1}^{j-1} [T_i f_{i+1} - f_i, 0]$ *for all* $j \geq 2$,
(c) $M_1 = [f_1, 0]$ *and*
(d) $M_j = [f_1, 0] + \sum_{i=1}^{j-1} [f_{i+1}, T_i f_{i+1}]$ *for all* $j \in \mathbb{N}$,

then the decomposition $[f_i, 0] = M_i + A_i$ *for all* $i \in \mathbb{N}$, *is the unique decomposition of* $([f_i, 0], \hat{T}_i)$ *with* (M_j, T_j) *a martingale in* $R(S)$, $(A_j) \subseteq R(S)$ *a positive and increasing sequence and* $A_{j+1} \in \mathcal{R}(T_j)$ *for all* $j \in \mathbb{N}$.

4. THE DAURES-NI-ZHANG VERSION OF DOOB'S DECOMPOSITION IN THE FUZZY SETTING

Let X be a Banach space. We first specialize our above discussion on the associated ordering to the ordered near vector space $(\mathrm{F_c}(X), +, \cdot, \leq)$.
The ordering $\sqsubseteq$ on $\mathrm{F_c}(X) \times \mathrm{F_c}(X)$ associated with $\leq$ is given by

$$A \sqsubseteq B \iff \exists\, C \in \mathrm{F_c}(X)\ (\chi_{\{0\}} \leq C \text{ and } A + C = B).$$

- $0 \in A \ominus B \Leftrightarrow B \subseteq A$ for all $A, B \in \mathcal{P}_0(X)$,
- if supp(A) is bounded, then $A \ominus A = \chi_{\{0\}}$,
- if $A \in \mathrm{F}(X)$, then $A \ominus B \in \mathrm{F}(X)$,
- if A is convex, so is $A \ominus B$ provided that $A \ominus B \neq \chi_\emptyset$,
- if supp(A) and supp(B) are bounded, then supp($A \ominus B$) is also bounded,

The following theorem and two subsequent corollaries follow from the results in [11].

Theorem 4.1. *Let* X *be a Banach space. If* $A, B \in \mathrm{F_c}(X)$, *then there exists* $C \in \mathrm{F_c}(X)$ *such that* $B + C = A$ *if and only if* $B + (A \ominus B) = A$. *Moreover, in this case,* $A \ominus B$ *is the unique* C *satisfying* $A = C + B$.

Corollary 4.1. *Let X be a Banach space and $A, B \in \mathrm{F_c}(X)$. Then the following statements are equivalent:*

(i) *There exists $C \in \mathrm{F_c}(X)$ such that $B + C = A$.*
(ii) *$B + (A \ominus B) = A$.*
(iii) *$[s(\,\cdot\,, A) - s(\,\cdot\,, B)]_\alpha$ is a convex function on X^* for each $\alpha \in I$.*

Corollary 4.2. *Let X be a Banach space. Then, for all $A, B \in \mathrm{F_c}(X)$, the following statements are equivalent:*

(i) *$B \sqsubseteq A$.*
(ii) *$\chi_{\{0\}} \leq A \ominus B$ and $B + A \ominus B = A$.*
(iii) *$B \leq A$ and $s(\,\cdot\,, A) - s(\,\cdot\,, B)$ is a convex function on X^*.*

Proof. From [12, p.159] we have that for each $\alpha \in (0, 1]$,

$$\{0\} = [\chi_{\{0\}}]_\alpha \subseteq A_\alpha \ominus B_\alpha \Leftrightarrow B_\alpha \subseteq A_\alpha$$
$$\Leftrightarrow B \leq A \text{ and } \chi_{\{0\}} \subseteq A \ominus B$$

by Lemma 2.2. By applying Theorem 4.1 we complete the proof. □

We use our main result Theorem 3.3 to obtain:

Theorem 4.2. *Let (F_i, Σ_i) be a fuzzy submartingale in $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$. If there exists $B \in \Sigma$ with $\mu(B) = 0$ such that for any $\omega \notin B$ and all $i \in \mathbb{N}$*

(i) *$[s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)]) - s(\,\cdot\,, F_{i-1}(\omega))]_\alpha$ and*
(ii) *$[s(\,\cdot\,, F_i(\omega)) - s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)])]_\alpha$*

are convex functions on X^* for all $\alpha \in I$, then (F_i, Σ_i) has a decomposition

$$F_i(\omega) = M_i(\omega) + A_i(\omega) \text{ for all } \omega \notin B,$$

where (M_i, Σ_i) is a fuzzy martingale and (A_i) is a predictable increasing sequence $\mathbf{M}[\Sigma, \mathrm{F_c}(X)]$ in such that for all $\omega \notin B$

(a) *$A_1(\omega) = \chi_{\{0\}}$,*
(b) *$A_j(\omega) = \overline{\left(\sum_{i=1}^{j-1} \mathcal{E}[F_{i+1}|\Sigma_i](\omega) \ominus F_i(\omega)\right)}$ for all $j \geq 2$,*
(c) *$M_1(\omega) = F_1(\omega)$, and*
(d) *$M_j(\omega) = \overline{\left(\sum_{i=2}^{j}[F_i(\omega) \ominus \mathcal{E}[F_i|\Sigma_{i-1}](\omega)\right) + F_1(\omega)}$ for all $j \geq 2$.*

Proof. The proof of this theorem is very similar to the proof of Theorem 4.4 from [11]. We want to apply Theorem 3.3 to the ordered near vector space $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$. It was noted earlier that $(\mathcal{E}[\,\cdot\,|\Sigma_i])$ is a commuting sequence of increasing $\mathbb{R}_+$-linear projections on $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ such that $\mathcal{R}(\mathcal{E}[\,\cdot\,|\Sigma_i]) = \mathcal{L}[\Sigma_i, \mathrm{cf}(E)]$ for all $i \in \mathbb{N}$. We first verify that (F_i, Σ_i) is a fuzzy $\sqsubseteq$-submartingale.
As (F_i, Σ_i) is a fuzzy submartingale, it follows from $F_i(\omega) \subseteq \mathcal{E}[F_{i+1}|\Sigma_i](\omega)$ a.e. for all $\omega \in \Omega$ and $i \in \mathbb{N}$ that

$$\chi_{\{0\}} \leq \mathcal{E}[F_{i+1}|\Sigma_i](\omega) \ominus F_i(\omega) \text{ a.e. for all } \omega \in \Omega \text{ and } i \in \mathbb{N}.$$

Also, $s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)]) - s(\,\cdot\,, F_{i-1}(\omega))$ for all $\omega \notin B$ and all $i \in \mathbb{N}$ means that

$$F_i(\omega) + \Big(\mathcal{E}[F_{i+1}|\Sigma_i](\omega) \ominus F_i(\omega)\Big) = \mathcal{E}[F_{i+1}|\Sigma_i](\omega)$$

for all $\omega \notin B$ and all $n \in \mathbb{N}$; consequently,

$$F_i(\omega) \sqsubseteq \mathcal{E}[F_{i+1}|\Sigma_i](\omega) \text{ a.e. for all } \omega \notin B \text{ and } i \in \mathbb{N}.$$

Hence, (F_i, Σ_i) be a set-valued $\sqsubseteq$-submartingale.

Let $\mathcal{A}_1(\omega) = 0$ for all $\omega \notin B$ and, for all $j \geq 2$,

$$\mathcal{A}_j = \sum_{i=1}^{j-1} \Big[\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i, \chi_{\{0\}}\Big] = \Big[\overline{\sum_{i=1}^{j-1} \Big(\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i\Big)}, \chi_{\{0\}}\Big],$$

$\mathcal{M}_1 = [F_1, 0]$ and, for all $j \geq 2$,

$$\mathcal{M}_j = [F_1, 0] + \sum_{i=1}^{j-1} \Big[F_{i+1}, \mathcal{E}[F_{i+1}|\Sigma_i]\Big].$$

Then it follows from Theorem 3.3 that in the Rådström's completion $R(\mathcal{L}[\Sigma, \mathrm{F_c}(X)])$ of $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$, we have that the submartingale $([F_i, \chi_{\{0\}}], \widehat{\mathcal{E}[\cdot\,|\Sigma_i]})$ has a unique decomposition

$$[F_i(\omega), \chi_{\{0\}}(\omega)] = \mathcal{M}_i(\omega) + \mathcal{A}_i(\omega) \text{ for all } \omega \notin B \text{ and } i \in \mathbb{N},$$

where with $(\mathcal{M}_j, \widehat{\mathcal{E}[\cdot\,|\Sigma_i]})$ a martingale in $R(\mathcal{L}[\Sigma, \mathrm{F_c}(X)])$, $(\mathcal{A}_j) \subseteq \mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ a positive and increasing sequence and $\mathcal{A}_{j+1} \in \mathcal{L}[\Sigma_j, \mathrm{F_c}(E)]$ for all $j \in \mathbb{N}$.
From the assumption $s(\,\cdot\,, F_n(\omega)) - s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)])$ for all $\omega \notin B$ and all $i \geq 2$, we get that $F_i = \mathcal{E}[F_i|\Sigma_{i-1}(\omega)] + \Big(F_i \ominus \mathcal{E}[F_i|\Sigma_{i-1}(\omega)]\Big)$. Hence, in $R(\mathcal{L}[\Sigma, \mathrm{F_c}(X)])$, it follows that

$$\Big[F_i, \mathcal{E}[F_i|\Sigma_{i-1}]\Big] = \Big[F_i \ominus \mathcal{E}[F_i|\Sigma_{i-1}], \chi_{\{0\}}\Big].$$

But then, for all $j \geq 2$,

$$\begin{aligned} \mathcal{M}_j &= [F_1, \chi_{\{0\}}] + \sum_{i=1}^{j-1} \Big[F_{i+1}, \mathcal{E}[F_{i+1}|\Sigma_i]\Big] \\ &= [F_1, \chi_{\{0\}}] + \sum_{i=1}^{j-1} \Big[F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i], \{0\}\Big] \\ &= \Big[\overline{\sum_{i=1}^{j-1} \Big(F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i]\Big) + F_1},\ \chi_{\{0\}}\Big] \end{aligned}$$

Let

$$A_1 = 0 \text{ and } A_j = \overline{\sum_{i=1}^{j-1} \Big(\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i\Big)} \text{ for all } j \geq 2,$$

$$M_1 = 0 \text{ and } M_j = \overline{\sum_{i=1}^{j-1} \Big(F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i]\Big) + F_1} \text{ for all } j \geq 2.$$

Then (F_i, Σ_i) has a decomposition

$$F_i = M_i + A_i \text{ for all } i \in \mathbb{N},$$

with the required properties. □

We are now in a position to prove Theorem 2.4, the fuzzy version of the Doob decomposition as noted by Daures, Ni and Zhang, using Theorem 4.2.
Proof of Theorem 2.4. As $\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)]$ is a sub ordered near vector space of $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ (see [9]), we may in Corollary 4.2 replace $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ by $\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)]$, which completes the proof of Theorem 2.4. □

5. THE SHEN-WANG VERSION OF DOOB'S DECOMPOSITION IN THE FUZZY SETTING

If $\{0\} \neq E$ is a Banach lattice, then the canonical embedding $E \hookrightarrow \mathrm{F_c}(E)$, given by $x \mapsto \chi_{\{x\}}$, is not order preserving if $\mathrm{cf}(E)$ is endowed with the usual fuzzy ordering. We want to relate the ordering on E to an appropriate ordering on $\mathrm{F_c}(E)$. We, therefore, consider

$$\mathrm{F_c}(E_+) : \quad = \quad \{A \in \mathrm{F_c}(E_+) : A \text{ is convex}\}.$$

For all $F, G \in \mathrm{F_c}(E)$, define

$$F \preceq G \Leftrightarrow \exists H \in \mathrm{F_c}(E_+)\,(F + H = G).$$

Direct verification yields that

- if $F \in \mathrm{F_c}(E)$, then $\chi_{\{0\}} \preceq F$ if and only if $0 \le f$ for all $f \in F$,
- $(\mathrm{F_c}(E), \preceq)$ is a partially ordered set and $(\mathrm{F_c}(E), +, \cdot, \preceq)$ is an ordered near vector space.

It is also clear that the ordering $\sqsubseteq$ associated with $\preceq$ on $\mathrm{F_c}(E)$ coincides with $\preceq$.
We extend the ordering $\preceq$ pointwise to the spaces $\mathcal{L}[\Sigma, \mathrm{F_c}(E)]$ and $\mathcal{L}^1[\Sigma, \mathrm{F_c}(E)]$. Then $(\mathcal{L}[\Sigma, \mathrm{F_c}(E)], +, \cdot, \preceq)$ and $(\mathcal{L}^1[\Sigma, \mathrm{F_c}(E)], +, \cdot, \preceq)$ are ordered near vector spaces.
The next result shows that conditional expectations are $\preceq$-preserving:

Lemma 5.1. *Let E be a Banach lattice, (Ω, Σ, μ) a finite measure space space and Σ_0 a sub σ-algebra of Σ. Then the following holds:*

(E3′) *If $F_1, F_2 \in \mathcal{L}[\Sigma, \mathrm{F_c}(E)]$, then $F_1 \preceq F_2$ implies $\mathcal{E}[F_1|\Sigma_0] \preceq \mathcal{E}[F_2|\Sigma_0]$.*

Proof. Once again the proof of this theorem is very similar to the proof of Theorem 5.2 from [11]. Let $F_1 \preceq F_2$. Select $H \in \mathcal{L}[\Sigma, \mathrm{F_c}(E)]$ for which $H(\omega) \in \mathrm{F_c}(E_+)$ a.e. and $F_1 + H = F_2$. Then $\mathcal{E}[F_1|\Sigma_0] + \mathcal{E}[H|\Sigma_0] = \mathcal{E}[F_2|\Sigma_0]$. To conclude that $\mathcal{E}[F_1|\Sigma_0] \preceq \mathcal{E}[F_2|\Sigma_0]$, it suffices to show that $\chi_{\{0\}} \preceq \mathcal{E}[H|\Sigma_0]$.
If $h \in L^1(\Omega, \Sigma, \mu)$ such that $h(\omega) \in H(\omega)$ a.e., then, as $H(\omega) \in \mathrm{F_c}(E_+)$ a.e., it follows that $h(\omega) \ge 0$ a.e.; consequently, $\mathbb{E}[h|\Sigma_0](\omega) \ge 0$ a.e. and

$$S_H^1(\Sigma_0) = \{h \in L^1(\Omega, \Sigma_0, \mu) : 0 \le h(\omega) \in H(\omega) \text{ a.e.}\}.$$

But then $\chi_{\{0\}} \preceq \overline{\{\mathbb{E}[h|\Sigma_0] : h \in S_H^1(\Sigma_0)\}}$. From the definition of $\mathcal{E}[H|\Sigma_0]$, it follows that $\chi_{\{0\}} \preceq \mathcal{E}[H|\Sigma_0]$, and the proof is complete. □

The following version of Doob's decomposition is similar to a result noted by Shen and Wang (see [16]). Their result differs from the one below mainly in the assumption (1) in Theorem 5.1. This assumption yields an explicit description of the martingale involved in the decomposition, which they do not obtain in their result.

Theorem 5.1. *Let E be a Banach lattice, (F_i, Σ_i) be a $\preceq$-submartingale in $\mathcal{L}[\Sigma, \mathrm{F_c}(E)]$ (alternatively, $\mathcal{L}^1[\Sigma, \mathrm{F_c}(E)]$). If there exists $B \in \Sigma$ with $\mu(B) = 0$ and, for each $i \ge 2$,*

$$s(\,\cdot\,, F_i(\omega)) - s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)]) \text{ for all } \omega \notin B$$

is a convex functions on X^*, then there is a decomposition of (F_i, Σ_i) as

$$F_i(\omega) = M_i(\omega) + A_i(\omega) \text{ for all } \omega \notin B$$

where (M_i, Σ_i) is a set-valued martingale in $\mathcal{L}[\Sigma, \mathrm{F_c}(E)]$ (alternatively, $\mathcal{L}^1[\Sigma, \mathrm{F_c}(E)]$) and (A_i) is a set-valued predictable $\preceq$-increasing sequence such that for all $\omega \notin B$

(a) $A_1(\omega) = 0$,
(b) $A_j(\omega) = \overline{\left(\sum_{i=1}^{j-1} \mathcal{E}[F_{i+1}|\Sigma_k](\omega) \ominus F_k(\omega)\right)}$ *for all* $j \geq 2$,
(c) $M_1(\omega) = F_1(\omega)$, *and*
(d) $M_j(\omega) = \overline{\left(\sum_{i=2}^{j} [F_k(\omega) \ominus \mathcal{E}[F_i|\Sigma_{i-1}](\omega)\right) + F_1(\omega)}$ *for all* $j \geq 2$.

Moreover, the decomposition is unique.

Proof. The proof is very similar to that of Theorem 4.2, although we are considering the ordering $\preceq$ instead of $\subseteq$. The details follow:
From Lemma 5.1, we know that $\mathcal{E}[\,\cdot\,|\Sigma_i]$ is $\preceq$-preserving. Hence, in $R(\mathcal{L}[\Sigma, \mathrm{F_c}(X)])$ we have that $([F_i, \chi_{\{0\}}], \widehat{\mathcal{E}[\cdot\,|\Sigma_i]})$ has a unique decomposition

$$[F_i, \chi_{\{0\}}] = \mathcal{M}_i + \mathcal{A}_i \text{ for all } i \in \mathbb{N},$$

where $\mathcal{A}_1 = 0$ and, for all $j \geq 2$,

$$\mathcal{A}_j = \sum_{i=1}^{j-1} \Big[\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i, \chi_{\{0\}}\Big] = \Big[\overline{\sum_{i=1}^{j-1} \Big(\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i\Big)}, \chi_{\{0\}}\Big],$$

$\mathcal{M}_1 = [F_1, 0]$ and, for all $j \geq 2$,

$$\mathcal{M}_j = [F_1, 0] + \sum_{i=1}^{j-1} \Big[F_{i+1}, \mathcal{E}[F_{i+1}|\Sigma_i]\Big]$$

with $(\mathcal{M}_j, \widehat{\mathcal{E}[\cdot\,|\Sigma_i]})$ a martingale in $R(\mathcal{L}[\Sigma, \mathrm{F_c}(X)])$, $(\mathcal{A}_j) \subseteq \mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ a positive and increasing sequence and $\mathcal{A}_{j+1} \in \mathcal{L}[\Sigma_j, \mathrm{F_c}(E)]$ for all $j \in \mathbb{N}$.
From the assumption $s(\,\cdot\,, F_n(\omega)) - s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)])$ for all $\omega \notin B$ and all $i \geq 2$, we get that $F_i = \mathcal{E}[F_i|\Sigma_{i-1}(\omega)] + \Big(F_i \ominus \mathcal{E}[F_i|\Sigma_{i-1}(\omega)]\Big)$. Hence, in $R(\mathcal{L}[\Sigma, \mathrm{F_c}(X)])$, it follows that

$$\Big[F_i, \mathcal{E}[F_i|\Sigma_{i-1}]\Big] = \Big[F_i \ominus \mathcal{E}[F_i|\Sigma_{i-1}], \chi_{\{0\}}\Big].$$

But then, for all $j \geq 2$,

$$\begin{aligned}
\mathcal{M}_j &= [F_1, \chi_{\{0\}}] + \sum_{i=1}^{j-1} \Big[F_{i+1}, \mathcal{E}[F_{i+1}|\Sigma_i]\Big] \\
&= [F_1, \chi_{\{0\}}] + \sum_{i=1}^{j-1} \Big[F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i], \chi_{\{0\}}\Big] \\
&= \Big[\overline{\sum_{i=1}^{j-1} \Big(F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i]\Big) + F_1},\ \chi_{\{0\}}\Big]
\end{aligned}$$

Let

$$A_1 = 0 \text{ and } A_j = \overline{\sum_{i=1}^{j-1} \Big(\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i\Big)} \text{ for all } j \geq 2,$$

$$M_1 = 0 \text{ and } M_j = \overline{\sum_{i=1}^{j-1} \Big(F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i]\Big) + F_1} \text{ for all } j \geq 2.$$

Then (F_i, Σ_i) has a decomposition

$$F_i = M_i + A_i \text{ for all } i \in \mathbb{N},$$

with the desired properties. □

References

[1] S.F. Cullender, C.C.A. Labuschagne, A description of norm-convergent martingales on vector valued L^p-spaces. *J. Math. Anal. Appl.,* **323** (2006), 119–130.

[2] S.F. Cullender, C.C.A. Labuschagne, Convergent martingales of operators and the Radon-Nikodým property in Banach spaces, *Proceedings of the American Mathematical Society,* **136** (2008), 3883–3893.

[3] J.P. Daures, Version multivoque du théorèm de Doob, *Ann. Inst. H. Poincaré*, **9** (1977), 167-176.

[4] R. DeMarr, A martingale convergence theorem in vector lattices. *Canad. J. Math.,* **18** (1966), 424–432.

[5] J. Diestel and J.J. Uhl, *Vector Measures*, A.M.S. Surveys, Volume 15, Providence, Rhode Island, 1977.

[6] F. Hiai, H. Umegaki, Integrals, conditional expectations, and martingales of multivalued functiones, *Journal of Multivariate Analysis*, **7** (1977), 149-182.

[7] W.-C. Kuo, C. C. A. Labuschagne, B. A. Watson, Discrete-time stochastic processes on Riesz spaces, *Indag. Math. (New Series),* **15** (2004), 435–451.

[8] C.C.A. Labuschagne, Join-semilattices of integrable set-valued martingales, *Thai Journal of Mathematics*, **5** (2007), 53–69.

[9] C.C.A. Labuschagne, A Banach lattice approach to convergent integrably bounded set-valued martingales and their positive parts, *Journal of Mathematical Analysis and Applications*, **342** (2008), 780–797.

[10] C.C.A. Labuschagne, A.L. Pinchuck, C.J. van Alten, A vector lattice version of Rådström's embedding theorem, Quaestiones Mathematicae, **30** (2007), 285-308.

[11] C.C.A. Labuschagne, A.L. Pinchuck, *Doob's decomposition of set-valued submartingales via ordered near vector spaces*, Quaest. Math. 32 (2009), Pages 247–264.

[12] S. Li, Y. Ogura, V. Kreinovich, *Limit theorems and applications of set-valued and fuzzy set-valued random variables*, Kluwer Acadenic Press, Dordrecht-Boston-London, 2002.

[13] Z. Ni, W. Zhang, Doob decomposition of sub-(super)martingales, *Acta Mathematica Sinica (Chinese Series),* **35** (1992), 53–62.

[14] H. Rådström, An embedding theorem for spaces of convex sets. *Proc. Amer. Math. Soc.* **3** (1952), 165–169.

[15] G. Stoica, Martingales in vector lattices. *Bull. Math. Soc. Sci. Math. Roumanie. (N.S.),* **34(82)** (1990), 357-362.

[16] L. Shen, Z. Wang A new semi-order in superspace $P_{bfc}(X)$ and it's applications. *Chinese Journal of Applied Probability and Statistics.* **12** (1996), 411–421.

[17] V. Troitsky, Martingales in Banach lattices. *Positivity*, **9** (2005), 437-456.