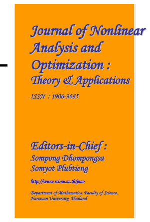
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 2, No. 1, (2011), 19-25
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



NIELSEN THEORY AND SOME SELECTED APPLICATIONS TO DIFFERENTIAL AND INTEGRAL EQUATIONS; A BRIEF SURVEY[◇]

NIRATTAYA KHAMSEMANAN[*]

School of Information, Computer, and Communication Technology (ICT),
Sirindhorn International Institute of Technology (SIIT), Thammasat University
P.O. Box 22, Pathum Thani 12121, Thailand

ABSTRACT. In this paper, we first give a quick introduction to Nielsen theory in the traditional topological setting. We then turn to the applications of Nielsen theory, and survey some results in nonlinear analysis.

KEYWORDS : Nielsen Theory; Nielsen Number; Nielsen Theory in Banach Space; Applications of Nielsen Theory.

1. INTRODUCTION

Nielsen theory occupies a prominent place within topological fixed point theory and is currently one of the most active research areas of algebraic topology. Nielsen theory, named in honor of its founder, Danish mathematician Jakob Nielsen (1890 - 1959), is concerned with finding the minimum number of solutions to certain equations involving maps, minimized among all the maps in a given homotopy class. The first part of this paper focuses on the traditional definition of Nielsen fixed point classes and the Nielsen number itself.

Not only has the Nielsen theory been widely studied in the topological setting, there are many areas of mathematics that have used the idea of Nielsen theory to solve existing problems. In the second part of the paper, we focus on the applications of Nielsen theory in nonlinear analysis.

[◇]This research was partially supported by Supported by The Thailand Research Fund, Grant MRG538024.

[*] *Corresponding author.*

Email address : nirattaya@gmail.com (N. Khamsemanan).

Article history : Received November 12 2010. Accepted January 28 2011.

This work originated with the suggestion of Professor Dr. Sompong Dhompongsa and people in his group when the author presented previous works concerned with calculating the Nielsen number on various topological spaces. They wanted to know if there is any way that the author's speciality in topological fixed point and Nielsen theory could be applied to their work in analysis. The author hopes that this brief survey will demonstrate the possibility of more collaborations between these two areas.

2. NIELSEN THEORY: THE INTRODUCTION

Let X be a connected compact polyhedron then X has a universal cover $p : \widetilde{X} \to X$. For any self-map $f : X \to X$, a *lift* $\tilde{f}$ of f is a map from $\widetilde{X}$ to itself such that $p \circ \tilde{f} = f \circ p$. Also, recall that a *deck transformation* is a homeomorphism $\alpha : \widetilde{X} \to \widetilde{X}$ such that $p \circ \alpha = p$.

2.1. **The Minimum Number.** The main object of study in topological (Nielsen) fixed point theory is concerned with finding the "minimum number" of the fixed points of $f : X \to X$.

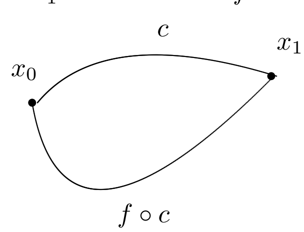
$$\mathrm{MF}[f] = \{ \text{ min } \sharp Fix(g) : g \simeq f\}$$

where $Fix(g) = \{x : g(x) = x\}$ and $\simeq$ denotes homotopy. Note that MF[f] = 0 would mean there is a map homotopic to f that has no fixed points.

Main Problem: Determine MF[f] from information about f.

2.2. **Fixed Point Classes.** Nielsen theory depends on the concept of the "fixed point class" which partitions $Fix(f)$ into equivalence classes. There are two ways of describing the equivalence relation.

Definition 2.1. Definition of Fixed Point Classes (via path homotopy): Two fixed points x_0 and x_1 of $f : X \to X$ are in the same fixed point class if and only if there exists a path c from x_0 to x_1 such that $c \simeq f \circ c$.



Definition 2.2. Definition of Fixed Point Classes (via lifting classes): Two lifts $\tilde{f}$ and $\tilde{f}'$ of f are said to be *conjugate* if there exists $\gamma \in \mathfrak{D}$ such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$. A lifting class is denoted as

$$[\tilde{f}] = \{\gamma \circ \tilde{f} \circ \gamma^{-1} | \gamma \in \mathfrak{D}\}.$$

Two fixed points x_0 and x_1 of $f : X \to X$ are in the same fixed point class if and only if $x_0, x_1 \in p(Fix(\tilde{f}))$. If $\tilde{f}, \tilde{f}'$ are conjugate lifts, then $p(Fix(\tilde{f})) = p(Fix(\tilde{f}'))$ so fixed point classes depend on conjugacy classes of lifts.

For a chosen base point x_0 and a chosen lift $\tilde{f}$ of f, the $\tilde{f}_\pi$-conjugacy class of $\alpha \in \pi_1(X, x_0)$ is called ***the coordinate of the lifting class*** (see [20] for more details) $[\alpha \circ \tilde{f}]$. This α is can be obtained geometrically.

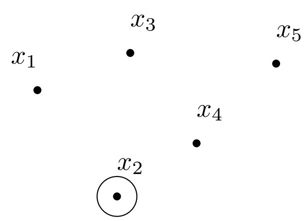
Suppose $x_0 \in p(Fix(\tilde{f}))$ and $\bar{x}_0 \in p^{-1}(x_0)$ and $\bar{x}_0 \in Fix(\tilde{f})$ is the constant path at

x_0 where $\tilde{f}$ is the chosen lift of f. Then the coordinate of the class of a fixed point x of f is the $\tilde{f}_\pi$-conjugacy class of $\alpha = [c * (f \circ c)^{-1}] \in \pi_1(X, x_0)$, where c is any path from x_0 to x. In other words, $x \in p(Fix(\alpha \circ \tilde{f}))$.

2.3. **The Nielsen Number.** The fixed point index is an indispensable tool of fixed point theory. The index of each fixed point allow us to algebraically count fixed points in an open set. Roughly, we can think of the index of $\mathbb{F}$, a fixed point class, with respect to f as follows. Suppose $\mathbb{F}$ is finite and for $x_2 \in \mathbb{F}$, pick a sphere centered at x_2 such that this sphere is small enough to exclude other fixed points and the sphere is mapped by f into a Euclidean neighborhood of x_2. The index of x_2 is the degree of the map

$$\frac{id - f}{|id - f|}$$

restricted to the sphere.



Definition 2.3. The *index* of a fixed point class $\mathbb{F}$ is the sum of the indices of all fixed points in $\mathbb{F}$.

A fixed point class $\mathbb{F}$ is said to be *essential* if the index of $\mathbb{F}$ with respect to f is not zero.

Definition 2.4. The *Nielsen Number* of f, denoted $N(f)$, is the number of essential fixed point classes of f.

Theorem 2.1. *If g is homotopic to f, then $N(g) = N(f)$ and therefore*

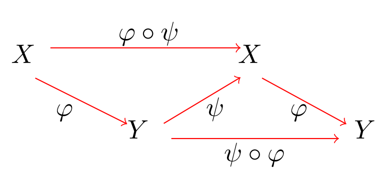
$$N(f) \leq MF[f]$$

In fact, if X is not a surface, then there exists a map $g \simeq f$ such that it has exactly $N(f)$ fixed points.

2.4. **Properties of the Nielsen number.**

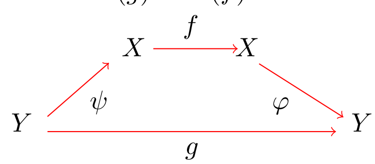
Homotopy invariant: The Nielsen number is a homotopy invariant.
Commutativity: Let φ be a map from X to Y and let ψ be a map from Y to X, then

$$N(\varphi \circ \psi) = N(\psi \circ \varphi)$$



Homotopy-type Invariant: Let φ be a map from X to Y and let ψ be a map from Y to X such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are homotopic to the identity. Let f be a self-map on X and suppose g be a self-map on Y is homotopic to $\varphi \circ f \circ \psi$ then

$$N(g) = N(f)$$



2.5. **Geometric and Algebraic Aspects of Nielsen Theory.** The interest in the Nielsen number $N(f)$ comes from the fact that it usually does solve the "minimum number" MF[f] problem, but not always. However it is difficult to calculate the Nielsen number $N(f)$ from the definition.

Thus the "classical" Nielsen fixed point theory has two aspects

Geometric Aspect: determining when $N(f) =$ MF[f].
Algebraic Aspect: calculating $N(f)$.

3. APPLICATIONS OF NIELSEN THEORY TO DIFFERENTIAL AND INTEGRAL EQUATIONS

This section of this review paper based on the subsection of the same title by Professor Robert F. Brown in [6].

At the 1950 International Congress of Mathematicians, Leray suggested that the work of Nielsen what gives a lower bound for the number of fixed points of a map should be extended to analytic problems because solutions to analytic problems can often be formulated as fixed points of functions and the existence of multiple solutions is often significant. However the setting of Nielsen theory was originally concerned with maps on finite polyhedra or compact ANRs, not the appropriate setting for analytic problems. In fact, Leray himself obtained a result in 1959, generalizing global Lefschetz fixed point theory that was the first step in extending Nielsen theory to the analytic setting [22].

3.1. **Selected applications of Nielsen Theory.**

(i) Let E, F be Banach spaces, $L : E \to F$ an isomorphism, $H : E \times \mathbb{R}^n \to F$ a completely continuous map and $B : E \to \mathbb{R}^n$ a continuous linear mapping onto a Euclidean space. Brown in [8] applied the Nielsen theory to help finding solutions to $Ly = H(y, \lambda)$ that satisfies $By = 0$. These solutions can be represented as the fixed points of a map $T : E \times \mathbb{R}^n \to E \times \mathbb{R}^n$. If T satisfies a condition called μ-retractibility, then Nielsen fixed point theory may be extended to produce lower bounds for the number of fixed points of such maps.

(ii) Brown and Zezza in [9] applied Nielsen theory to some problems in control theory. They studied the controllability of perturbed linear control processes whose control space can be reduced to a finite-dimensional one. Their techniques produce a lower bound on the number of controls that achieve a given target, using Nielsen theory to detect when there is more than one solution.

(iii) Fečkan [11] used Nielsen theory to establish the existence of multiple periodic solutions to problems in which the operator L is not an isomorphism. Suppose $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is 1-periodic in the first variable and bounded. For the system $y' = \varepsilon h(t, y)$, with small ε, if the map $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ is μ-retractible onto a compact, locally contractible subset S by a retraction ρ, then $N(\rho\Phi|_S)$, the Nielsen number of the map $\rho\Phi|_S$, is a lower bound for the number of 1-periodic solution to the system.
Fečkan also used this approach to study some nth order boundary value problems in [12].

(iv) Suppose that $L : X \to Y$ be a linear map between two Banach spaces X, Y. We say that L is a Fredholm operator if the image of L is closed in Y and the kernel and cokernel of L are finite dimensional. The index of a Fredholm map is defined as

$$i(L) = dimKer(L) - dim(Y/Im(L))$$

Feckan [15] applied Nielsen theory to the problem of the form $Lu = F(u)$ for L Fredholm with positive index.
Many of his results applying Nielsen theory to ordinary differential equations on Banach spaces are summarized in [10]. He also mentions there that his techniques could be used for some problems concerning partial differential equations as well.

(v) Borisovich, Kucharski and Marzantowicz [4] applied Nielsen theory to nonlinear integration equations of the *Urysohn type*.
Let X be the Banach space of pairs of continuous functions on $[0, 1]$. Let $G : X \to X$ be defined by

$$G(u, v)(t) = (u(t), v(t))$$

where $u(t)$ and $v(t)$ are of Urysohn type, i.e.

$$\begin{aligned} u(t) &= \int_0^1 K_1(t, s, u(s), v(s))v^\beta(s)ds \\ v(t) &= \int_0^1 K_2(t, s, u(s), v(s))u^\alpha(s)ds. \end{aligned}$$

Let A be the subset of K consisting of pairs (u, v) of functions, each of which takes only non-negative or only non-positive values and are not both the zero function. They found that the Nielsen number of G restricted to A was 2 which means that the system has at least two non-zero solutions.

(vi) There has been extensive work on the application of Nielsen theory to nonlinear analysis focusing on differential inclusions, multivalued functions and boundary value problems due mainly to Andres, Gorniewicz and Jeierski. See [1] for more in-depth information on this area.

(vii) In 2003, Andres and Vath [2] developed two definitions of the Nielsen numbers for the more general setting of noncompact maps. One definition is based on Nielsen's original idea for fixed point classes. The second definition is based on the idea of classes due to Wecken. They focused on a Nielsen number for coincidence points of two continuous maps $p, q : \Gamma \to X$ which is the lower bound of the number of coincidence points ($p(x) = q(x)$ for $x \in \Gamma$). Notice that if q is the identity map, this gives the classical Nielsen number. In particular, their definition can be used in the Banach space setting.

(viii) In 2007, Andres and Vath [3] used the definitions that they defined in [2] to calculate the Lefschetz and the Nielsen numbers of iterated function systems or dynamical systems in hyperspaces. Their success was due to the fact that hyperspaces are topologically simple. Their result stated that the Lefschetz and the Nielsen numbers can be calculated as easily as just counting fixed points of a map of a finite set of, typically, very small cardinality.

Acknowledgements. The author thanks Robert F. Brown who has given great support, suggestions, advice, and insights—both in this work and over the years. The author also would like to thank Professor Dr. Sompong Dhompongsa who suggested the idea of this work. This project has been made possible by the Thai Research Fund (TRF) grant ID MRG5380241. Many thanks to TRF as well.

References

1. Andres, J. and Gorniewicz, L. *Topological Fixed Point Principles for Boundary Value Problems,* Kluwer. 2003.
2. Andres, J. and Vath, M. *Calculation of Lefschetz and Nielsen Numbers in Hyperspaces for Fractals and Dynamical Systems.* Proc. Amer. Math. Soc. 135 (2007), 479-487.
3. Andres, J. and Vath, M. *Two Topological Definitions of Nielsen Numbers for Coincidences of Noncompact maps.* Fixed Point Theory and Applications 2004:1 (2004) 49-69
4. Borisovich, A., Kucharski, Z. and Marzantowicz, W. *A multiplicity result for a system of real integral equations by use of the Nielsen number,* Banach Center Publ. 49 (1999), 9-18.
5. Brown, R.F. *Fixed point theory* History of Topology Elsevier (1999), 271-299
6. Brown, R. F. *More about Nielsen theories and their applications* The Handbook of Topological Fixed Point Theory. Springer (2005).
7. Brown, R. F. *A Topological Introduction to Nonlinear Analysis* Second Edition, Birkhauser,2004.
8. Brown, R. F. *Topological identification of multiple solutions to parametrized nonlinear equations* Pacific J. Math. 131 (1988), 51-69.
9. Brown, R. and Zezza P. *Multiple local solutions to nonlinear control processes* J. Optim. Theory Appl. 67 (1990), 463-485.
10. Fečkan, M. *Multiple solutions of nonlinear equations via Nielsen fixed-point theory:a survey* Nonlinear Analysis in Geometry and Topology, Hadronic Press, 2000, pp. 77-97.
11. Fečkan, M. *Nielsen fixed point theory and nonlinear equations* J. Differential Equations 106 (1993), 312-331.
12. Fečkan, M. *Parametrized singularly perturbed boundary value problems* J. Math. Anal.Appl. 188 (1994), 417-425.
13. Fečkan, M. *Multiple periodic solutions of small vector fields on differential equations* J. Differential Equations 113 (1994), 189-200.
14. Fečkan, M. *Parametrized singular boundary value problems* J. Math. Anal. Appl. 188 (1994), 426-435.
15. Fečkan, M. *Differential equations with nonlinear boundary conditions* Proc. Amer. Math. Soc. 121 (1994), 103-111.
16. Fečkan, M. *Note on weakly inward mappings* Ann. Polon. Math. 63 (1996), 1-5.

17. Fečkan, M. *The interaction of linear boundary value and nonlinear functional conditions* Ann. Polon. Math. 58 (1993), 103-111.
18. Hart, E. *The Reidemeister trace and the calculation of the Nielsen number*, Nielsen Theory and Reidemeister Torsion, Banach Center Publication, Polish Academy of Science, 49 (1999), 151-157.
19. Hatcher, A. *Algebraic Topology* Cambridge University Press, New York, NY (2002).
20. Jiang, B. *Lectures on Nielsen Fixed Point Theory,* Contemp. Math. Vol. 14, Amer. Math Soc. Providence, RI (1983).
21. Kiang, T. *The Theory of Fixed Point Classes* Springer, Berlin (1989).
22. Leray, J. *Theorie des pointes fixes: indice total et nombre de Lefschetz*, Bull. Soc. Math. France 87 (1959), 221-233.
23. McCord, D. *An estimate of the Nielsen number and an example concerning the Lefschetz fixed point theorem*, Pacific J. Math **66** (1976), 195-203.
24. Massey, W. S. *Algebraic Topopogy: An Introduction,* Harcount, Brace, and World, New York, 1967.