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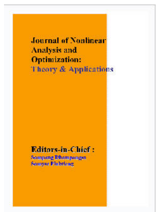

Common Fixed Points of a New Three-Step Iteration with Errors of Asymptotically Quasi-Nonexpansive Nonself-Mappings in Banach spaces

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ABSTRACT: In this paper, we study a new three-step iterative scheme for approximating a common fixed point of three asymptotically quasi-nonexpansive nonself-mappings with errors and prove several strong and weak convergence results of the iterative sequences with errors in a uniformly convex Banach space. We also extend and improve some recent corresponding results in the literature.

1. **Introduction**

We assume that X is a normed space and C is a nonempty subset of X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* [3] if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all $x, y \in C$ and each $n \geq 1$. The class of asymptotically nonexpansive mappings is a natural generalization of the important class of nonexpansive mappings. Goebel and Kirk [3] proved that if C is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self-mapping has a fixed point. A mapping $T : C \to C$ is called *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - q\| \leq k_n\|x - q\|$ for all $x \in C, q \in F(T), n \geq 1$, where $F(T)$ is the set of fixed points of T. The mapping T is called *uniformly L-Lipschitzian* if there exists a positive constant L such that $\|T^n x - T^n y\| \leq L\|x - y\|$ for all $x, y \in C$ and each $n \geq 1$. It is easy to see that an asymptotically nonexpansive mapping must be uniformly L-Lipschitzian as well as asymptotically quasi-nonexpansive but the converse does not hold.

In 2000, Noor [9] introduced a three-step iterative sequence and studied the approximate solutions of variational inclusions in Hilbert spaces. Glowinski and Le Tallec [4] applied three-step iterative sequences for finding the approximate solutions of the elastoviscoplasticity problem, eigenvalue problems and in the liquid crystal theory. It has been shown in [1], that three-step method performs better than two-step and one-step methods for solving variational inequalities. The three-step schemes are natural generalization of the splitting methods to solve partial differential equations; see, Noor [9, 10, 11]. This signifies that Noor three-step methods are robust and more efficient than the Mann (one-step) and Ishikawa (two-step) type

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iterative methods to solve problems of pure and applied sciences.

In 2001, Khan and Takahashi [5] have approximated common fixed points of two asymptotically nonexpansive mappings by the modified Ishikawa iteration. Recently Shahzad and Udomene [15] established convergence theorems for the modified Ishikawa iteration process of two asymptotically quasi-non expansive mappings to a common fixed point of the mappings. For related results with error terms, we refer to [2, 6, 13] and [15].

The purpose of this paper is to establish strong and weak convergence theorems of a new three-step iteration for three asymptotically quasi-nonexpan sive non-self mappings in a uniformly convex Banach space. This scheme can be viewed as an extension of Xu and Noor [18], Suantai [16] and Nilsrakoo and Saejung [8].

Let X be a normed space. A subset C of X is said to be a *retract* of X if there exists a continuous map $P : X \to C$ such that $Px = x$ for all $x \in C$. Every closed convex set of a uniformly convex Banach space is a retract. A map $P : X \to C$ is said to be a *retraction* if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P. A mapping $T : C \to X$ is said to be *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim\limits_{n\to\infty} k_n = 1$ such that

$$\|T(PT)^{n-1}x - q\| \leq k_n\|x - q\|$$

for all $x \in C, q \in F(T), n \geq 1$, where $F(T)$ is the set of fixed points of T and $(PT)^0 = I$, the identity operator on C.

The mapping $T : C \to X$ is called *uniformly L-Lipschitzian* if there exists a positive constant L such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Let C be a nonempty closed convex subset of X and $P : X \to C$ a nonexpansive retraction of X onto C, and let $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive mappings and F is the set of all common fixed points of T_i i.e., $F = \bigcap\limits_{i=1}^{3} F(T_i)$, where $F(T_i) = \{x \in C : T_i x = x\}$ for all $i = 1, 2, 3$. Then, for arbitrary $x_1 \in C$, compute the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= P[a_n T_1(PT_1)^{n-1}x_n + (1 - a_n - \delta_n)x_n + \delta_n u_n], \\ y_n &= P[b_n T_2(PT_2)^{n-1}z_n + c_n T_1(PT_1)^{n-1}x_n + (1 - b_n - c_n - \sigma_n)x_n + \sigma_n v_n], \\ x_{n+1} &= P[\alpha_n T_3(PT_3)^{n-1}y_n + \beta_n T_2(PT_2)^{n-1}z_n + \gamma_n T_1(PT_1)^{n-1}x_n \\ &\qquad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)x_n + \rho_n w_n] \end{aligned} \tag{1}$$

for all $n \geq 1$, where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ are appropriate sequences in $[0, 1]$ and $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C.

Without errors ($\delta_n = \sigma_n = \rho_n \equiv 0$), and T_1, T_2, T_3 are self-maps of C, the iterative scheme (1) reduces to the following iterative scheme:

$$\begin{aligned} z_n &= a_n T_1^n x_n + (1 - a_n)x_n, \\ y_n &= b_n T_2^n z_n + c_n T_1^n x_n + (1 - b_n - c_n)x_n, \\ x_{n+1} &= \alpha_n T_3^n y_n + \beta_n T_2^n z_n + \gamma_n T_1^n x_n + (1 - \alpha_n - \beta_n - \gamma_n)x_n, \quad n \geq 1, \end{aligned} \tag{2}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0, 1]$.

If $T := T_1 = T_2 = T_3$, then (2) reduces to the iterative scheme defined by Nilsrakoo and Saejung [8].

If $\gamma_n \equiv 0$ and $T := T_1 = T_2 = T_3$, then (2) reduces to the iterative scheme defined by Suantai [16].

If $c_n = \beta_n = \gamma_n \equiv 0$ and $T := T_1 = T_2 = T_3$, then (2) reduces to the iterative scheme defined by Xu and Noor [18].

If $a_n = b_n = c_n \equiv 0$, then (2) reduces to the following iterative scheme:

$$x_{n+1} = \alpha_n T_3^n x_n + \beta_n T_2^n x_n + \gamma_n T_1^n x_n + (1 - \alpha_n - \beta_n - \gamma_n) x_n \tag{3}$$

for all $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0, 1]$.

To study strong and weak convergence theorems of the iterative scheme 1, we recall some useful well-known concepts and results.

Recall that a Banach space X is said to satisfy *Opial's condition* [12] if for each sequence $\{x_n\}$ and $x, y \in X$ with $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that

$$\lim_{n\to\infty} \sup \|x_n - x\| < \lim_{n\to\infty} \sup \|x_n - y\|.$$

In what follows, we shall make use of the following lemmas.

Lemma 1.1. [17, Lemma 1]. *Let $\{a_n\}, \{b_n\}, \{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n \qquad \text{for all } n = 1, 2, \ldots.$$

If $\sum\limits_{n=1}^{\infty} \delta_n < \infty$ and $\sum\limits_{n=1}^{\infty} b_n < \infty$, then

(i) $\lim\limits_{n\to\infty} a_n$ *exists, and*

(ii) $\lim\limits_{n\to\infty} a_n = 0$ *whenever* $\lim\limits_{n\to\infty} \inf a_n = 0$.

Lemma 1.2. [7, Lemma 1.4]. *Let X be a uniformly convex Banach space and let $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$ be a closed ball of X. Then there exists a continuous, strictly increasing convex function $g : [0, \infty) \to [0, \infty), g(0) = 0$ such that*

$$\|\lambda x + \mu y + \xi z + \vartheta w\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \xi \|z\|^2 + \vartheta \|w\|^2 - \lambda \mu g\big(\|x - y\|\big)$$

for all $x, y, z, w \in B_r$ and all $\lambda, \mu, \xi, \vartheta \in [0, 1]$ with $\lambda + \mu + \xi + \vartheta = 1$.

Similar to Lemma 1.2, we can prove the next lemma.

Lemma 1.3. *Let X be a uniformly convex Banach space and let B_r be a closed ball of X. Then there exists a continuous, strictly increasing convex function $g : [0, \infty) \to [0, \infty), g(0) = 0$ such that*

$$\|\lambda x + \mu y + \xi z + \vartheta w + \zeta s\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \xi \|z\|^2 + \vartheta \|w\|^2 + \zeta \|s\|^2 - \lambda \mu g\big(\|x - y\|\big)$$

for all $x, y, z, w, s \in B_r$ and all $\lambda, \mu, \xi, \vartheta, \zeta \in [0, 1]$ with $\lambda + \mu + \xi + \vartheta + \zeta = 1$.

Lemma 1.4. [16, Lemma 2.7]. *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be so that $\lim\limits_{n\to\infty} \|x_n - u\|$ and $\lim\limits_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

2. Main Results

In this section, we prove strong and weak convergence theorems for the iterative scheme (1) for asymptotically quasi-nonexpansive nonself-mappings in a Banach space. In order to prove our main results, the following lemmas are needed.

Lemma 2.1. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive mappings with respect to sequences $\{k_n\}, \{l_n\}, \{m_n\}$, respectively, such that $F \neq \emptyset$, $k_n \geq 1$, $l_n \geq 1, m_n \geq 1$, $\sum\limits_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum\limits_{n=1}^{\infty} (l_n - 1) < \infty$ $\sum\limits_{n=1}^{\infty} (m_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$, $\{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ be real sequences in $[0, 1]$ such that $a_n + \delta_n$, $b_n + c_n + \sigma_n$ and $\alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0, 1]$ for all $n \geq 1$, $\sum\limits_{n=1}^{\infty} \delta_n < \infty$, $\sum\limits_{n=1}^{\infty} \sigma_n < \infty$, $\sum\limits_{n=1}^{\infty} \rho_n < \infty$ and let $\{u_n\}, \{v_n\}, \{w_n\}$ be bounded sequences in C. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the* *sequences defined as in* (1). *Then*

(i) $\lim_{n\to\infty} \|x_n - q\|$ *exists for all* $q \in F$.

(ii) *If one of the following conditions* $(a), (b), (c)$ *and* (d) *holds, then* $\lim_{n\to\infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0$.

(a) $\lim_{n\to\infty} \inf \beta_n > 0$ *and* $0 < \lim_{n\to\infty} \inf a_n \le \lim_{n\to\infty} \sup(a_n + \delta_n) < 1$.

(b) $\lim_{n\to\infty} \inf \alpha_n, \lim_{n\to\infty} \inf b_n > 0$ *and* $0 < \lim_{n\to\infty} \inf a_n \le \lim_{n\to\infty} \sup(a_n + \delta_n) < 1$.

(c) $0 < \lim_{n\to\infty} \inf \gamma_n \le \lim_{n\to\infty} \sup(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$.

(d) $0 < \lim_{n\to\infty} \inf \alpha_n$ *and* $0 < \lim_{n\to\infty} \inf c_n \le \lim_{n\to\infty} \sup(b_n + c_n + \sigma_n) < 1$.

(iii) *If either* (a) $0 < \lim_{n\to\infty} \inf \beta_n \le \lim_{n\to\infty} \sup(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$ *or* (b) $\lim_{n\to\infty} \inf \alpha_n > 0$ *and* $0 < \lim_{n\to\infty} \inf b_n \le \lim_{n\to\infty} \sup(b_n + c_n + \sigma_n) < 1$, *then* $\lim_{n\to\infty} \|T_2(PT_2)^{n-1}z_n - x_n\| = 0$.

(iv) *If* $0 < \lim_{n\to\infty} \inf \alpha_n \le \lim_{n\to\infty} \sup(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$, *then* $\lim_{n\to\infty} \|T_3(PT_3)^{n-1}y_n - x_n\| = 0$.

Proof. (i) Let $q \in F$. By (1), we obtain

$$
\begin{aligned}
\|z_n - q\| &= \|P[a_nT_1(PT_1)^{n-1}x_n + (1 - a_n - \delta_n)x_n + \delta_n u_n] - P(q)\| \\
&\le a_n\|T_1(PT_1)^{n-1}x_n - q\| + (1 - a_n - \delta_n)\|x_n - q\| + \delta_n\|u_n - q\| \\
&\le (1 + a_n(k_n - 1) - \delta_n)\|x_n - q\| + \delta_n\|u_n - q\| \qquad (4)
\end{aligned}
$$

and

$$
\begin{aligned}
\|y_n - q\| &= \|P[b_nT_2(PT_2)^{n-1}z_n + c_nT_1(PT_1)^{n-1}x_n + (1 - b_n - c_n - \sigma_n)x_n \\
&\quad + \sigma_n v_n] - P(q)\| \\
&\le b_n\|T_2(PT_2)^{n-1}z_n - q\| + c_n\|T_1(PT_1)^{n-1}x_n - q\| \\
&\quad + (1 - b_n - c_n - \sigma_n)\|x_n - q\| + \sigma_n\|v_n - q\| \\
&\le b_nl_n\|z_n - q\| + c_nk_n\|x_n - q\| + (1 - b_n - c_n - \sigma_n)\|x_n - q\| \\
&\quad + \sigma_n\|v_n - q\|. \qquad (5)
\end{aligned}
$$

By (4) and (5), we obtain

$$
\begin{aligned}
\|x_{n+1} - q\| &= \|P[\alpha_nT_3(PT_3)^{n-1}y_n + \beta_nT_2(PT_2)^{n-1}z_n + \gamma_nT_1(PT_1)^{n-1}x_n \\
&\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)x_n + \rho_n w_n] - P(q)\| \\
&\le \alpha_n\|T_3(PT_3)^{n-1}y_n - q\| + \beta_n\|T_2(PT_2)^{n-1}z_n - q\| \\
&\quad + \gamma_n\|T_1(PT_1)^{n-1}x_n - q\| + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\| \\
&\quad + \rho_n\|w_n - q\| \\
&\le \alpha_nm_n\|y_n - q\| + \beta_nl_n\|z_n - q\| + \gamma_nk_n\|x_n - q\| \\
&\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\| + \rho_n\|w_n - q\| \\
&\le (\alpha_nm_nb_nl_n + \beta_nl_n)\|z_n - q\| + \alpha_nm_nc_nk_n\|x_n - q\| \\
&\quad + (\alpha_nm_n - \alpha_nm_nb_n - \alpha_nm_nc_n - \alpha_nm_n\sigma_n)\|x_n - q\| \\
&\quad + \alpha_nm_n\sigma_n\|v_n - q\| + \gamma_nk_n\|x_n - q\| \\
&\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\| + \rho_n\|w_n - q\| \\
&\le \|x_n - q\| + ((l_n - 1)(\alpha_nm_nb_n + \beta_n) + (k_n - 1)(\gamma_n + \alpha_nm_nc_n \\
&\quad + (\alpha_nm_nb_nl_n + \beta_nl_n)a_n) + \alpha_n(m_n - 1))\|x_n - q\| \\
&\quad + (m_nl_n + l_n)\delta_n\|u_n - q\| + m_n\sigma_n\|v_n - q\| + \rho_n\|w_n - q\|.
\end{aligned}
$$

Since $\{l_n\}, \{m_n\}, \{u_n\}, \{v_n\}, \{w_n\}$ are bounded, there exists a constant $K > 0$ such that $\alpha_n m_n b_n + \beta_n \le K$, $\gamma_n + \alpha_n m_n c_n + (\alpha_n m_n b_n l_n + \beta_n l_n)a_n \le K$, $(m_n l_n + l_n)\|u_n - q\| \le K, m_n\|v_n - q\| \le K, \|w_n - q\| \le K$ and $\alpha_n \le K$ for all $n \ge 1$. Then

$$\begin{aligned} \|x_{n+1} - q\| \le &\Big(1 + K\big((k_n - 1) + (l_n - 1) + (m_n - 1)\big)\Big)\|x_n - q\| \\ &+ K(\delta_n + \sigma_n + \rho_n) \end{aligned} \tag{6}$$

By Lemma 1.1, we obtain $\lim_{n\to\infty} \|x_n - q\|$ exists.

Next, we want to prove (ii), (iii) and (iv). It follows from (i) that $\{x_n - q\}, \{T_1(PT_1)^{n-1}x_n - q\}, \{y_n - q\}, \{T_3(PT_3)^{n-1}y_n - q\}, \{z_n - q\}$ and $\{T_2(PT_2)^{n-1}z_n - q\}$ are all bounded. Let

$$\begin{aligned} M = \max\Big\{ &\sup_{n\ge1} \|x_n - q\|, \sup_{n\ge1} \|T_1(PT_1)^{n-1}x_n - q\|, \sup_{n\ge1} \|y_n - q\|, \\ &\sup_{n\ge1} \|T_3(PT_3)^{n-1}y_n - q\|, \sup_{n\ge1} \|z_n - q\|, \sup_{n\ge1} \|u_n - q\|, \\ &\sup_{n\ge1} \|T_2(PT_2)^{n-1}z_n - q\|, \sup_{n\ge1} \|v_n - q\|, \sup_{n\ge1} \|w_n - q\| \Big\}. \end{aligned}$$

By Lemma 1.3, there exists a continuous, strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\begin{aligned} \|\lambda x + \mu y + \xi z + \vartheta w + \zeta s\|^2 \le &\lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 + \vartheta\|w\|^2 + \zeta\|s\|^2 \\ &- \lambda\mu g\big(\|x - y\|\big) \end{aligned} \tag{7}$$

for all $x, y, z, w, s \in B_r$ and all $\lambda, \mu, \xi, \vartheta, \zeta \in [0, 1]$ with $\lambda + \mu + \xi + \vartheta + \zeta = 1$. By (7), we have

$$\begin{aligned} \|z_n - q\|^2 &= \|P[a_n T_1(PT_1)^{n-1}x_n + (1 - a_n - \delta_n)x_n + \delta_n u_n] - P(q)\|^2 \\ &\le \|a_n(T_1(PT_1)^{n-1}x_n - q) + (1 - a_n - \delta_n)(x_n - q) + \delta_n(u_n - q)\|^2 \\ &\le a_n\|T_1(PT_1)^{n-1}x_n - q\|^2 + (1 - a_n - \delta_n)\|x_n - q\|^2 \\ &\quad + \delta_n\|u_n - q\|^2 - a_n(1 - a_n - \delta_n)g\big(\|T_1(PT_1)^{n-1}x_n - x_n\|\big) \\ &\le a_n k_n^2\|x_n - q\|^2 + (1 - a_n - \delta_n)\|x_n - q\|^2 + \delta_n\|u_n - q\|^2 \\ &\quad - a_n(1 - a_n - \delta_n)g\big(\|T_1(PT_1)^{n-1}x_n - x_n\|\big) \\ &\le (1 + a_n(k_n^2 - 1) - \delta_n)\|x_n - q\|^2 + \delta_n\|u_n - q\|^2 \end{aligned} \tag{8}$$

and

$$\begin{aligned} \|y_n - q\|^2 &= \|P[b_n T_2(PT_2)^{n-1}z_n + c_n T_1(PT_1)^{n-1}x_n + (1 - b_n - c_n - \sigma_n) \\ &\quad x_n + \sigma_n v_n] - P(q)\|^2 \\ &\le \|b_n(T_2(PT_2)^{n-1}z_n - q) + c_n(T_1(PT_1)^{n-1}x_n - q) \\ &\quad + (1 - b_n - c_n - \sigma_n)(x_n - q) + \sigma_n(x_n - q)\|^2 \\ &\le b_n\|T_2(PT_2)^{n-1}z_n - q\|^2 + (1 - b_n - c_n - \sigma_n)\|x_n - q\|^2 \\ &\quad + c_n\|T_1(PT_1)^{n-1}x_n - q\|^2 + \sigma_n\|v_n - q\|^2 \\ &\quad - b_n(1 - b_n - c_n - \sigma_n)g\big(\|T_2(PT_2)^{n-1}z_n - x_n\|\big) \\ &\le b_n l_n^2\|z_n - q\|^2 + (1 - b_n - c_n - \sigma_n)\|x_n - q\|^2 + c_n k_n^2\|x_n - q\|^2 \\ &\quad + \sigma_n\|v_n - q\|^2 - b_n(1 - b_n - c_n - \sigma_n)g\big(\|T_2(PT_2)^{n-1}z_n - x_n\|\big) \end{aligned} \tag{9}$$

By (7) , (8) and (9), we obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|P[\alpha_n T_3(PT_3)^{n-1}y_n+\beta_n T_2(PT_2)^{n-1}z_n+\gamma_n T_1(PT_1)^{n-1}x_n \\
&\quad +(1-\alpha_n-\beta_n-\gamma_n-\rho_n)x_n+\rho_n w_n]-P(q)\|^2 \\
&\le \alpha_n\|T_3(PT_3)^{n-1}y_n-q\|^2+\beta_n\|T_2(PT_2)^{n-1}z_n-q\|^2 \\
&\quad +\gamma_n\|T_1(PT_1)^{n-1}x_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2 \\
&\quad -\alpha_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_3(PT_3)^{n-1}y_n-x_n\|\big) \\
&\quad +\rho_n\|w_n-q\|^2 \\
&\le \alpha_n m_n^2\|y_n-q\|^2+\beta_n l_n^2\|z_n-q\|^2+\gamma_n k_n^2\|x_n-q\|^2+\rho_n\|w_n-q\|^2 \\
&\quad +(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2 \\
&\quad -\alpha_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_3(PT_3)^{n-1}y_n-x_n\|\big) \\
&\le \alpha_n m_n^2 c_n k_n^2\|x_n-q\|^2+\alpha_n m_n^2(1-b_n-c_n-\sigma_n)\|x_n-q\|^2 \\
&\quad +\gamma_n k_n^2\|x_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2 \\
&\quad +(\alpha_n m_n^2 b_n l_n^2+\beta_n l_n^2)\|z_n-q\|^2+\alpha_n m_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2 \\
&\quad -\alpha_n m_n^2 b_n(1-b_n-c_n-\sigma_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) \\
&\quad -\alpha_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_3(PT_3)^{n-1}y_n-x_n\|\big) \\
&\le \alpha_n m_n^2 c_n k_n^2\|x_n-q\|^2+\alpha_n m_n^2(1-b_n-c_n)\|x_n-q\|^2 \\
&\quad +\gamma_n k_n^2\|x_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n)\|x_n-q\|^2+(\alpha_n m_n^2 b_n l_n^2 \\
&\quad +\beta_n l_n^2)\|x_n-q\|^2+(\alpha_n m_n^2 b_n l_n^2+\beta_n l_n^2)(a_n(k_n^2-1))\|x_n-q\|^2 \\
&\quad -(\alpha_n m_n^2 b_n l_n^2+\beta_n l_n^2)a_n(1-a_n-\delta_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) \\
&\quad +(m_n^2 l_n^2+l_n^2)\delta_n\|u_n-q\|^2+m_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2 \\
&\quad -\alpha_n m_n^2 b_n(1-b_n-c_n-\sigma_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) \\
&\quad -\alpha_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_3(PT_3)^{n-1}y_n-x_n\|\big) \\
&= \|x_n-q\|^2+\big((k_n^2-1)(\alpha_n m_n^2 c_n+\gamma_n+(\alpha_n m_n^2 b_n l_n^2+\beta_n l_n^2)a_n) \\
&\quad +(l_n^2-1)(\alpha_n m_n^2 b_n+\beta_n)+\alpha_n(m_n^2-1)\big)\|x_n-q\|^2 \\
&\quad +(m_n^2 l_n^2+l_n^2)\delta_n\|u_n-q\|^2+m_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2 \\
&\quad -\alpha_n m_n^2 b_n l_n^2 a_n(1-a_n-\delta_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) \\
&\quad -\beta_n l_n^2 a_n(1-a_n-\delta_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) \\
&\quad -\alpha_n m_n^2 b_n(1-b_n-c_n-\sigma_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) \\
&\quad -\alpha_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_3(PT_3)^{n-1}y_n-x_n\|\big)
\end{aligned}$$

Since $\{k_n\},\{l_n\},\{m_n\},\{u_n\},\{v_n\},\{w_n\}$ are bounded and $\{x_n\}$ is bounded, there exist constants $K_0>0$ such that
$(\alpha_n m_n^2 c_n+\gamma_n+(\alpha_n m_n^2 b_n l_n^2+\beta_n l_n^2)a_n)\|x_n-q\|^2\le K_0$,
$(\alpha_n m_n^2 b_n+\beta_n)\|x_n-q\|^2\le K_0, \alpha_n\|x_n-q\|^2\le K_0, (m_n^2 l_n^2+l_n^2)\|u_n-q\|^2\le K_0$,
$m_n^2\|v_n-q\|^2\le K_0$ and $\|w_n-q\|^2\le K_0$ for all $n\ge 1$. Thus

$$\begin{aligned}
\alpha_n m_n^2 b_n l_n^2 a_n(1-a_n-\delta_n)g\big(||T_1(PT_1)^{n-1}x_n-x_n||\big) &\le ||x_n-q||^2-||x_{n+1}-q||^2 \\
&\quad +K_0\big((k_n^2-1)+(l_n^2-1)+(m_n^2-1)\big) \\
&\quad +K_0(\delta_n+\sigma_n+\rho_n). \qquad (10)
\end{aligned}$$

$$\begin{aligned}
\beta_n l_n^2 a_n(1-a_n-\delta_n)g\big(||T_1(PT_1)^{n-1}x_n-x_n||\big) &\le ||x_n-q||^2-||x_{n+1}-q||^2 \\
&\quad +K_0\big((k_n^2-1)+(l_n^2-1)+(m_n^2-1)\big) \\
&\quad +K_0(\delta_n+\sigma_n+\rho_n). \qquad (11)
\end{aligned}$$

$$
\begin{aligned}
\alpha_n m_n^2 b_n(1-b_n-c_n-\sigma_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) &\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2 \\
&\quad +K_0\big((k_n^2-1)+(l_n^2-1)+(m_n^2-1)\big) \\
&\quad +K_0(\delta_n+\sigma_n+\rho_n). \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
\alpha_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_3(PT_3)^{n-1}y_n-x_n\|\big) &\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2 \\
&\quad +K_0\big((k_n^2-1)+(l_n^2-1)+(m_n^2-1)\big) \\
&\quad +K_0(\delta_n+\sigma_n+\rho_n). \qquad (13)
\end{aligned}
$$

Again (7), we obtain

$$
\begin{aligned}
\|y_n-q\|^2 &= \|P[b_nT_2(PT_2)^{n-1}z_n+c_nT_1(PT_1)^{n-1}x_n+(1-b_n-c_n-\sigma_n)x_n \\
&\quad +\sigma_nv_n]-P(q)\|^2 \\
&\leq c_n\|T_1(PT_1)^{n-1}x_n-q\|^2+(1-b_n-c_n-\sigma_n)\|x_n-q\|^2 \\
&\quad +b_n\|T_2(PT_2)^{n-1}z_n-q\|^2+\sigma_n\|v_n-q\|^2 \\
&\quad -c_n(1-b_n-c_n-\sigma_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) \\
&\leq c_nk_n^2\|x_n-q\|^2+(1-b_n-c_n-\sigma_n)\|x_n-q\|^2+b_nl_n^2\|z_n-q\|^2 \\
&\quad +\sigma_n\|v_n-q\|^2-c_n(1-b_n-c_n-\sigma_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) \qquad (14)
\end{aligned}
$$

By (7), (8) and (14), we have

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &= \|P[\alpha_nT_3(PT_3)^{n-1}y_n+\beta_nT_2(PT_2)^{n-1}z_n+\gamma_nT_1(PT_1)^{n-1}x_n \\
&\quad +(1-\alpha_n-\beta_n-\gamma_n-\rho_n)x_n+\rho_nw_n]-P(q)\|^2 \\
&\leq \alpha_n\|T_3(PT_3)^{n-1}y_n-q\|^2+\beta_n\|T_2(PT_2)^{n-1}z_n-q\|^2+\rho_n\|w_n-q\|^2 \\
&\quad +\gamma_n\|T_1(PT_1)^{n-1}x_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2 \\
&\quad -\beta_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) \\
&\leq \alpha_nm_n^2\|y_n-q\|^2+\beta_nl_n^2\|z_n-q\|^2+\gamma_nk_n^2\|x_n-q\|^2 \\
&\quad +\rho_n\|w_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2 \\
&\quad +\beta_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) \\
&\leq \alpha_nm_n^2c_nk_n^2\|x_n-q\|^2+\alpha_nm_n^2(1-b_n-c_n-\alpha_n)\|x_n-q\|^2 \\
&\quad +\gamma_nk_n^2\|x_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2 \\
&\quad +(\alpha_nm_n^2b_nl_n^2+\beta_nl_n^2)\|z_n-q\|^2+\alpha_nm_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2 \\
&\quad -\alpha_nm_n^2c_n(1-b_n-c_n-\sigma_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) \\
&\quad -\beta_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) \\
&\leq \alpha_nm_n^2c_nk_n^2\|x_n-q\|^2+\alpha_nm_n^2(1-b_n-c_n)\|x_n-q\|^2 \\
&\quad +\gamma_nk_n^2\|x_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n)\|x_n-q\|^2+(\alpha_nm_n^2b_nl_n^2 \\
&\quad +\beta_nl_n^2)\|x_n-q\|^2+(\alpha_nm_n^2b_nl_n^2+\beta_nl_n^2)(a_n(k_n^2-1))\|x_n-q\|^2 \\
&\quad +(m_n^2l_n^2+l_n^2)\delta_n\|u_n-q\|^2+m_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2 \\
&\quad -\alpha_nm_n^2c_n(1-b_n-c_n-\sigma_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) \\
&\quad -\beta_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) \\
&= \|x_n-q\|^2+\big((k_n^2-1)(\alpha_nm_n^2c_n+\gamma_n+(\alpha_nm_n^2b_nl_n^2+\beta_nl_n^2)a_n) \\
&\quad +(l_n^2-1)(\alpha_nm_n^2b_n+\beta_n)+\alpha_n(m_n^2-1)\big)\|x_n-q\|^2 \\
&\quad +(m_n^2l_n^2+l_n^2)\delta_n\|u_n-q\|^2+m_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2 \\
&\quad -\alpha_nm_n^2c_n(1-b_n-c_n-\sigma_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) \\
&\quad -\beta_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big)
\end{aligned}
$$

Thus

$$\begin{aligned}\alpha_n m_n^2 c_n(1-b_n-c_n-\sigma_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) &\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2\\ &\quad +K_0\big((k_n^2-1)+(l_n^2-1)+(m_n^2-1)\big)\\ &\quad +K_0(\delta_n+\sigma_n+\rho_n). \qquad (15)\end{aligned}$$

$$\begin{aligned}\beta_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) &\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2\\ &\quad +K_0\big((k_n^2-1)+(l_n^2-1)+(m_n^2-1)\big)\\ &\quad +K_0(\delta_n+\sigma_n+\rho_n). \qquad (16)\end{aligned}$$

By (7) , (8) and (9), we obtain

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|P[\alpha_n T_3(PT_3)^{n-1}y_n+\beta_n T_2(PT_2)^{n-1}z_n+\gamma_n T_1(PT_1)^{n-1}x_n\\ &\quad +(1-\alpha_n-\beta_n-\gamma_n-\rho_n)x_n+\rho_n w_n]-P(q)\|^2\\ &\leq \gamma_n\|T_1(PT_1)^{n-1}x_n-q\|^2+\alpha_n\|T_3(PT_3)^{n-1}y_n-q\|^2+\rho_n\|w_n-q\|^2\\ &\quad +(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2+\beta_n\|T_2(PT_2)^{n-1}z_n-q\|^2\\ &\quad -\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big)\\ &\leq \gamma_n k_n^2\|x_n-q\|^2+\alpha_n m_n^2\|y_n-q\|^2+\beta_n l_n^2\|z_n-q\|^2\\ &\quad +\rho_n\|w_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2\\ &\quad -\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big)\\ &\leq \gamma_n k_n^2\|x_n-q\|^2+\beta_n l_n^2\|z_n-q\|^2+\alpha_n m_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2\\ &\quad +\alpha_n m_n^2 b_n l_n^2\|z_n-q\|^2+\alpha_n m_n^2(1-b_n-c_n-\sigma_n)\|x_n-q\|^2\\ &\quad +\alpha_n m_n^2 c_n k_n^2\|x_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2\\ &\quad -\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big)\\ &\leq \gamma_n k_n^2\|x_n-q\|^2+m_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2\\ &\quad +(\alpha_n m_n^2 b_n l_n^2+\beta_n l_n^2)\|x_n-q\|^2+(m_n^2 l_n^2+l_n^2)\delta_n\|u_n-q\|^2\\ &\quad +\big((\alpha_n m_n^2 b_n l_n^2+\beta_n l_n^2)a_n(k_n^2-1)\big)\|x_n-q\|^2+\alpha_n m_n^2 c_n k_n^2\|x_n-q\|^2\\ &\quad +(1-\alpha_n-\beta_n-\gamma_n)\|x_n-q\|^2+\alpha_n m_n^2(1-b_n-c_n)\|x_n-q\|^2\\ &\quad -\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big)\\ &= \|x_n-q\|^2+\big((k_n^2-1)(\alpha_n m_n^2 c_n+\gamma_n+(\alpha_n m_n^2 b_n l_n^2+\beta_n l_n^2)a_n)\\ &\quad +(l_n^2-1)(\alpha_n m_n^2 b_n+\beta_n)+\alpha_n(m_n^2-1)\big)\|x_n-q\|^2\\ &\quad +(m_n^2 l_n^2+l_n^2)\delta_n\|u_n-q\|^2+m_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2\\ &\quad -\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big).\end{aligned}$$

Thus

$$\begin{aligned}\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) &\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2\\ &\quad +K_0\big((k_n^2-1)+(l_n^2-1)+(m_n^2-1)\big)\\ &\quad +K_0(\delta_n+\sigma_n+\rho_n). \qquad (17)\end{aligned}$$

(ii) (a) Let $\liminf_{n\to\infty}\beta_n>0$ and $0<\liminf_{n\to\infty}a_n\leq\limsup_{n\to\infty}(a_n+\delta_n)<1$. Then there exists a positive integer n_0 and $\eta,\eta^{'}\in(0,1)$ such that $0<\eta<\beta_n$, $0<\eta<a_n$ and $a_n+\delta_n<\eta^{'}<1$

for all $n \geq n_0$. This implies by (11) that

$$\begin{aligned}\eta^2(1-\eta')g\big(\|T_1(PT_1)^{n-1}x_n - x_n\|\big) \quad &\leq \quad \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\ &\quad + K_0\big((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1)\big) \\ &\quad + K_0(\delta_n + \sigma_n + \rho_n) \end{aligned} \tag{18}$$

for all $n \geq n_0$. It follows from (18) that for $r \geq n_0$,

$$\begin{aligned}\sum_{n=n_0}^{r} g\Big(\|T_1(PT_1)^{n-1}x_n - x_n\|\Big) &\leq \frac{1}{\eta^2(1-\eta')}\Big(\sum_{n=n_0}^{r} (\|x_n - q\|^2 - \|x_{n+1} - q\|^2) \\ &\quad + K_0 \sum_{n=n_0}^{r} \big((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1) \\ &\quad + \delta_n + \sigma_n + \rho_n)\Big) \\ &\leq \frac{1}{\eta^2(1-\eta')}\Big(\|x_{n_0} - q\|^2 + K_0 \sum_{n=n_0}^{r} \big((k_n^2 - 1) \\ &\quad + (l_n^2 - 1) + (m_n^2 - 1)\big)\Big). \end{aligned} \tag{19}$$

Since $0 \leq t^2 - 1 \leq 2t(t-1)$ for all $t \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, we get $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n^2 - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n^2 - 1) < \infty$. By inequality (19), let $r \to \infty$. We get $\sum_{n=n_0}^{\infty} g\big(\|T_1(PT_1)^{n-1}x_n - x_n\|\big) < \infty$. Thus $\lim_{n\to\infty} g\big(\|T_1(PT_1)^{n-1}x_n - x_n\|\big) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty}\|T_1(PT_1)^{n-1}x_n - x_n\| = 0$.

By using a similar method as in (ii) part (a) together with (10), (17), (15), (16), (12) and (13), the results in (ii) (b,c,d), (iii) (a,b) and (iv), respectively, can be proved. □

Lemma 2.2. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive mappings with respect to sequences $\{k_n\}, \{l_n\}, \{m_n\}$, respectively, such that $F \neq \emptyset, k_n \geq 1$, $l_n \geq 1, m_n \geq 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ be real sequences in $[0,1]$ such that $a_n + \delta_n$, $b_n + c_n + \sigma_n, \alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0,1]$ for all $n \geq 1$, $\sum_{n=1}^{\infty}\delta_n < \infty$, $\sum_{n=1}^{\infty}\sigma_n < \infty$, $\sum_{n=1}^{\infty}\rho_n < \infty$ and let $\{u_n\}, \{v_n\}, \{w_n\}$ be bounded sequences in C. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences defined as in (1). Suppose T_1, T_2, T_3 are uniformly L-Lipschitzian. If $\lim_{n\to\infty}\|T_1(PT_1)^{n-1}x_n - x_n\| = 0$, $\lim_{n\to\infty}\|T_2(PT_2)^{n-1}z_n - x_n\| = 0$, $\lim_{n\to\infty}\|T_3(PT_3)^{n-1}y_n - x_n\| = 0$, then*

(i) $\lim_{n\to\infty}\|T_1x_n - x_n\| = 0$,
(ii) $\lim_{n\to\infty}\|T_2x_n - x_n\| = 0$, *and*
(iii) $\lim_{n\to\infty}\|T_3x_n - x_n\| = 0$.

Proof. Since

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \alpha_n\|T_3(PT_3)^{n-1}y_n - x_n\| + \beta_n\|T_2(PT_2)^{n-1}z_n - x_n\| \\ &\quad + \gamma_n\|T_1(PT_1)^{n-1}x_n - x_n\| + \rho_n\|w_n - x_n\| \\ &\longrightarrow 0 \text{ as } n \to \infty,\end{aligned}$$

we obtain

$$\begin{aligned}\|T_1(PT_1)^{n-1}x_{n+1} - x_{n+1}\| &\le \|T_1(PT_1)^{n-1}x_{n+1} - T_1(PT_1)^{n-1}x_n\| \\ &\quad + \|T_1(PT_1)^{n-1}x_n - x_n\| + \|x_{n+1} - x_n\| \\ &\le L\|x_{n+1} - x_n\| + \|T_1(PT_1)^{n-1}x_n - x_n\| \\ &\quad + \|x_{n+1} - x_n\| \longrightarrow 0 \text{ as } n \to \infty. \end{aligned} \tag{20}$$

By (20), we get

$$\begin{aligned}\|T_1x_n - x_n\| &\le \|T_1(PT_1)^{n-1}x_n - x_n\| + \|T_1(PT_1)^{n-1}x_n - T_1x_n\| \\ &\le \|T_1(PT_1)^{n-1}x_n - x_n\| + L\|T_1(PT_1)^{n-2}x_n - x_n\| \\ &\longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

Thus $\lim_{n\to\infty} \|T_1x_n - x_n\| = 0$. Next, we want to prove (ii). Since

$$\|z_n - x_n\| \le a_n\|T_1(PT_1)^{n-1}x_n - x_n\| + \delta_n\|u_n - x_n\| \longrightarrow 0 \text{ as } n \to \infty,$$

we obtain

$$\begin{aligned}\|T_2(PT_2)^{n-1}x_{n+1} - x_{n+1}\| &\le \|T_2(PT_2)^{n-1}x_{n+1} - T_2(PT_2)^{n-1}x_n\| \\ &\quad + \|T_2(PT_2)^{n-1}z_n - T_2(PT_2)^{n-1}x_n\| \\ &\quad + \|T_2(PT_2)^{n-1}z_n - x_n\| + \|x_{n+1} - x_n\| \\ &\le L\|x_{n+1} - x_n\| + L\|z_n - x_n\| + \|T_2(PT_2)^{n-1}z_n - x_n\| \\ &\quad + \|x_{n+1} - x_n\| \longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|T_2x_n - x_n\| &\le \|T_2(PT_2)^{n-1}x_n - x_n\| + \|T_2(PT_2)^{n-1}x_n - T_2x_n\| \\ &\le \|T_2(PT_2)^{n-1}z_n - T_2(PT_2)^{n-1}x_n\| + \|T_2(PT_2)^{n-1}z_n - x_n\| \\ &\quad + L\|T_2(PT_2)^{n-2}x_n - x_n\| \longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

Thus $\lim_{n\to\infty} \|T_2x_n - x_n\| = 0$, so (ii) is obtained. Since

$$\begin{aligned}\|y_n - x_n\| &\le b_n\|T_2(PT_2)^{n-1}z_n - x_n\| + c_n\|T_1(PT_1)^{n-1}x_n - x_n\| \\ &\quad + \sigma_n\|v_n - x_n\| \longrightarrow 0\end{aligned}$$

and $\|T_3(PT_3)^{n-1}y_n - x_n\| \longrightarrow 0$ as $n \to \infty$, we obtain

$$\begin{aligned}\|T_3(PT_3)^{n-1}x_n - x_n\| &\le \|T_3(PT_3)^{n-1}y_n - T_3(PT_3)^{n-1}x_n\| \\ &\quad + \|T_3(PT_3)^{n-1}y_n - x_n\| \\ &\le L\|y_n - x_n\| + \|T_3(PT_3)^{n-1}y_n - x_n\| \\ &\longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

Thus

$$\begin{aligned}\|T_3(PT_3)^{n-1}x_{n+1} - x_{n+1}\| &\le \|T_3(PT_3)^{n-1}x_{n+1} - T_3(PT_3)^{n-1}x_n\| \\ &\quad + \|T_3(PT_3)^{n-1}y_n - T_3(PT_3)^{n-1}x_n\| \\ &\quad + \|T_3(PT_3)^{n-1}y_n - x_n\| + \|x_{n+1} - x_n\| \\ &\le L\|x_{n+1} - x_n\| + L\|y_n - x_n\| + \|T_3(PT_3)^{n-1}y_n - x_n\| \\ &\quad + \|x_{n+1} - x_n\| \longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

It follows that,

$$\begin{aligned}\|T_3x_n - x_n\| &\leq \|T_3(PT_3)^{n-1}x_n - x_n\| + \|T_3(PT_3)^{n-1}x_n - T_3x_n\|\\ &\leq \|T_3(PT_3)^{n-1}x_n - x_n\| + L\|T_3(PT_3)^{n-2}x_n - x_n\|\\ &\longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

Hence (iii) is satisfied. $\square$

Theorem 2.3. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive mappings with respect to sequences $\{k_n\}, \{l_n\}, \{m_n\}$, respectively, such that $F \neq \emptyset, k_n \geq 1$, $l_n \geq 1, m_n \geq 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$ and $\sum_{n=1}^{\infty}(m_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ be real sequences in $[0, 1]$ such that $a_n + \delta_n$, $b_n + c_n + \sigma_n$ and $\alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0, 1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty}\delta_n < \infty$, $\sum_{n=1}^{\infty}\sigma_n < \infty$, $\sum_{n=1}^{\infty}\rho_n < \infty$ and let $\{u_n\}, \{v_n\}, \{w_n\}$ be bounded sequences in C. Assume that T_1, T_2, T_3 are uniformly L-Lipschitzian. If one of $T_i (i = 1, 2, 3)$ is a completely continuous and one of the following conditions (C1)-(C5) is satisfied:*

(C1) $0 < \lim_{n\to\infty}\inf a_n \leq \lim_{n\to\infty}\sup(a_n + \delta_n) < 1$,
$0 < \lim_{n\to\infty}\inf b_n \leq \lim_{n\to\infty}\sup(b_n + c_n + \sigma_n) < 1$, *and*
$0 < \lim_{n\to\infty}\inf \alpha_n \leq \lim_{n\to\infty}\sup(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$.

(C2) $0 < \lim_{n\to\infty}\inf b_n, \lim_{n\to\infty}\inf c_n \leq \lim_{n\to\infty}\sup(b_n + c_n + \sigma_n) < 1$, *and*
$0 < \lim_{n\to\infty}\inf \alpha_n \leq \lim_{n\to\infty}\sup(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$.

(C3) $0 < \lim_{n\to\infty}\inf b_n \leq \lim_{n\to\infty}\sup(b_n + c_n + \sigma_n) < 1$, *and*
$0 < \lim_{n\to\infty}\inf \alpha_n, \lim_{n\to\infty}\inf \gamma_n \leq \lim_{n\to\infty}\sup(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$.

(C4) $\lim_{n\to\infty}\inf b_n > 0$, *and* $0 < \lim_{n\to\infty}\inf a_n \leq \lim_{n\to\infty}\sup(a_n + \delta_n) < 1$, *and*
$0 < \lim_{n\to\infty}\inf \alpha_n, \lim_{n\to\infty}\inf \beta_n \leq \lim_{n\to\infty}\sup(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$.

(C5) $0 < \lim_{n\to\infty}\inf \alpha_n, \lim_{n\to\infty}\inf \beta_n, \lim_{n\to\infty}\inf \gamma_n \leq \lim_{n\to\infty}\sup(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$.

Then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ defined as in (1) converge strongly to a common fixed point of T_1, T_2 and T_3.

Proof. Suppose one of the conditions (C1)-(C5) is satisfied. By Lemma 2.2, we obtain $\lim_{n\to\infty}\|T_ix_n - x_n\| = 0$ for $i = 1, 2, 3$. Assume one of T_1, T_2 and T_3 says T_1 is completely continuous. Since $\{x_n\}$ is a bounded sequence in C, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{T_1x_{n_k}\}$ converges to $q \in C$. Since $\|x_{n_k} - q\| \leq \|T_1x_{n_k} - x_{n_k}\| + \|T_1x_{n_k} - q\|$, we get $\lim_{k\to\infty}\|x_{n_k} - q\| = 0$. Thus $\{x_{n_k}\}$ converges to $q \in C$. By continuity of T_i, we have $T_ix_{n_k} \to T_iq$ as $k \to \infty$. Since

$$\|T_iq - q\| \leq \|T_ix_{n_k} - T_iq\| + \|T_ix_{n_k} - x_{n_k}\| + \|x_{n_k} - q\| \to 0 \text{ as } k \to \infty,$$

we obtain $T_iq = q$ $(i = 1, 2, 3)$. Thus $q \in F$. By Lemma 2.1 (i), $\lim_{n\to\infty}\|x_n - q\|$ exists. This implies $\lim_{n\to\infty}\|x_n - q\| = 0$. By Lemma 2.1, we have

$$\|T_1(PT_1)^{n-1}x_n - x_n\| \to 0 \text{ and } \|T_2(PT_2)^{n-1}z_n - x_n\| \to 0 \text{ as } n \to \infty.$$

It follows that

$$\|y_n - x_n\| \leq b_n\|T_2(PT_2)^{n-1}z_n - x_n\| + c_n\|T_1(PT_1)^{n-1}x_n - x_n\| + \sigma_n\|v_n - x_n\|$$

$\longrightarrow 0$ and $\|z_n - x_n\| \leq a_n\|T_1(PT_1)^{n-1}x_n - x_n\| + \delta_n\|u_n - x_n\| \to 0$ as $n \to \infty$. These imply $\lim_{n\to\infty} y_n = q$ and $\lim_{n\to\infty} z_n = q$. $\square$

Remark 2.4. In Theorem 2.3, assume that T_1, T_2 and T_3 are asymptotically nonexpansive self-mappings of C such that one of them is completely continuous and thus T_1, T_2, T_3 are uniformly

L-Lipschitzian and $\delta_n = \sigma_n = \rho_n \equiv 0$. We obtain the following results.

(1) If one of the conditions $(C1)-(C5)$ is satisfied, then the sequences $\{x_n\}$, $\{y_n\}, \{z_n\}$ defined as in (2) converge strongly to a common fixed point of T_1, T_2 and T_3.

(2) If $T := T_1 = T_2 = T_3$ and one of the conditions $(C1)-(C5)$ is satisfied, then we obtain the results of Nilsrakoo and Saejung [8].

(3) If $T := T_1 = T_2 = T_3$ and one of the conditions $(C1), (C2), (C4)$ is satisfied and $\gamma_n \equiv 0$, then we obtain the results of Suantai [16].

(4) If $T := T_1 = T_2 = T_3$ with condition $(C1)$ is satisfied and $c_n = \beta_n = \gamma_n \equiv 0$, then we obtain the results of Xu and Noor [18].

(5) If the condition $(C5)$ is satisfied and $a_n = b_n = c_n \equiv 0$, then the sequence $\{x_n\}$ defined as in (3) converges strongly to a common fixed point of T_1, T_2 and T_3.

The mapping $T : C \to X$ with $F(T) \neq \emptyset$ is said to satisfy *Condition A* [14] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf\{\|x-q\| : q \in F(T)\}$. As Tan and Xu [17] pointed out, the Condition A is weaker than the compactness of C.

The following result gives a strong convergence theorem for asymptotically quasi-nonexpansive nonself-mappings in a uniformly convex Banach space satisfying Condition A.

Theorem 2.5. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive mappings with respect to sequences $\{k_n\}, \{l_n\}, \{m_n\}$, respectively, such that $F \neq \emptyset, k_n \geq 1$, $l_n \geq 1, m_n \geq 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$ and $\sum_{n=1}^{\infty}(m_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ be real sequences in $[0,1]$ such that $a_n + \delta_n$, $b_n + c_n + \sigma_n$ and $\alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty}\delta_n < \infty$, $\sum_{n=1}^{\infty}\sigma_n < \infty$, $\sum_{n=1}^{\infty}\rho_n < \infty$ and let $\{u_n\}, \{v_n\}, \{w_n\}$ be bounded sequences in C. Suppose T_1 satisfies Condition A and T_2, T_3 are uniformly L-Lipschitzian and one of the conditions (C1)-(C5) in Theorem 2.3 is satisfied. Then the sequence $\{x_n\}$ defined as in (1) converges strongly to a common fixed point of T_1, T_2 and T_3.*

Proof. Let $q \in F$. By Lemma 2.1, $\lim_{n\to\infty}\|x_n - q\|$ exists. Thus $\{x_n - q\}$ is bounded. Then there is a constant H such that $\|x_n - q\| \leq H$ for all $n \geq 1$. This together with (6), we have

$$\|x_{n+1} - q\| \leq \|x_n - q\| + D_n, \tag{21}$$

where $D_n = KH\big((k_n - 1) + (l_n - 1) + (m_n - 1)\big) + K(\delta_n + \sigma_n + \rho_n) < \infty$ for all $n \geq 1$. By Lemma 2.2, we have $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ $(i = 1,2,3)$. Since T_1 satisfies Condition A, we obtain $\lim_{n\to\infty} d(x_n, F(T_1)) = 0$. Next, we want to show $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, F(T_1)) = 0$ and $\sum_{n=1}^{\infty} D_n < \infty$, for any $\epsilon > 0$, there exists a positive integer n_0 such that $d(x_n, F(T_1)) < \epsilon/4$ and $\sum_{k=n_0}^{n} D_k < \epsilon/2$ for all $n \geq n_0$. Now, let $n \in \mathbb{N}$ be such that $n \geq n_0$. Then we can find $q^* \in F$ such that $\|x_n - q^*\| < \epsilon/4$. This implies by (21) that for $m \geq 1$,

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - q^*\| + \|x_n - q^*\| \\ &\leq 2\|x_n - q^*\| + \sum_{k=n}^{n+m-1} D_k \\ &= 2\|x_n - q^*\| + \sum_{k=n_0}^{n+m-1} D_k < 2\big(\frac{\epsilon}{4}\big) + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence and so it is convergent. Let $\lim_{n\to\infty} x_n = p$. Since $d(x_n, F(T_1)) \to 0$ as $n \to \infty$, it follows that $d(p, F(T_1)) = 0$ and hence $p \in F(T_1)$. Next, we want to show $p \in F(T_2) \cap F(T_3)$. Since T_2, T_3 are uniformly L-Lipschitzian and by Lemma 2.2, we obtain

$$\begin{aligned} \|T_i p - p\| &\le \|T_i x_n - T_i p\| + \|T_i x_n - x_n\| + \|x_n - p\| \\ &\le L\|x_n - p\| + \|T_i x_n - x_n\| + \|x_n - p\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus $T_i p = p$ $(i = 2, 3)$. Therefore $p \in F$. □

In the next result, we prove weak convergence for the iterative scheme (1) for asymptotically quasi-nonexpansive nonself-mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.6. *Let X be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive mappings with respect to sequences $\{k_n\}, \{l_n\}, \{m_n\}$, respectively, such that $F \neq \emptyset, k_n \ge 1, l_n \ge 1, m_n \ge 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$ and $\sum_{n=1}^{\infty}(m_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ be real sequences in $[0, 1]$ such that $a_n + \delta_n$, $b_n + c_n + \sigma_n$ and $\alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0, 1]$ for all $n \ge 1$ and $\sum_{n=1}^{\infty}\delta_n < \infty$, $\sum_{n=1}^{\infty}\sigma_n < \infty$, $\sum_{n=1}^{\infty}\rho_n < \infty$ and let $\{u_n\}, \{v_n\}, \{w_n\}$ be bounded sequences in C. Suppose T_1, T_2, T_3 are uniformly L-Lipschitzian and $I - T_i$ $(i = 1, 2, 3)$ is demiclosed at 0. If one of the following conditions (C1)-(C5) in Theorem 2.3 is satisfied, then the sequence $\{x_n\}$ defined as in (1) converges weakly to a common fixed point of T_1, T_2 and T_3.*

Proof. Assume one of the conditions (C1)-(C5) is satisfied. By Lemma 2.1 and Lemma 2.2, we have $\lim_{n\to\infty}\|T_i x_n - x_n\| = 0$ $(i = 1, 2, 3)$. Since X is uniformly convex and $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_n \to u$ weakly as $n \to \infty$. Since $I - T_i$ is demiclosed at 0, we obtain $u \in F$. Suppose subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. Also, since $I - T_i (i = 1, 2, 3)$ is demiclosed at 0, we have u and $v \in F$. By Lemma 2.1, we obtain $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. It follows from Lemma 1.4 that $u = v$. Therefore $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3. □

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