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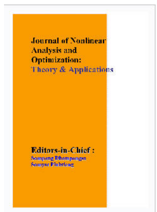


# An iterative method for finding common solutions of generalized mixed equilibrium problems and fixed point problems

**Benjawan Rodjanadid**

**ABSTRACT**: In this paper, we introduce an iterative method for finding a common element of the set of solutions of a generalized mixed equilibrium problem and the set of common fixed points of a finite family of nonexpansive mappings in a real Hilbert space. Then, we prove that the sequence converges strongly to a common element of the above two sets. Furthermore, we apply our result to prove three new strong convergence theorems in fixed point problems, mixed equilibrium problems, generalized equilibrium problems and equilibrium problems.

## 1. Introduction

Let $H$ be a real Hilbert space, $C$ a nonempty closed convex subset of $H$, $\varphi : C \to \mathbb{R}$ a real value function, $A : C \to H$ a nonlinear mapping and let $\Phi : C \times C \to \mathbb{R}$ be a bifunction, i.e., $\Phi(x, x) = 0$ for each $x \in C$. Then, we consider the following mixed equilibrium problem :
Find $x^* \in C$ such that

$$(1) \qquad (GMEP): \qquad \Phi(x^*, y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^* \rangle \geq 0, \qquad \forall y \in C.$$

The set of solutions for problem (1) is denoted by $\Omega$, i.e.,

$$(2) \qquad \Omega = \{x^* \in C : \Phi(x^*, y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C\}.$$

If $A \equiv 0$ in (1), then (GMEP) (1) reduces to the classical mixed equilibrum problem (for short, MEP) and $\Omega$ is denoted by $MEP(\Phi, \varphi)$, that is,

$$(3) \qquad MEP(\Phi, \varphi) = \{x^* \in C : \Phi(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C\}.$$

If $\varphi \equiv 0$ in (1), then (GMEP) (1) reduces to the generalized equilibrium problem (for short, GEP) and $\Omega$ is denoted by $EP$, that is,

$$(4) \qquad EP = \{x^* \in C : \Phi(x^*, y) + \langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C\}.$$

If $\varphi \equiv 0$ and $A \equiv 0$ in (1), then (GMEP) (1) reduces to the classical equilibrium problem (for short, EP) and $\Omega$ is denoted by $EP(\Phi)$, that is,

$$(5) \qquad EP(\Phi) = \{x^* \in C : \Phi(x^*, y) \geq 0, \quad \forall y \in C\}.$$

*Corresponding author: benjawan@sut.ac.th (Benjawan Rodjanadid)
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If $\Phi \equiv 0$ and $\varphi \equiv 0$ in (1), then (GMEP) (1) reduces to the classical variational inequality and $\Omega$ is denoted by $VI(A,C)$, that is,

$$VI(A,C) = \{x^* \in C : \langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C\}. \tag{6}$$

In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to the initial data when $EP(\Phi) \neq \emptyset$ and proved a strong convergence theorem.

In 2006, Takahashi and Takahashi [14] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of nonexpansive mapping in a Hilbert space and proved a strong convergence theorem.

In 2007, Tada and Takahashi [12] introduced two iterative schemes for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space and obtained both strong convergence theorem and weak convergence theorem. In 2008, Takahashi and Takahashi [13] introduced an iterative method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space and then obtain that the sequence converges strongly to a common element of two sets. Moreover they proved three new strong convergence theorems in fixed point problems, variational inequalities and equilibrium problems.

Recently, Ceng and Yao [2] introduced a hybrid iterative scheme for finding a common element of the set of solutions of mixed equilibrium problem (3) and the set of common fixed points of finitely many nonexpansive mappings and they proved that the sequences generated by the hybrid iterative scheme converge strongly to a common element of the set of solutions of mixed equilibrium problem and the set of common fixed points of finitely many nonexpansive mappings.

In 2008, Peng and Yao [9] obtained some strong convergence theorems for iterative schemes based on the hybrid method and the extragradient method for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality.

In this paper, we introduced another iterative method for finding an element of the set of solutions of problem (1) and the set of common fixed points of finitely many nonexpansive mappings in real Hilbert space, where $A : C \to H$ is also an $\alpha-$inverse strongly monotone mapping and then obtain a strong convergence theorem. Moreover we using this theorem to the problem for finding a common elements of $\cap_{i=1}^{N} F(T_i) \cap MEP(\Phi, \varphi)$ , $\cap_{i=1}^{N} F(T_i) \cap EP$ and $\cap_{i=1}^{N} F(T_i) \cap EP(\Phi)$, respectively.

## 2. **Preliminaries**

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let symbols $\rightharpoonup$ and $\to$ denote weak and strong convergence, respectively. Let $C$ be a nonempty closed convex subset of $H$. Then, for any $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C(x)$ such that $\|x - P_C(x)\| \leq \|x - y\|, \ \forall y \in C$. The mapping $P_C : x \to P_C(x)$ is called the *metric projection* of $H$ onto $C$. We know that $P_C$ is nonexpansive.

The following characterizes the projection $P_C$.

**Lemma 2.1.** *(See* [11]*) Given $x \in H$ and $y \in C$. Then $P_C(x) = y$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \ \ \forall z \in C.$$

Recall that the following definitions.

(1) A mapping $T : C \to C$ is called **nonexpansive** if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Next, we denote by $F(T)$ the set of fixed points of $T$, i.e., $F(T) = \{x \in C : Tx = x\}$.

(2) A mapping $f : H \to H$ is said to be a **contraction** if there exists a constant $\rho \in (0,1)$ such that $\|f(x) - f(y)\| \leq \rho\|x - y\|$ for all $x, y \in H$.

(3) A mapping $A : C \to H$ is called **monotone** if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in C$ and it is called $\alpha-$**inverse strongly monotone** if there exists a positive real number $\alpha$ such that $\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C$. We can see that if $A$ is $\alpha-$**inverse strongly monotone**, then $A$ is monotone mapping.

The following lemmas will be useful for proving our main results.

**Lemma 2.2.** *(See [11]) For all $x, y \in H$, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

**Lemma 2.3.** *(See [11]) In a strictly convex Banach space $E$, if*

$$\|x\| = \|y\| = \|\lambda x + (1 - \lambda)y\|,$$

*for all $x, y \in E$ and $\lambda \in (0, 1)$, then $x = y$.*

**Lemma 2.4.** *(See [16]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying $a_{n+1} = (1 - \alpha_n)a_n + \alpha_n\beta_n, \ \forall n \geq 0$ where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions*

(i) $\{\alpha_n\} \subset [0, 1], \quad \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$.

*Then $\lim_{n\to\infty} a_n = 0$.*

**Lemma 2.5.** *(See [10]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $X$ and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n,$$

*for all integer $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

*Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

**Lemma 2.6.** *(See [15]) Let $C$ be a nonempty closed convex subset of $H$ , $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function and let $\Phi$ be a bifunction of $C \times C$ in to $\mathbb{R}$ satisfy*

(A1) *$\Phi(x, x) = 0$ for all $x \in C$;*

(A2) *$\Phi$ is monotone, i.e., $\Phi(x, y) + \Phi(y, x) \leq 0, \quad \forall x, y \in C$;*

(A3) *for all $x, y, z \in C, \quad \lim_{t\to 0} \Phi(tz + (1 - t)x, y) \leq \Phi(x, y)$;*

(A4) *for all $x \in C, \quad y \mapsto \Phi(x, y)$ is convex and lower semicontinuous;*

(B1) *for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C\backslash D_x$,*

$$\Phi(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z).$$

(B2) *$C$ is bounded set.*

*Assume that either (B1) or (B2) holds. For $x \in C$ and $r > 0$, define a mapping $T_r : H \to C$ as follows.*

$$T_r(x) := \{z \in C : \Phi(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z), \ \forall y \in C\}$$

*for all $x \in H$. Then , the following conditions hold:*

(i) *For each $x \in H, T_r(x) \neq \emptyset$;*

(ii) *$T_r$ is single-valued;*

(iii) *$T_r$ is firmly nonexpansive, i.e.,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \ \forall x, y \in H;$$

(iv) *$F(T_r) = MEP(\Phi, \varphi)$ ;*

(v) *$MEP(\Phi, \varphi)$ is closed and convex.*

**Lemma 2.7.** *(see [1]) Let $C$ be a nonempty closed convex subset of $H$, and let $\Phi$ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\Phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

**Lemma 2.8.** *(see [3]) Assume that $\Phi : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$, define a mapping $S_r : H \to C$ as follows:*

$$S_r(x) = \{z \in C : \Phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}$$

*for all $x \in H$. Then, the following hold:*

(i) *$S_r$ is single-valued;*
(ii) *$S_r$ is firmly nonexpansive;*
(iii) *$F(S_r) = EP(\Phi)$;*
(iv) *$EP(\Phi)$ is closed and convex.*

Let $X$ be a real Hilbert space and $C$ a nonempty closed convex subset of $X$. For a finite family of nonexpansive mappings $T_1, T_2, \ldots, T_N$ and sequence $\{\lambda_{n,i}\}_{i=1}^N$ in $[0,1]$, Kangtunyakarn and Suantai [6] defined the mapping $K_n : C \to C$ as follows:

$$\begin{aligned}
U_{n,1} &= \lambda_{n,1}T_1 + (1-\lambda_{n,1})I,\\
U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1-\lambda_{n,2})U_{n,1},\\
U_{n,3} &= \lambda_{n,2}T_3U_{n,2} + (1-\lambda_{n,3})U_{n,2},\\
&\vdots\\
U_{n,N-1} &= \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1-\lambda_{n,N-1})U_{n,N-2},\\
K_n &= U_{n,N} = \lambda_{n,N}T_NU_{n,N-1} + (1-\lambda_{n,N})U_{n,N-1}
\end{aligned} \tag{7}$$

Such a mapping $K_n$ is called the $K-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$.

**Definition 2.9.** (See [6]) Let $C$ be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mapping of $C$ into itself, and let $\lambda_1, \ldots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, \ldots, N$. They define a mapping $K : C \to C$ as follows:

$$\begin{aligned}
U_1 &= \lambda_1T_1 + (1-\lambda_1)I,\\
U_2 &= \lambda_2T_2U_1 + (1-\lambda_2)U_1,\\
U_3 &= \lambda_3T_3U_2 + (1-\lambda_3)U_2,\\
&\vdots\\
U_{N-1} &= \lambda_{N-1}T_{N-1}U_{N-2} + (1-\lambda_{N-1})U_{N-2},\\
K &= U_N = \lambda_NT_NU_{N-1} + (1-\lambda_N)U_{N-1}.
\end{aligned}$$

Such a mapping $K$ is called the *$K-$mapping* generated by $T_1, \ldots, T_N$ and $\lambda_1, \ldots, \lambda_N$.

**Lemma 2.10.** *(See [6]) Let $C$ be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of $C$ into itself with $\cap_{i=1}^N F(T_i) \neq \emptyset$ and let $\lambda_1, \ldots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, \ldots, N-1$ and $0 < \lambda_N \leq 1$. Let $K$ be the $K-$mapping generated by $T_1, \ldots, T_N$ and $\lambda_1, \ldots, \lambda_N$. Then $F(K) = \cap_{i=1}^N F(T_i)$.*

**Lemma 2.11.** *(See [6]) Let $C$ be a nonempty closed convex subset of a Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of $C$ into itself and $\{\lambda_{n,i}\}_{i=1}^N$ sequences in $[0,1]$ such that $\lambda_{n,i} \to \lambda_i$, as $n \to \infty$ $(i = 1, 2, \ldots, N)$. Moreover, for every $n \in \mathbb{N}$, let $K$ and $K_n$ be the $K-$mappings generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$, respectively. Then, for every $x \in C$,*

$$\lim_{n\to\infty} \|K_nx - Kx\| = 0.$$

**Lemma 2.12.** *Let $\{x_n\}$ be a bounded sequence in a Hilbert space $H$. Then there exits $L > 0$ such that*

$$\|K_{n+1}x_{n+1} - K_n x_n\| \leq \|x_{n+1} - x_n\| + L\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|, \ \ \forall n \geq 0. \tag{8}$$

*Proof.* From (7) and the nonexpansivity of $T_N$ and $U_{n,N}$, we obtain

$$\begin{aligned}
\|K_{n+1}x_n - K_n x_n\| &= \|\lambda_{n+1,N}T_N U_{n+1,N-1}x_n + (1-\lambda_{n+1,N})U_{n+1,N-1}x_n \\
&\quad - \lambda_{n,N}T_N U_{n,N-1}x_n - (1-\lambda_{n,N})U_{n,N-1}x_n\| \\
&= \|\lambda_{n+1,N}T_N U_{n+1,N-1}x_n + U_{n+1,N-1}x_n - \lambda_{n+1,N}U_{n+1,N-1}x_n \\
&\quad - \lambda_{n,N}T_N U_{n,N-1}x_n - U_{n,N-1}x_n + \lambda_{n,N}U_{n,N-1}x_n\| \\
&\leq \|\lambda_{n+1,N}T_N U_{n+1,N-1}x_n - \lambda_{n,N}T_N U_{n,N-1}x_n\| + \|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
&\quad + \|\lambda_{n+1,N}U_{n+1,N-1}x_n - \lambda_{n,N}U_{n,N-1}x_n\| \\
&= \|\lambda_{n+1,N}T_N U_{n+1,N-1}x_n - \lambda_{n+1,N}T_N U_{n,N-1}x_n + \lambda_{n+1,N}T_N U_{n,N-1}x_n \\
&\quad - \lambda_{n,N}T_N U_{n,N-1}x_n\| + \|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| + \|\lambda_{n+1,N}U_{n+1,N-1}x_n \\
&\quad - \lambda_{n+1,N}U_{n,N-1}x_n + \lambda_{n+1,N}U_{n,N-1}x_n - \lambda_{n,N}U_{n,N-1}x_n\| \\
&\leq \lambda_{n+1,N}\|T_N U_{n+1,N-1}x_n - T_N U_{n,N-1}x_n\| + |\lambda_{n+1,N} - \lambda_{n,N}|\|T_N U_{n,N-1}x_n\| \\
&\quad + \|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| + \lambda_{n+1,N}\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
&\quad + |\lambda_{n+1,N} - \lambda_{n,N}|\|U_{n,N-1}x_n\| \\
&\leq \lambda_{n+1,N}\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| + \|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
&\quad + \lambda_{n+1,N}\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| + |\lambda_{n+1,N} - \lambda_{n,N}|\|U_{n,N-1}x_n\| \\
&\quad + |\lambda_{n+1,N} - \lambda_{n,N}|\|T_N U_{n,N-1}x_n\| \\
&\leq (2\lambda_{n+1,N} + 1)\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| + 2L_1|\lambda_{n+1,N} - \lambda_{n,N}|, \qquad (9)
\end{aligned}$$

where $L_1 = \sup_{n\geq 0}\{\|U_{n,j-1}x_n\|, \|T_N U_{n,j-1}x_n\|\}, \ \ j = 1, 2, \ldots, N$.
Again, from (7), we have

$$\begin{aligned}
\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| &= \|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}x_n + (1-\lambda_{n+1,N-1})U_{n+1,N-2}x_n \\
&\quad - \lambda_{n,N-1}T_{N-1}U_{n,N-2}x_n - (1-\lambda_{n,N-1})U_{n,N-2}x_n\| \\
&\leq \|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}x_n - \lambda_{n,N-1}T_{N-1}U_{n,N-2}x_n\| \\
&\quad + \|U_{n+1,N-2}x_n - U_{n,N-2}x_n\| + \|\lambda_{n+1,N-1}U_{n+1,N-2}x_n \\
&\quad - \lambda_{n,N-1}U_{n,N-2}x_n\|
\end{aligned}$$

$$\begin{aligned}
&\leq \lambda_{n+1,N-1}\|U_{n+1,N-2}x_n - U_{n,N-2}x_n\| + |\lambda_{n+1,N-1} - \lambda_{n,N-1}| \\
&\quad \times \|T_{N-1}U_{n,N-2}x_n\| + \|U_{n+1,N-2}x_n - U_{n,N-2}x_n\| \\
&\quad + \lambda_{n+1,N-1}\|U_{n+1,N-2}x_n - U_{n,N-2}x_n\| + |\lambda_{n+1,N-1} - \lambda_{n,N-1}| \\
&\quad \times \|U_{n,N-2}x_n\| \\
&\leq (2\lambda_{n+1,N-1} + 1)\|U_{n+1,N-2}x_n - U_{n,N-2}x_n\| + 2L_1|\lambda_{n+1,N-1} - \lambda_{n,N-1}|. \qquad (10)
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
&\quad \leq (2\lambda_{n+1,N-1}+1)(2\lambda_{n+1,N-2}+1)\|U_{n+1,N-3}x_n - U_{n,N-3}\| \\
&\quad\quad + (2\lambda_{n+1,N-1}+1)2L_1|\lambda_{n+1,N-2}-\lambda_{n,N-2}| + 2L_1|\lambda_{n+1,N-1}-\lambda_{n,N-1}| \\
&\quad \leq \prod_{i=N-1}^{2}(2\lambda_{n+1,i}+1)\|U_{n+1,1}x_n - U_{n,1}x_n\| + \prod_{i=N-1}^{3}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,2}-\lambda_{n,2}| \\
&\quad\quad + \prod_{i=N-1}^{4}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,3}-\lambda_{n,3}| + \prod_{i=N-1}^{5}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,4}-\lambda_{n,4}| \\
&\quad + \ldots + \prod_{i=N-1}^{N-1}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,N-2}-\lambda_{n,N-2}| + 2L_1|\lambda_{n+1,N-1}-\lambda_{n,N-1}| \\
&\quad = \prod_{i=N-1}^{2}(2\lambda_{n+1,i}+1)\|\lambda_{n+1,1}T_1x_n + (1-\lambda_{n+1,1})x_n - \lambda_{n,1}T_1x_n - (1-\lambda_{n,1})x_n\| \\
&\quad\quad + \prod_{i=N-1}^{3}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,2}-\lambda_{n,2}| + \prod_{i=N-1}^{4}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,3}-\lambda_{n,3}| \\
&\quad\quad + \prod_{i=N-1}^{5}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,4}-\lambda_{n,4}| + \ldots + \\
&\quad\quad + \prod_{i=N-1}^{N-1}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,N-2}-\lambda_{n,N-2}| + 2L_1|\lambda_{n+1,N-1}-\lambda_{n,N-1}|,
\end{aligned}
$$

then

$$
\begin{aligned}
&\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
&\quad \leq \prod_{i=N-1}^{2}(2\lambda_{n+1,i}+1)(|\lambda_{n+1,1}-\lambda_{n,1}|\|T_1x_n\| + |\lambda_{n+1,1}-\lambda_{n,1}|\|x_n\|) \\
&\quad\quad + \prod_{i=N-1}^{3}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,2}-\lambda_{n,2}| + \prod_{i=N-1}^{4}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,3}-\lambda_{n,3}| \\
&\quad\quad + \prod_{i=N-1}^{5}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,4}-\lambda_{n,4}| + \ldots + \\
&\quad + \prod_{i=N-1}^{N-1}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,N-2}-\lambda_{n,N-2}| + 2L_1|\lambda_{n+1,N-1}-\lambda_{n,N-1}| \\
&\leq \prod_{i=N-1}^{2}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,1}-\lambda_{n,1}| \\
&\quad + \prod_{i=N-1}^{3}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,2}-\lambda_{n,2}| + \prod_{i=N-1}^{4}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,3}-\lambda_{n,3}| \\
&\quad + \prod_{i=N-1}^{5}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,4}-\lambda_{n,4}| + \ldots + \\
&\quad + \prod_{i=N-1}^{N-1}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,N-2}-\lambda_{n,N-2}| + 2L_1|\lambda_{n+1,N-1}-\lambda_{n,N-1}| \qquad (11)
\end{aligned}
$$

Substituting (11) in (9), we have

$$
\begin{aligned}
&\|K_{n+1}x_n - K_n x_n\| \\
&\quad \leq \prod_{i=N}^{2}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,1}-\lambda_{n,1}| + \prod_{i=N}^{3}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,2}-\lambda_{n,2}| \\
&\qquad + \prod_{i=N}^{4}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,3}-\lambda_{n,3}| + \prod_{i=N}^{5}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,4}-\lambda_{n,4}| \\
&\qquad + \ldots + \prod_{i=N}^{N-1}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,N-2}-\lambda_{n,N-2}| \\
&\qquad + \prod_{i=N}^{N}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,N-1}-\lambda_{n,N-1}| + 2L_1|\lambda_{n+1,N}-\lambda_{n,N}| \\
&\quad \leq L\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|, \qquad (12)
\end{aligned}
$$

where $L = \prod_{i=N}^{2}(2\lambda_{n+1,i}+1)2L_1$. It follows that

$$
\begin{aligned}
\|K_{n+1}x_{n+1} - K_n x_n\| &\leq \|K_{n+1}x_{n+1} - K_{n+1}x_n\| + \|K_{n+1}x_n - K_n x_n\| \\
&\leq \|x_{n+1} - x_n\| + L\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|.
\end{aligned}
$$

□

## 3. **Main Results**

In this section, we deal with an iterative scheme by the approximation method for finding a common element of the set of common fixed points of finite family of nonexpansive mappings and the set of solutions of GMEP (1) in real Hilbert spaces.

**Theorem 3.1.** *Let $H$ be a Hilbert space, $C$ a closed convex nonempty subset of $H$, $\varphi : C \to \mathbb{R}$ a proper lower semicontinuous and convex functional, $A$ an $\alpha-$inverse strongly monotone mapping of $C$ into $H$, $\Phi : C \times C \to \mathbb{R}$ a bifunction satisfying (A1)-(A4), $\{T_i\}_{i=1}^{N}$ a finite family of nonexpansive mappings of $C$ into itself such that $\cap_{i=1}^{N} F(T_i) \cap \Omega \neq 0$ and $f$ a $\rho-$contraction of $C$ into itself. Moreover, let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\{\lambda_{n,i}\}_{i=1}^{N}$ a sequence in $[a,b]$ with $0 < a \leq b < 1$ and $\{r_n\}$ a sequence in $[0, 2\alpha]$ for all $n \in \mathbb{N}$. Assume that:*

- (i) *either (B1) or (B2) holds;*
- (ii) *the sequence $\{r_n\}$ satisfies*
  - (C1) $0 < c \leq r_n \leq d < 2\alpha$; *and*
  - (C2) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;
- (iii) *the sequence $\{\alpha_n\}$ satisfies*
  - (D1) $\lim_{n\to\infty} \alpha_n = 0$; *and*
  - (D2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (iv) *the sequence $\{\beta_n\}$ satisfies*
  - (E1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
- (v) *the finite family of sequences $\{\lambda_{n,i}\}_{i=1}^{N}$ satisfies*
  - (F1) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every* $i \in \{1, 2, \ldots, N\}$.

*For every $n \in \mathbb{N}$, let $K_n$ be a $K-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$ and let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0 \in C$ and*

$$\begin{cases} \Phi(y_n, x) + \varphi(x) - \varphi(y_n) + \langle Ax_n, x - y_n\rangle + \frac{1}{r_n}\langle x - y_n, y_n - x_n\rangle \geq 0, \ \ \forall x \in C \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n y_n. \end{cases} \tag{13}$$

*Then both $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^* = P_\Gamma f(x^*)$ where $\Gamma = \cap_{i=1}^N F(T_i) \cap \Omega$*

*Proof.* Let $x, y \in C$. Since $A$ is $\alpha-$strongly monotone and $r_n \in (0, 2\alpha)\ \ \forall n \in \mathbb{N}$, we have

$$\begin{aligned} \|(I - r_nA)x - (I - r_nA)y\|^2 &= \|x - y - r_n(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2r_n\alpha\|Ax - Ay\|^2 + r_n^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 + r_n(r_n - 2\alpha)\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2, \end{aligned}$$

which implies that $I - r_nA$ is nonexpansive.

Next we prove that the sequences $\{x_n\}, \{y_n\}, \{Ax_n\}, \{f(x_n)\}$ and $\{K_ny_n\}$ are bounded. Since

$$\Phi(y_n, x) + \varphi(x) - \varphi(y_n) + \langle Ax_n, x - y_n\rangle + \frac{1}{r_n}\langle x - y_n, y_n - x_n\rangle \geq 0, \ \ \forall x \in C,$$

we have

$$\Phi(y_n, x) + \varphi(x) - \varphi(y_n) + \frac{1}{r_n}\langle x - y_n, y_n - (x_n - r_nAx_n)\rangle \geq 0, \ \ \forall x \in C.$$

It follows from Lemma 2.6 that $y_n = T_{r_n}(x_n - r_nAx_n), \ \ \forall n \in \mathbb{N}$.
Let $p \in \cap_{i=1}^N F(T_i) \cap \Omega$. Then we have

$$\Phi(p, y) + \varphi(y) - \varphi(p) + \langle Ap, y - p\rangle \geq 0, \ \ \forall y \in C,$$

so

$$\Phi(p, y) + \varphi(y) - \varphi(p) + \frac{1}{r_n}\langle y - p, p - (p - r_nAp)\rangle \geq 0, \ \ \forall y \in C.$$

By Lemma 2.6, we have $p = T_{r_n}(p - r_nAp)$.
Since $T_{r_n}$ and $(I - r_nA)$ are nonexpansive, we have

$$\begin{aligned} \|y_n - p\| &= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp)\| \\ &\leq \|(x_n - r_nAx_n) - (p - r_nAp)\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{14}$$

From (13) and (14), we deduce that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n y_n - p\| \\ &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n y_n - (\alpha_n + \beta_n + \gamma_n)p\| \\ &\leq \alpha_n\|f(x_n) - p\| + \beta_n\|x_n - p\| + \gamma_n\|K_ny_n - p\| \\ &\leq \alpha_n(\|f(x_n) - f(p)\| + \|f(p) - p\|) + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\| \\ &\leq \alpha_n(\rho\|x_n - p\| + \|f(p) - p\|) + \beta_n\|x_n - p\| + \gamma_n\|x_n - p\| \\ &= \alpha_n\rho\|x_n - p\| + \beta_n\|x_n - p\| + \gamma_n\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &= \alpha_n\rho\|x_n - p\| + (1 - \alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &= (1 - \alpha_n(1 - \rho))\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &= (1 - \alpha_n(1 - \rho))\|x_n - p\| + \alpha_n(1 - \rho) \cdot \frac{1}{1 - \rho}\|f(p) - p\| \end{aligned} \tag{15}$$

It follows from (15) induction that

$$\|x_n - p\| \leq M, \quad \forall n \geq 0$$

where $M = \max\{\|x_0 - p\|, \frac{1}{1-\rho}\|f(p) - p\|\}$. So $\{x_n\}$ is bounded. Therefore $\{y_n\}, \{Ax_n\}, \{f(x_n)\}$ and $\{K_n y_n\}$ are also bounded.
Next we shall show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
Define

$$z_n = \frac{\alpha_n}{1-\beta_n} f(x_n) + \frac{\gamma_n}{1-\beta_n} K_n y_n, \tag{16}$$

we have

$$x_{n+1} = \beta_n x_n + (1-\beta_n) z_n, \quad \forall n \geq 0. \tag{17}$$

Consider

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|\frac{\alpha_{n+1}}{1-\beta_{n+1}} f(x_{n+1}) + \frac{\gamma_{n+1}}{1-\beta_{n+1}} K_{n+1} y_{n+1} - \frac{\alpha_n}{1-\beta_n} f(x_n) - \frac{\gamma_n}{1-\beta_n} K_n y_n\| \\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \|f(x_{n+1}) - f(x_n)\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}| \|f(x_n)\| \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \|K_{n+1} y_{n+1} - K_n y_n\| + |\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}| \|K_n y_n\| \\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \rho \|x_{n+1} - x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}| (\|f(x_n)\| + \|K_n y_n\|) \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \|K_{n+1} y_{n+1} - K_n y_n\|. \end{aligned} \tag{18}$$

Substituting (8) from Lemma 2.12 into (18), we have

$$\begin{aligned} \|z_{n+1} - z_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \rho \|x_{n+1} - x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}| (\|f(x_n)\| + \|K_n y_n\|) \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (\|y_{n+1} - y_n\| + L \sum_{i=1}^{N} |\lambda_{n+1,i} - \lambda_{n,i}|). \end{aligned} \tag{19}$$

Putting $u_n = x_n - r_n A x_n$. Then we have $y_{n+1} = T_{r_{n+1}} u_{n+1}, \quad y_n = T_{r_n} u_n$. Hence from the nonexpansivity of $T_{r_{n+1}}$ we have

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|T_{r_{n+1}} u_{n+1} - T_{r_n} u_n\| \\ &\leq \|T_{r_{n+1}} u_{n+1} - T_{r_{n+1}} u_n\| + \|T_{r_{n+1}} u_n - T_{r_n} u_n\| \\ &\leq \|u_{n+1} - u_n\| + \|T_{r_{n+1}} u_n - T_{r_n} u_n\|. \end{aligned} \tag{20}$$

Since $I - r_n A$ is nonexpansive for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \|u_{n+1} - u_n\| &= \|x_{n+1} - r_{n+1} A x_{n+1} - x_n + r_n A x_n\| \\ &\leq \|(I - r_{n+1} A) x_{n+1} - (I - r_{n+1} A) x_n\| + |r_n - r_{n+1}| \|A x_n\| \\ &\leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n| \|A x_n\|. \end{aligned} \tag{21}$$

By Lemma 2.6, we obtain

$$\Phi(T_{r_n} u_n, y) + \varphi(y) - \varphi(T_{r_n} u_n) + \frac{1}{r_n} \langle y - T_{r_n} u_n, T_{r_n} u_n - u_n \rangle \geq 0, \;\; \forall y \in C, \tag{22}$$

and

$$\Phi(T_{r_{n+1}} u_n, y) + \varphi(y) - \varphi(T_{r_{n+1}} u_n) + \frac{1}{r_{n+1}} \langle y - T_{r_{n+1}} u_n, T_{r_{n+1}} u_n - u_n \rangle \geq 0, \;\; \forall y \in C. \tag{23}$$

Putting $y = T_{r_{n+1}} u_n$ in (22) and $y = T_{r_n} u_n$ in (23), we have

$$\Phi(T_{r_n} u_n, T_{r_{n+1}} u_n) + \varphi(T_{r_{n+1}} u_n) - \varphi(T_{r_n} u_n) + \frac{1}{r_n} \langle T_{r_{n+1}} u_n - T_{r_n} u_n, T_{r_n} u_n - u_n \rangle \geq 0, \tag{24}$$

and

$$\Phi(T_{r_{n+1}} u_n, T_{r_n} u_n) + \varphi(T_{r_n} u_n) - \varphi(T_{r_{n+1}} u_n) + \frac{1}{r_{n+1}} \langle T_{r_n} u_n - T_{r_{n+1}} u_n, T_{r_{n+1}} u_n - u_n \rangle \geq 0. \tag{25}$$

Summing up (24) and (25) and using (A2), we have

$$\frac{1}{r_{n+1}}\langle T_{r_n}u_n - T_{r_{n+1}}u_n, T_{r_{n+1}}u_n - u_n\rangle + \frac{1}{r_n}\langle T_{r_{n+1}}u_n - T_{r_n}u_n, T_{r_n}u_n - u_n\rangle \geq 0,$$

and

$$\langle T_{r_n}u_n - T_{r_{n+1}}u_n, \frac{T_{r_{n+1}}u_n - u_n}{r_{n+1}} - \frac{T_{r_n}u_n - u_n}{r_n}\rangle \geq 0,$$

and hence

$$\begin{aligned} 0 &\leq \langle T_{r_{n+1}}u_n - T_{r_n}u_n, T_{r_n}u_n - u_n - \frac{r_n}{r_{n+1}}(T_{r_{n+1}}u_n - u_n)\rangle \\ &= \langle T_{r_{n+1}}u_n - T_{r_n}u_n, T_{r_n}u_n - T_{r_{n+1}}u_n + (1 - \frac{r_n}{r_{n+1}})(T_{r_{n+1}}u_n - u_n)\rangle \\ &\leq \|T_{r_{n+1}}u_n - T_{r_n}u_n\|(\|T_{r_{n+1}}u_n - T_{r_n}u_n\| + |1 - \frac{r_n}{r_{n+1}}|\|T_{r_{n+1}}u_n - u_n\|). \end{aligned}$$

From (C1), we can find a real number $a$ such that $r_n \geq a > 0$ for all $n \in \mathbb{N}$.
Then, we have

$$\|T_{r_{n+1}}u_n - T_{r_n}u_n\|^2 \leq |1 - \frac{r_n}{r_{n+1}}|\|T_{r_{n+1}}u_n - T_{r_n}u_n\|(\|T_{r_{n+1}}u_n\| + \|u_n\|),$$

and hence

$$\begin{aligned} \|T_{r_{n+1}}u_n - T_{r_n}u_n\| &\leq |1 - \frac{r_n}{r_{n+1}}|(\|T_{r_{n+1}}u_n\| + \|u_n\|) \\ &\leq \frac{1}{a}|r_{n+1} - r_n|\hat{L}, \end{aligned} \tag{26}$$

where $\hat{L} = \sup\{\|T_{r_{n+1}}u_n\| + \|u_n\| \ : n \in \mathbb{N}\}$.
By (20), (21) and (26), we have

$$\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n|\|Ax_n\| + \frac{1}{a}|r_{n+1} - r_n|\hat{L}. \tag{27}$$

Combining (19) and (27), we deduce

$$\begin{aligned} \|z_{n+1} - z_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\rho\|x_{n+1} - x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\| + \|K_n y_n\|) \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(\|x_{n+1} - x_n\| + |r_{n+1} - r_n|\|Ax_n\| + \frac{1}{a}|r_{n+1} - r_n|\hat{L} \\ &\quad + L\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|) \\ &\leq \|x_{n+1} - x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\| + \|K_n y_n\|) \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}|r_{n+1} - r_n|\|Ax_n\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\cdot\frac{1}{a}|r_{n+1} - r_n|\hat{L} \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}L\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|. \end{aligned}$$

Therefore

$$\begin{aligned} \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\| + \|K_n y_n\|) \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}|r_{n+1} - r_n|\|Ax_n\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\cdot\frac{1}{a}|r_{n+1} - r_n|\hat{L} \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}L\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|. \end{aligned} \tag{28}$$

Applying the conditions (C2), (D1), (E1) and (F1) and taking the superior limit as $n \to \infty$ to (28), we have

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) = 0.$$

Hence, by Lemma 2.5, we have $\lim_{n\to\infty} \|z_n - x_n\| = 0$. This implies that

$$\lim_{n\to\infty} \|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{29}$$

Using (C2), (27) and (29), we have

$$\lim_{n\to\infty} \|y_{n+1}-y_n\| = 0. \tag{30}$$

Next we show that $\lim_{n\to\infty} \|x_n - y_n\| = 0$ and $\lim_{n\to\infty} \|K_n y_n - y_n\| = 0$.
Since $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n y_n$, we obtain

$$\begin{aligned}
\|x_n - K_n y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - K_n y_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n y_n - K_n y_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n f(x_n) + \beta_n x_n - (1-\gamma_n) K_n y_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n f(x_n) + \beta_n x_n - (\alpha_n+\beta_n) K_n y_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - K_n y_n\| + \beta_n\|x_n - K_n y_n\|
\end{aligned}$$

and hence

$$\|x_n - K_n y_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - K_n y_n\|. \tag{31}$$

Since $\alpha_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, (31) implies that

$$\lim_{n\to\infty} \|x_n - K_n y_n\| = 0. \tag{32}$$

From (14) and monotonicity of $A$ and nonexpansivity of $T_{r_n}$, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n y_n - p\|^2 \\
&\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|K_n y_n - p\|^2 \\
&\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2 \\
&= \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|T_{r_n}(x_n - r_n Ax_n) - T_{r_n}(p - r_n Ap)\|^2 \\
&\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|(x_n - r_n Ax_n) - (p - r_n Ap)\|^2 \\
&= \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\
&= \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n(\|x_n - p\|^2 - 2r_n\langle x_n - p, Ax_n - Ap\rangle \\
&\quad + r_n^2\|Ax_n - Ap\|^2) \\
&\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n(\|x_n - p\|^2 - 2r_n\alpha\|Ax_n - Ap\|^2 \\
&\quad + r_n^2\|Ax_n - Ap\|^2) \\
&= \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 - 2r_n\gamma_n\alpha\|Ax_n - Ap\|^2 \\
&\quad + \gamma_n r_n^2\|Ax_n - Ap\|^2 \\
&= \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 + \gamma_n r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2. \qquad (33)
\end{aligned}$$

By (33), we have

$$\begin{aligned}
\gamma_n r_n(2\alpha - r_n)\|Ax_n - Ap\|^2 &\le \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&= \alpha_n\|f(x_n) - p\|^2 - \alpha_n\|x_n - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\le \alpha_n\|f(x_n) - p\|^2 - \alpha_n\|x_n - p\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|). \qquad (34)
\end{aligned}$$

Since, $0 < c \le r_n \le d < 2\alpha$, we have
(35)
$$\gamma_n c(2\alpha - d)\|Ax_n - Ap\|^2 \le \alpha_n\|f(x_n) - p\|^2 - \alpha_n\|x_n - p\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|).$$
From $\alpha_n \to 0$ , $\|x_n - x_{n+1}\| \to 0$ and the boundedness of $\{x_n\}$ and $\{f(x_n)\}$, we have

(36)
$$\lim_{n\to\infty} \|Ax_n - Ap\| = 0.$$

Since $T_{r_n}$ is a firmly nonexpansive, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp)\|^2 \\
&\le \langle T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp), (x_n - r_nAx_n) - (p - r_nAp)\rangle \\
&= \langle y_n - p, (x_n - r_nAx_n) - (p - r_nAp)\rangle \\
&= \frac{1}{2}(\|y_n - p\|^2 + \|(x_n - r_nAx_n) - (p - r_nAp)\|^2 - \|(y_n - p) \\
&\quad - ((x_n - r_nAx_n) - (p - r_nAp))\|^2) \\
&\le \frac{1}{2}(\|y_n - p\|^2 + \|x_n - p\|^2 - \|(x_n - y_n) - r_n(Ax_n - Ap)\|^2) \\
&= \frac{1}{2}(\|y_n - p\|^2 + \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2r_n\langle x_n - y_n, Ax_n - Ap\rangle \\
&\quad - r_n^2\|Ax_n - Ap\|^2) \qquad (37)
\end{aligned}$$

and hence

(38)
$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2r_n\|x_n - y_n\|\|Ax_n - Ap\|.$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2 \\
&\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n(\|x_n - p\|^2 - \|x_n - y_n\|^2 \\
&\quad + 2r_n\|x_n - y_n\|\|Ax_n - Ap\|) \\
&= \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 - \gamma_n\|x_n - y_n\|^2 \\
&\quad + 2\gamma_n r_n\|x_n - y_n\|\|Ax_n - Ap\| \\
&= \alpha_n\|f(x_n) - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - \gamma_n\|x_n - y_n\|^2 \\
&\quad + 2\gamma_n r_n\|x_n - y_n\|\|Ax_n - Ap\|.
\end{aligned}$$

This implies

$$\begin{aligned}
\gamma_n\|x_n - y_n\|^2 &\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 - \alpha_n\|x_n - p\|^2 \\
&\quad + 2\gamma_n r_n\|x_n - y_n\|\|Ax_n - Ap\| \\
&\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) \\
&\quad - \alpha_n\|x_n - p\|^2 + 2\gamma_n r_n\|x_n - y_n\|\|Ax_n - Ap\|. \qquad (39)
\end{aligned}$$

Since $\alpha_n \to 0$, $\|x_n - x_{n+1}\| \to 0$, $\|Ax_n - Ap\| \to 0$ and the sequences $\{x_n\}, \{y_n\}$ and $\{f(x_n)\}$ are bounded, it follows from (39) that

(40)
$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

From $\|K_ny_n - y_n\| \le \|K_ny_n - x_n\| + \|x_n - y_n\|$
by (32) and (40), we have

(41)
$$\lim_{n\to\infty} \|K_ny_n - y_n\| = 0.$$

Next, we show that

(42)
$$\limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^*\rangle \le 0,$$

where $x^* = P_{\cap_{i=1}^N F(T_i)\cap\Omega} f(x^*)$. To show this inequality, we can choose a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\lim_{i\to\infty}\langle f(x^*) - x^*, y_{n_i} - x^*\rangle = \limsup_{n\to\infty}\langle f(x^*) - x^*, y_n - x^*\rangle. \tag{43}$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{ij}}\}$ of $\{y_{n_i}\}$ which converges weakly to $\omega$. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup \omega$. From $\|K_n y_n - y_n\| \to 0$, so we have $K_n y_{n_i} \rightharpoonup \omega$. Let us show $\omega \in \cap_{i=1}^N F(T_i) \cap \Omega$.
First, we show $\omega \in \Omega$. Since $y_n = T_{r_n}(x_n - r_n A x_n)$, for any $z \in C$ we have

$$\Phi(y_n, z) + \varphi(z) - \varphi(y_n) + \langle Ax_n, z - y_n\rangle + \frac{1}{r_n}\langle z - y_n, y_n - x_n\rangle \geq 0.$$

From (A2) we have

$$\varphi(z) - \varphi(y_n) + \langle Ax_n, z - y_n\rangle + \frac{1}{r_n}\langle z - y_n, y_n - x_n\rangle \geq -\Phi(y_n, z) \geq \Phi(z, y_n),$$

and hence

$$\varphi(z) - \varphi(y_{n_i}) + \langle Ax_{n_i}, z - y_{n_i}\rangle + \langle z - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq \Phi(z, y_{n_i}), \tag{44}$$

Put $y_t = tz + (1-t)\omega$ for all $t \in (0,1]$ and $z \in C$. Then we have $y_t \in C$. From (44) we have

$$\begin{aligned}
&\varphi(y_t) - \varphi(y_{n_i}) + \langle y_t - y_{n_i}, Ay_t\rangle \\
&\quad\geq \langle y_t - y_{n_i}, Ay_t\rangle - \langle y_t - y_{n_i}, Ax_{n_i}\rangle - \langle y_t - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\rangle + \Phi(y_t, y_{n_i}) \\
&\quad= \langle y_t - y_{n_i}, Ay_t - Ay_{n_i}\rangle + \langle y_t - y_{n_i}, Ay_{n_i} - Ax_{n_i}\rangle - \langle y_t - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\rangle + \Phi(y_t, y_{n_i}).
\end{aligned}$$

Since $\|y_{n_i} - x_{n_i}\| \to 0$, we have $\|Ay_{n_i} - Ax_{n_i}\| \to 0$. Further, from monotonicity of $A$, we have $\langle y_t - y_{n_i}, Ay_t - Ay_{n_i}\rangle \geq 0$.
Thus from the weakly semicontinuity of $\varphi$ and (A4), we have

$$\varphi(y_t) - \varphi(\omega) + \langle y_t - \omega, Ay_t\rangle \geq \Phi(y_t, \omega) \ \text{ as } \ i \to \infty. \tag{45}$$

From (A1), (A4), (45) and the convexity of $\varphi$ , we also have

$$\begin{aligned}
0 &= \Phi(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&= \Phi(y_t, (tz + (1-t)\omega)) + \varphi(tz + (1-t)\omega) - \varphi(y_t) \\
&\leq t\Phi(y_t, z) + (1-t)\Phi(y_t, \omega) + t\varphi(z) + (1-t)\varphi(\omega) - \varphi(y_t) \\
&\leq t\Phi(y_t, z) + (1-t)(\varphi(y_t) - \varphi(\omega) + \langle y_t - \omega, Ay_t\rangle) + t\varphi(z) + (1-t)\varphi(\omega) - \varphi(y_t) \\
&= t\Phi(y_t, z) - t\varphi(y_t) + (1-t)\langle y_t - \omega, Ay_t\rangle + t\varphi(z) \\
&= t[\Phi(y_t, z) - \varphi(y_t) + \varphi(z)] + (1-t)t\langle z - \omega, Ay_t\rangle \qquad (46)
\end{aligned}$$

Dividing by $t$, we have

$$\Phi(y_t, z) - \varphi(y_t) + \varphi(z) + (1-t)\langle z - \omega, Ay_t\rangle \geq 0, \quad \forall z \in C.$$

Letting $t \to 0$, it follows from (A3) and the weakly semicontinuity of $\varphi$ that

$$\Phi(\omega, z) - \varphi(\omega) + \varphi(z) + \langle z - \omega, A\omega\rangle \geq 0, \quad \forall z \in C. \tag{47}$$

Therefore $\omega \in \Omega$. Next, we show that $\omega \in \cap_{i=1}^N F(T_i)$. Assume that there exists $j \in \{1, 2, \dots, N\}$ such that $\omega \neq T_j\omega$. By Lemma 2.10, we have $\omega \neq K\omega$.
Since $y_{n_i} \rightharpoonup \omega$ and $\omega \neq K\omega$, by Opial's condition [8] and (41) and Lemma 2.11, we have

$$\begin{aligned}
\liminf_{i\to\infty}\|y_{n_i} - \omega\| &< \liminf_{i\to\infty}\|y_{n_i} - K\omega\| \\
&\leq \liminf_{i\to\infty}(\|y_{n_i} - K_{n_i}y_{n_i}\| + \|K_{n_i}y_{n_i} - K_{n_i}\omega\| + \|K_{n_i}\omega - K\omega\|) \\
&\leq \liminf_{i\to\infty}\|y_{n_i} - \omega\|,
\end{aligned}$$

which is a contradiction. Thus $\omega = K\omega$ and $\omega \in F(K) = \cap_{i=1}^{N} F(T_i)$. Hence $\omega \in \cap_{i=1}^{N} F(T_i) \cap \Omega$. Since $x^* = P_{\cap_{i=1}^{N} F(T_i)\cap\Omega} f(x^*)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle &= \lim_{i\to\infty}\langle f(x^*) - x^*, x_{n_i} - x^*\rangle \\ &= \lim_{i\to\infty}\langle f(x^*) - x^*, y_{n_i} - x^*\rangle \\ &= \langle f(x^*) - x^*, \omega - x^*\rangle \le 0. \end{aligned} \tag{48}$$

Finally, we prove that $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^*$. From (13), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n y_n - x^*, x_{n+1} - x^*\rangle \\ &= \alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle + \beta_n\langle x_n - x^*, x_{n+1} - x^*\rangle + \gamma_n\langle K_n y_n - x^*, x_{n+1} - x^*\rangle \\ &\le \alpha_n\langle f(x_n) - f(x^*), x_{n+1} - x^*\rangle + \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\quad + \frac{1}{2}\beta_n(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \frac{1}{2}\gamma_n(\|K_n y_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\ &\le \frac{1}{2}(1-\alpha_n)(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\quad + \frac{1}{2}\alpha_n(\|f(x_n) - f(x^*)\|^2 + \|x_{n+1} - x^*\|^2) \\ &\le \frac{1}{2}(1-\alpha_n)(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\quad + \frac{1}{2}\alpha_n\rho^2\|x_n - x^*\|^2 + \frac{1}{2}\alpha_n\|x_{n+1} - x^*\|^2 \\ &= \frac{1}{2}(1 - \alpha_n(1-\rho^2))\|x_n - x^*\|^2 + \frac{1}{2}\|x_{n+1} - x^*\|^2 + \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle, \end{aligned} \tag{49}$$

which implies that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le (1 - \alpha_n(1-\rho^2))\|x_n - x^*\|^2 + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &= (1 - \alpha_n(1-\rho^2))\|x_n - x^*\|^2 + \alpha_n(1-\rho^2)\cdot\frac{2}{(1-\rho^2)}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &= (1-\delta_n)\|x_n - x^*\|^2 + \delta_n\sigma_n, \end{aligned} \tag{50}$$

where $\delta_n = \alpha_n(1-\rho^2)$ and $\sigma_n = \frac{2}{(1-\rho^2)}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle$. It is easy to see that $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\sigma_n \le 0$. Applying Lemma 2.4 to (50), we conclude that $x_n \to x^*$ as $n \to \infty$. Consequently, $\{y_n\}$ converge strongly to $x^*$. This completes the proof. □

**Corollary 3.2.** *Let $H$ be a Hilbert space, $C$ a closed convex nonempty subset of $H$, $\varphi : C \to \mathbb{R}$ a proper lower semicontinuous and convex functional, $\Phi : C \times C \to \mathbb{R}$ a bifunction satisfying (A1)-(A4), $\{T_i\}_{i=1}^{N}$ a finite family of nonexpansive mappings of $C$ into itself such that $\cap_{i=1}^{N} F(T_i) \cap MEP(\Phi,\varphi) \neq 0$ and $f$ a $\rho-$contraction of $C$ into itself. Moreover, let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\{\lambda_{n,i}\}_{i=1}^{N}$ a sequence in $[a,b]$ with $0 < a \le b < 1$ and $\{r_n\}$ a sequence in $[0,2\alpha]$ for all $n \in \mathbb{N}$. Assume that:*

(i) *either (B1) or (B2) holds;*
(ii) *the sequence $\{r_n\}$ satisfies*
 (C1) $0 < c \le r_n \le d < 2\alpha$; *and*
 (C2) $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$;
(iii) *the sequence $\{\alpha_n\}$ satisfies*
 (D1) $\lim_{n\to\infty}\alpha_n = 0$; *and*
 (D2) $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(iv) *the sequence $\{\beta_n\}$ satisfies*

(E1) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(v) *the finite family of sequences* $\{\lambda_{n,i}\}_{i=1}^N$ *satisfies*

(F1) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every* $i \in \{1, 2, \dots, N\}$.

*For every* $n \in \mathbb{N}$, *let* $K_n$ *be a* $K-$*mapping generated by* $T_1, T_2, \dots, T_N$ *and* $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$ *and let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by* $x_0 \in C$ *and*

$$\begin{cases} \Phi(y_n, x) + \varphi(x) - \varphi(y_n) + \frac{1}{r_n}\langle x - y_n, y_n - x_n\rangle \ge 0, \ \ \forall x \in C \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n y_n. \end{cases}$$

*Then both* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $x^* = P_{\cap_{i=1}^N F(T_i)\cap MEP(\Phi,\varphi)} f(x^*)$.

*Proof.* Put $A \equiv 0$. Then, for all $\alpha \in (0, \infty)$, we have that

$$\langle x - y, Ax - Ay\rangle \ge \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Hence all the conditions of Theorem 3.1 are satisfied. Therefore the corollary is obtained by Theorem 3.1. □

**Corollary 3.3.** *Let* $H$ *be a Hilbert space,* $C$ *a closed convex nonempty subset of* $H$, $A$ *an* $\alpha-$*inverse strongly monotone mapping of* $C$ *into* $H$, $\Phi : C \times C \to \mathbb{R}$ *a bifunction satisfying (A1)-(A4),* $\{T_i\}_{i=1}^N$ *a finite family of nonexpansive mappings of* $C$ *into itself such that* $\cap_{i=1}^N F(T_i) \cap EP \neq 0$ *and* $f$ *a* $\rho-$*contraction of* $C$ *into itself. Moreover, let* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are three sequences in* $(0, 1)$ *with* $\alpha_n + \beta_n + \gamma_n = 1$, $\{\lambda_{n,i}\}_{i=1}^N$ *a sequence in* $[a, b]$ *with* $0 < a \le b < 1$ *and* $\{r_n\}$ *a sequence in* $[0, 2\alpha]$ *for all* $n \in \mathbb{N}$. *Assume that:*

(i) *the sequence* $\{r_n\}$ *satisfies*

(C1) $0 < c \le r_n \le d < 2\alpha$; *and*

(C2) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(ii) *the sequence* $\{\alpha_n\}$ *satisfies*

(D1) $\lim_{n\to\infty} \alpha_n = 0$; *and*

(D2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) *the sequence* $\{\beta_n\}$ *satisfies*

(E1) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iv) *the finite family of sequences* $\{\lambda_{n,i}\}_{i=1}^N$ *satisfies*

(F1) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every* $i \in \{1, 2, \dots, N\}$.

*For every* $n \in \mathbb{N}$, *let* $K_n$ *be a* $K-$*mapping generated by* $T_1, T_2, \dots, T_N$ *and* $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$ *and let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by* $x_0 \in C$ *and*

$$\begin{cases} \Phi(y_n, x) + \langle Ax_n, x - y_n\rangle + \frac{1}{r_n}\langle x - y_n, y_n - x_n\rangle \ge 0, \ \ \forall x \in C \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n y_n. \end{cases}$$

*Then both* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $x^* = P_{\cap_{i=1}^N F(T_i)\cap EP} f(x^*)$.

*Proof.* Put $\varphi \equiv 0$ in Theorem 3.1. Hence all the conditions of Theorem 3.1 are satisfied. Therefore the corollary is obtained by Theorem 3.1. □

**Corollary 3.4.** *Let* $H$ *be a Hilbert space,* $C$ *a closed convex nonempty subset of* $H$, $\Phi : C \times C \to \mathbb{R}$ *a bifunction satisfying (A1)-(A4),* $\{T_i\}_{i=1}^N$ *a finite family of nonexpansive mappings of* $C$ *into itself such that* $\cap_{i=1}^N F(T_i) \cap EP(\Phi) \neq 0$ *and* $f$ *a* $\rho-$*contraction of* $C$ *into itself. Moreover, let* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are three sequences in* $(0, 1)$ *with* $\alpha_n + \beta_n + \gamma_n = 1$, $\{\lambda_{n,i}\}_{i=1}^N$ *a sequence in* $[a, b]$ *with* $0 < a \le b < 1$ *and* $\{r_n\}$ *a sequence in* $[0, 2\alpha]$ *for all* $n \in \mathbb{N}$. *Assume that:*

(i) *the sequence* $\{r_n\}$ *satisfies*

(C1) $0 < c \le r_n \le d < 2\alpha$; *and*

(C2) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(ii) *the sequence* $\{\alpha_n\}$ *satisfies*

(D1) $\lim_{n\to\infty} \alpha_n = 0$; *and*

(D2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) *the sequence* $\{\beta_n\}$ *satisfies*

(E1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) *the finite family of sequences* $\{\lambda_{n,i}\}_{i=1}^{N}$ *satisfies*

(F1) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every* $i \in \{1, 2, \ldots, N\}$.

*For every* $n \in \mathbb{N}$, *let* $K_n$ *be a* $K-$*mapping generated by* $T_1, T_2, \ldots, T_N$ *and* $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$ *and let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by* $x_0 \in C$ *and*

$$\begin{cases} \Phi(y_n, x) + \frac{1}{r_n}\langle x - y_n, y_n - x_n\rangle \geq 0, \quad \forall x \in C \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n y_n. \end{cases}$$

*Then both* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $x^* = P_{\cap_{i=1}^{N} F(T_i) \cap EP(\Phi)} f(x^*)$.

*Proof.* Put $\varphi \equiv 0$ and $A \equiv 0$ in Theorem3.1. Hence the corollary is obtained by Theorem 3.1. □

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School of Mathematics, Institute of Science, Suranaree University of Technology, Nakhon Ratchasima, 30000, Thailand.
Email address: benjawan@sut.ac.th