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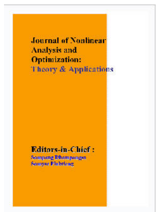

A general iterative algorithm for the solution of variational inequalities for a nonexpansive semigroup in Banach spaces

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ABSTRACT: Let X be a uniformly convex and smooth Banach space which admits a weakly sequentially continuous duality mapping, C a nonempty bounded closed convex subset of X. Let $\mathcal{S} = \{T(s) : 0 \leq 0 < \infty\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$ and $f : C \to C$ is a contraction mapping with coefficient $\alpha \in (0,1)$, A a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. We prove that the sequences $\{x_t\}$ and $\{x_n\}$ are generated by the following iterative algorithms, respectively

$$x_t = t\gamma f(x_t) + (I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds$$

and

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds$$

where $\{t\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$ and $\{\lambda_t\}_{0<t<1}$, $\{t_n\}$ are positive real divergent sequences, converging strongly to a common fixed point $x^* \in F(\mathcal{S})$, which solves variational inequality $\langle (\gamma f - A)x^*, J(x - x^*)\rangle \leq 0$ for $x \in F(\mathcal{S})$. Our results presented in this paper extend and improve the corresponding results announced by many others.

1. Introduction

Let X be a real Banach space, and let C a nonempty closed convex subset of X. Mapping T of C into itself is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x,\ y \in C$. We denote $F(T)$ as the set of fixed points of T. We know that $F(T)$ is nonempty if C is bounded; for more detail see [3]. A one-parameter family $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ from C of X into itself is said to be a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s+t) = T(s) \circ T(t)$ for all $s, t \geq 0$;
(iii) for each $x \in C$ the mapping $t \mapsto T(t)x$ is continuous; and
(iv) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$.

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We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is $F(\mathcal{S}) = \bigcap_{s\geq 0} F(T(s))$. We know that $F(\mathcal{S})$ is nonempty if C is bounded, see [4]. Recall that a self mapping $f : C \to C$ is a *contraction* if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for each $x,\ y \in C$.

Iterative methods for nonexpansive mappings have recently been applied to solve minimization problems; see, e.g. [8, 20, 21, 23, 24]. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle, \tag{1}$$

where F is the fixed point set of a nonexpansive mapping T on H, and u is a given point in H.

Assume A is strongly positive; that is, there is a constant $\bar{\gamma}$ with the property such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2 \tag{2}$$

for all $x \in H$.

In 2003, Xu [20] proved that the sequence $\{x_n\}$ generated by

$$x_{n+1} = \alpha_n u + (I - \alpha_n A)Tx_n,\ \forall n \geq 0, \tag{3}$$

converges strongly to the unique solution of the minimization problem (1), provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

On the other hand, Moudafi [15] introduced the viscosity approximation method for nonexpansive mappings (see [22] for further developments in both Hilbert and Banach spaces). Starting with an arbitrary initial $x_0 \in H$, defined the sequence $\{x_n\}$ recursively by

$$x_{n+1} = \sigma_n f(x_n) + (1 - \sigma_n)Tx_n,\ \forall n \geq 0, \tag{4}$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$. It is proved in [15, 22] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (4) strongly converges to the unique solution x^* of the variational inequality

$$\langle (f - I)x^*, x - x^*\rangle \leq 0,\ \forall x \in F(T). \tag{5}$$

Recently, Marino and Xu [14] combined the iterative method (3) with the viscosity approximation method (4) considering the following general iterative process:

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)Tx_n,\ \forall n \geq 0, \tag{6}$$

where $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. They proved that the sequence $\{x_n\}$ generated by (6) converges strongly to a unique solution x^* of the variational inequality

$$\langle (\gamma f - A)x^*, x - x^*\rangle \leq 0,\ \forall x \in F(T). \tag{7}$$

On the other hand, Browder [2] proved that if X is a Hilbert space for a nonexpansive mapping from C into itself, then the net sequence $\{x_t\}$ with $t \in (0,1)$, generated by

$$x_t = tu + (1 - t)Tx_t, \tag{8}$$

converges strongly to the element of $F(T)$, which is nearest to $x \in F(T)$ as $t \to 0$. Moudafi [15] and Xu [22] used the viscosity approximation method for a nonexpanive mapping T. It proved that the net sequence $\{x_t\}$ with $t \in (0,1)$, generated by

$$x_t = tf(x_t) + (1 - t)Tx_t, \tag{9}$$

converges strongly to the element in $F(T)$ which is the unique solution to the variational inequality (5). Later, Bailon and Brezis [1] proved that if $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ is a nonexpansive semigroup on C, then the continuous scheme with $t \in (0,1)$

$$x_t = \frac{1}{t}\int_0^t T(s)x_t ds, \tag{10}$$

converges weakly to a common fixed point of $\mathcal{S}$. Those results have been generalized by many authors; see, for instance Takahashi [19]. Shioji and Takahashi [18] introduced the implicit iteration

(11)
$$x_n = \alpha_n u + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \ \forall n \in \mathbb{N}.$$

In 2007, Chen and Song [6] proposed the explicit iterative process $\{x_n\}$ in a Banach space, as follows:

(12)
$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds,$$

where $\{t_n\}$ is a positive real divergent sequence. They proved, under certain appropriate conditions $\{\alpha_n\}$ be a real sequence in $(0,1)$, that $\{x_n\}$ converges strongly to a unique solution x^* of the variational inequality

(13)
$$\langle (f-I)x^*, J(x-x^*)\rangle, \ \forall x \in F(T).$$

Recently, Li et al [12] and Plubtieng and Wangkeeree [16] considered the iterative process $\{x_n\}$, in a Hilbert space H, $x_0 \in H$ is arbitrary and

(14)
$$x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \ \forall n \geq 0,$$

where A is a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$, $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $\{t_n\}$ is a positive real divergent sequence. They proved, under certain appropriate conditions $\{\alpha_n\} \subset (0,1)$ and $\{t_n\}$ is a positive real divergent sequence, that $\{x_n\}$ converges strongly to a unique solution x^* of the variational inequality (7). Moreover, Plubtieng and Wangkeeree [16], also considered and studied the continuous scheme $\{x_t\}$ with $t \in (0,1)$ defined as follows:

(15)
$$x_t = t\gamma f(x_t) + (I-tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds,$$

where A is a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$, $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $\{\lambda_t\}$ is a positive real divergent net. They proved, under certain appropriate conditions $\{\lambda_t\} \subset (0,1)$, that $\{x_t\}$ converges strongly to a unique solution x^* of the variational inequality (7).

Very recently, Kang et al.[11] considered the iterative process $\{x_n\}$ in a Hilbert space as follows:

(16)
$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \ \forall n \geq 0,$$

They proved, under certain appropriate condition $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequence in $(0,1)$, that $\{x_n\}$ converges strongly to a unique solution of the variational inequality (7).

Question 1.1. Can Theorem of Kang et al. [11] and Plubtieng and Wangkeeree [16] be extend from Hilbert spaces to a general Banach space? such as uniformly convex Banach space.

Question 1.2. Can we extend the iterative method of algorithm (14) to a general iterative process?

The purpose of this paper is to give affirmative answer to these questions mentioned above. In this paper, motivated and inspired by Chen and Song [6] and Kang et al.[11], we consider the iterative schemes defined by (15) and (16) for a nonexpansive semigroup in a Banach space. We proved that both schemes converge strongly to a common fixed point of $\mathcal{S}$. The results in this paper extend and improve the main results of Kang et al.[11], Li et al. [12] and Plubtieng and Wangkeeree [16] and some others to Banach spaces.

2. **Preliminaries**

Throughout this paper, let X be a real Banach space, C be a closed convex subset of X. Let $J : X \to 2^{X^*}$ be a normalized duality mapping by $J(x) = \{ f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \}$, where X^* denotes the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the generalized duality paring. In the following, the notation $\rightharpoonup$ and $\to$ denote the weak and strong convergence, respectively. Also, a mapping $I : C \to C$ denotes the identity mapping.

The norm of a Banach space X is said to be *Gâteaux differentiable* if the $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in C$ on the unit sphere $S(X)$ of X. In this case X is smooth. Moreover, if for each y in $S(X)$ the limit above is uniformly attained for $x \in S(X)$, we say that the norm X is *uniformly Gâteaux differentiable*.

Recall that the Banach space X is said to be *smooth* if duality mapping J is single valued. In a smooth Banach space, we always assume that A is strongly positive (see [5]), that is, a constant $\bar{\gamma} > 0$ with the property

$$(17) \qquad \langle Ax, J(x) \rangle \geq \bar{\gamma}\|x\|^2, \; \|aI - bA\| = \sup_{\|x\|\leq 1} \|\langle (aI - bA)x, J(x) \rangle\| \; a \in [0,1], \; b \in [-1,1].$$

A Banach space X is said to be *strictly convex* if $\|x\| = \|y\| = 1$, $x \neq y$ implies $\frac{\|x+y\|}{2} < 1$. A Banach space X is said to be *uniformly convex* if $\delta_X(\epsilon) > 0$ for all $\epsilon > 0$, where $\delta_X(\epsilon)$ is *modulus of convexity* of X defined by $\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \; \|y\| \leq 1, \; \|x+y\| \geq \varepsilon\right\}$, $\forall \epsilon \in [0,2]$. A uniformly convex Banach space X is reflexive and strictly convex (see Theorem 4.1.6, Theorem 4.1.2 of [19]).

In the sequel we will use the following lemmas, which will be used in the proofs for the main results in the next section.

Lemma 2.1. (Cai and Hu [5]) *Assume that A is a strongly positive linear bounded operator on a smooth Banach space X with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \bar{\gamma}$.*

Lemma 2.2. (Chen and Song [6]) *Let C be a closed convex subset of a uniformly convex Banach space X and let $\Gamma = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S})$ is nonempty. Then for each $r > 0$ and $h \geq 0$,*

$$\lim_{t\to\infty} \sup_{x\in C\cap B_r} \|\frac{1}{t}\int_0^t T(s)xds - T(h)(\frac{1}{t}\int_0^t T(s)xds)\| = 0.$$

Lemma 2.3. (Liu [13]) *Let X be a real Banach space and $J : X \to 2^{X^*}$ be the normalized duality mapping. Then, for any $x, y \in X$, we have*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y) \rangle$$

for all $j(x+y) \in J(x+y)$ with $x \neq y$.

If a Banach space X admits a sequentially continuous duality mapping J from weak topology to weak star topology, then by Lemma 1 of [9], we have that duality mapping J is a single value. In this case, the duality mapping J is said to be a weakly sequentially continuous duality mapping, i.e. for each $\{x_n\} \subset X$ with $x_n \rightharpoonup x$, we have $J(x_n) \rightharpoonup^* J(x)$ (see [9, 10, 17] for more detail).

A Banach space X is said to be satisfying Opial's condition if for any sequence $x_n \rightharpoonup x$ for all $x \in X$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \; \forall y \in X, \text{ with } x \neq y.$$

By Theorem 1 in [9], it is well known that if X admits a weakly sequentially continuous duality mapping, then X satisfies Opial's condition, and X is smooth.

Lemma 2.4. ([10] Demiclosed Principle*) Let C be a nonempty closed convex subset of a reflexive Banach space X which satisfies Opial's condition, and suppose $T : C \to X$ is nonexpansive. Then the mapping $I - T$ is demiclosed at zero, i.e., $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ implies $x = Tx$.*

Lemma 2.5. (Xu [20]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} = (1-\gamma_n)a_n + \delta_n,\ n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that:

(i) $\lim_{n\to\infty} \gamma_n = 0$;
(ii) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(iii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. **Main Results**

In this section, we prove our main results.

Theorem 3.1. *Let C be a nonempty bounded closed convex subset of a uniformly convex, smooth Banach space X which admits a weakly sequentially continuous duality mapping J from X into X^*, $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup such that $F(\mathcal{S}) \neq \emptyset$, $f : C \to C$ is a contraction mapping with coefficient $\alpha \in (0,1)$, A a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$. Let $t \in (0,1)$ such that $t \leq \|A\|^{-1}$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ which satisfies $t \to 0$. Then the sequence $\{x_t\}$ defined by* (15) *converges strongly to the common fixed point x^* as $t \to 0$, where x^* is a unique solution in $F(\mathcal{S})$ of the variational inequality*

$$\langle (\gamma f - A)x^*, J(x - x^*)\rangle \leq 0,\ \forall x \in F(\mathcal{S}). \tag{18}$$

Proof. First, we show the uniqueness of a solution of the variational inequality. Supposing $\tilde{x},\ x^* \in F(\mathcal{S})$ satisfy the inequality, we have

$$\langle (\gamma f - A)\tilde{x}, J(x^* - \tilde{x})\rangle \leq 0, \tag{19}$$

and

$$\langle (\gamma f - A)x^*, J(\tilde{x} - x^*)\rangle \leq 0. \tag{20}$$

Adding up (19) and (20), we get that

$$\begin{aligned} 0 &\geq \langle (\gamma f - A)\tilde{x} - (\gamma f - A)x^*, J(x^* - \tilde{x})\rangle \\ &= \langle A(x^* - \tilde{x}), J(x^* - \tilde{x})\rangle - \gamma\langle f(x^*) - f(\tilde{x}), J(x^* - \tilde{x})\rangle \\ &\geq \bar{\gamma}\|x^* - \tilde{x}\|^2 - \gamma\|f(x^*) - f(\tilde{x})\|\|J(x^* - \tilde{x})\| \\ &\geq \bar{\gamma}\|x^* - \tilde{x}\|^2 - \gamma\alpha\|x^* - \tilde{x}\|^2 \\ &= (\bar{\gamma} - \gamma\alpha)\|x^* - \tilde{x}\|^2. \end{aligned}$$

Since $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ this implies that $\bar{\gamma} - \gamma\alpha > 0$, which is a contradiction. Hence $\tilde{x} = x^*$ and the uniqueness is proved.

Next, we show that $\{x_t\}$ is bounded. Indeed, for any $p \in F(\mathcal{S})$, we have

$$\begin{aligned}
\|x_t - p\| &= \|t\gamma f(x_t) + (I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - p\| \\
&= \|t(\gamma f(x_t) - Ap) + (I - tA)(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - p)\| \\
&\leq t\|\gamma f(x_t) - Ap\| + \|I - tA\|\frac{1}{\lambda_t}\|\int_0^{\lambda_t} T(s)x_t - p\|ds \\
&\leq t\|\gamma f(x_t) - Ap\| + (1 - t\bar{\gamma})\|x_t - p\| \\
&\leq t\|\gamma(f(x_t) - f(p)) + \gamma f(p) - Ap\| + (1 - t\bar{\gamma})\|x_t - p\| \\
&\leq t(\gamma\alpha\|x_t - p| + \|\gamma f(p) - Ap\|) + (1 - t\bar{\gamma})\|x_t - p\| \\
&= (1 - t(\bar{\gamma} - \gamma\alpha))\|x_t - p\| + t\|\gamma f(p) - Ap\|.
\end{aligned}$$

It follows that $\|x_t - p\| \leq \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha}$. Hence $\{x_t\}$ is bounded.
Next, we show that $\|x_t - T(h)x_t\| \to 0$ as $t \to 0$. We observe that

$$\begin{aligned}
\|x_t - T(h)x_t\| &= \|x_t - \tfrac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\| + \|\tfrac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - T(h)\tfrac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\| \\
&\quad + \|T(h)\tfrac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - T(h)x_t\| \\
&\leq 2\|x_t - \tfrac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\| + \|\tfrac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - T(h)\tfrac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\|
\end{aligned} \tag{21}$$

for every $0 \leq h \leq \infty$. On the other hand, we note that

$$\|\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - x_t\| = t\|A(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds) - \gamma f(x_t)\| \tag{22}$$

for every $t > 0$. Define the set $K = \{\|z - p\| \leq \frac{1}{\bar{\gamma} - \gamma\alpha}\|\gamma f(p) - Ap\|\}$, then K is a nonempty bounded closed convex subset of C which is $T(s)-$ invariant for each $s \in [0, \infty]$. Since $\{x_t\} \subset K$ and K is bounded, there exists $r > 0$ such that $K \subset B_r$, and it follows by Lemma 2.2 that

$$\lim_{\lambda_t \to \infty} \|\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - T(h)(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds)\| = 0 \tag{23}$$

for every $0 \leq h < \infty$. From (21)-(23) and let $t \to 0$, then

$$\|x_t - T(h)x_t\| \to 0, \tag{24}$$

for every $0 \leq h < \infty$. Assume $\{t_n\}_{n=1}^{\infty} \subset (0, 1)$ is such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$ and $\lambda_n := \lambda_{t_n}$. We will show that $\{x_n\}$ contains a subsequence converging strongly to x^*, where $x^* \in F(\mathcal{S})$. Since $\{x_n\}$ is bounded sequence and Banach space X is uniformly convex, hence it is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to $x^* \in C$ as $n \to \infty$. Again since Banach space X has a weakly sequentially continuous duality mapping satisfying Opial's condition. It follows by Lemma 2.4 and noting 24, we have $x^* \in F(\mathcal{S})$. For each $n \geq 1$, we note that

$$\begin{aligned} x_n - x^* &= t_n\gamma f(x_n) + (I - t_nA)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds - x^* \\ &= t_n(\gamma f(x_n) - Ax^*) + (I - t_nA)(\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds - x^*). \end{aligned}$$

Thus, we have

$$\begin{aligned} \|x_n - x^*\|^2 &= t_n\langle \gamma f(x_n) - Ax^*, J(x_n - \tilde{x})\rangle + \langle (I - t_nA)(\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds), J(x_n - x^*)\rangle \\ &\leq t_n\langle \gamma f(x_n) - Ax^*, J(x_n - x^*)\rangle + \|I - t_nA\|\|\frac{1}{\lambda_n}\int_0^{\lambda_n}(T(s)x_n - \tilde{x})ds\|\|J(x_n - x^*)\| \\ &\leq t_n\langle \gamma f(x_n) - Ax^*, J(x_n - x^*)\rangle + (1 - t_n\bar{\gamma})\|x_n - z\|(\frac{1}{\lambda_n}\int_0^{\lambda_n}\|T(s)x_n - x^*\|ds) \\ &\leq t_n\langle \gamma f(x_n) - Ax^*, J(x_n - x^*)\rangle + (1 - t_n\bar{\gamma})\|x_n - \tilde{x}\|(\frac{1}{\lambda_n}\int_0^{\lambda_n}\|x_n - x^*\|ds) \\ &\leq t_n\langle \gamma f(x_n) - Az, J(x_n - x^*)\rangle + (1 - t_n\bar{\gamma})\|x_n - x^*\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_n - x^*\|^2 &\leq \frac{1}{\bar{\gamma}}\langle \gamma f(x_n) - Ax^*, J(x_n - x^*)\rangle \\ &= \frac{1}{\bar{\gamma}}[\langle \gamma f(x_n) - \gamma f(x^*), J(x_n - x^*)\rangle + \langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle] \\ &\leq \frac{1}{\bar{\gamma}}[\gamma\alpha\|x_n - x^*\|^2 + \langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle]. \end{aligned}$$

This implies that

$$\|x_n - x^*\|^2 \leq \frac{1}{\bar{\gamma} - \gamma\alpha}\langle \gamma f(x^*) - Ax^*, J(x_n - \tilde{x})\rangle.$$

In particular, we have

$$\|x_{n_j} - x^*\|^2 \leq \tfrac{1}{\bar{\gamma}-\gamma\alpha}\langle \gamma f(x^*) - Ax^*, J(x_{n_j} - x^*)\rangle. \tag{25}$$

Since $\{x_n\}$ is bounded and the duality mapping J is single-valued and weakly sequentially continuous from X into X^*, it follows (25), we have that $x_{n_j} \to x^*$ as $j \to \infty$. Next, we show that x^* solves the variational inequality (18). Since $x_t = t\gamma f(x_t) + (I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds$. Thus, we have

$$(\gamma f - A)x_t = -\frac{1}{t}(I - tA)(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - x_t).$$

We notice that

$$\begin{aligned}
\langle \frac{1}{\lambda_t}\int_0^{\lambda_t}(I-T(s))xds - \frac{1}{\lambda_t}\int_0^{\lambda_t}(I-T(s))x_tds, J(x-x_t)\rangle &\geq \|x-x_t\|^2 \\
&\quad -\|\frac{1}{\lambda_t}\int_0^{\lambda_t}(T(s)x_t - T(s)x)ds\|\|J(x-x_t)\| \\
&\geq \|x-x_t\|^2 - \|x-x_t\|\|x-x_t\| \\
&= \|x-x_t\|^2 - \|x-x_t\|^2 \\
&= 0,
\end{aligned}$$

for each $x \in F(\mathcal{S})$ and for all $t > 0$,

$$\begin{aligned}
\langle(\gamma f - A)x_t, J(x-x_t)\rangle &= -\tfrac{1}{t}\langle(I-tA)(\tfrac{1}{\lambda_t}\int_0^{\lambda_t}T(s)x_tds - x_t), J(x-x_t)\rangle \\
&= -\tfrac{1}{t}\langle(I-tA)(\tfrac{1}{\lambda_t}\int_0^{\lambda_t}T(s)x_tds - \tfrac{1}{\lambda_t}\int_0^{\lambda_t}x_tds), J(x-x_t)\rangle \\
&= -\tfrac{1}{t}\langle\tfrac{1}{\lambda_t}\int_0^{\lambda_t}(I-T(s))xds - \tfrac{1}{\lambda_t}\int_0^{\lambda_t}(I-T(s))x_tds, J(x-x_t)\rangle \\
&\quad +\langle A(\tfrac{1}{\lambda_t}\int_0^{\lambda_t}(I-T(s))x_tds), J(x-x_t)\rangle \\
&\leq \langle A(\tfrac{1}{\lambda_t}\int_0^{\lambda_t}(T(s)-I)x_tds), J(x-x_t)\rangle.
\end{aligned} \tag{26}$$

Now replacing t and λ_t with t_{n_j} and λ_{n_j}, respectively in (26), and letting $j \to \infty$, we notice that $(T(s)-I)x_{n_j} \to (T(s)-I)x^* = 0$ for $x^* \in F(\mathcal{S})$, we obtain $\langle(\gamma f - A)x^*, J(x-x^*)\rangle \leq 0$. That is, x^* is a solution of variational inequality (18). By uniqueness, as $x^* = \tilde{x}$, we have shown that each cluster point of the net sequence $\{x_t\}$ is equal to x^*. Then, we conclude that $x_t \to x^*$ as $t \to 0$. This proof is completes.

If X is a Hilbert space, we can get the following corollary easily.

Corollary 3.2. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup such that $F(\mathcal{S}) \neq \emptyset$. Let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and let $t \in (0,1)$ such that $t \leq \|A\|^{-1}$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, which satisfies $t \to 0$. Then the sequence $\{x_t\}$ defined by* (15) *converges strongly to the common fixed point x^* as $t \to 0$, where x^* is a unique solution in $F(\mathcal{S})$ of the variational inequality* (7).

Remark 3.3. Theorem 3.1 improves and extends Theorem 3.1 of Plubtieng and Wangkeeree [16] from a Hilbert space to a Banach space.

Theorem 3.4. *Let C be a nonempty bounded closed convex subset of a uniformly convex, smooth Banach space X which admits a weakly sequentially continuous duality mapping J from X into X^*, $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup such that $F(\mathcal{S}) \neq \emptyset$, $f : C \to C$ is a contraction mapping with coefficient $\alpha \in (0,1)$, A a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $(0,1)$. Assume the following control conditions are hold:*

(C1) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$;
(C2) $\sum_{n=0}^{\infty}\alpha_n = \infty$.

Then the sequence $\{x_n\}$ defined by (16) *converges strongly to the common fixed point x^* as $n \to \infty$, where x^* is a unique solution in $F(\mathcal{S})$ of the variational inequality* (18).

Proof. First, we show $\{x_n\}$ is bounded. By the control condition $(C1)$, we may assume, with no loss of generality, that $\alpha_n \leq (1-\beta_n)\|A\|^{-1}$. Since A is a linear bounded operator on X, by

(17), we have $\|A\| = \sup\{|\langle Au, J(u)\rangle| : u \in X,\ \|u\| = 1\}$. Observe that

$$\begin{aligned}\langle((1-\beta_n)I - \alpha_n A)u, J(u)\rangle &= 1-\beta_n - \alpha_n\langle Au, J(u)\rangle\\ &\geq 1-\beta_n-\alpha_n\|A\|\\ &\geq 0.\end{aligned}$$

It follows that

$$\begin{aligned}\|(1-\beta_n)I-\alpha_n A\| &= \sup\{\langle((1-\beta_n)I-\alpha_n A)u, J(u)\rangle : u\in X,\ \|u\|=1\}\\ &= \sup\{1-\beta_n-\alpha_n\langle Au, J(u)\rangle : u\in X,\ \|u\|=1\}\\ &\leq 1-\beta_n-\alpha_n\bar{\gamma}.\end{aligned}$$

Taking, $p \in F(\mathcal{S})$ we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n\gamma f(x_n)+\beta_n x_n + ((1-\beta_n)I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - p\|\\ &= \|\alpha_n(\gamma f(x_n)-Ap)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_n A)(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - p)\|\\ &\leq \alpha_n[\gamma\|f(x_n)-f(p)\|+\|\gamma f(p)-Ap\|]+\beta_n\|x_n-p\| +\\ &\quad \|(1-\beta_n)I-\alpha_n A\|\frac{1}{t_n}\int_0^{t_n}\|T(s)x_n-p\|ds\\ &\leq \alpha_n[\gamma\alpha\|x_n-p\|+\|\gamma f(p)-Ap\|]+\beta_n\|x_n-p\|+((1-\beta_n)-\alpha_n\bar{\gamma})\|x_n-p\|\\ &= \alpha_n\|\gamma f(p)-Ap\|+[1-(\bar{\gamma}-\gamma\alpha)\alpha_n]\|x_n-p\|\\ &= (\bar{\gamma}-\gamma\alpha)\alpha_n\frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}+[1-(\bar{\gamma}-\gamma\alpha)\alpha_n]\|x_n-p\|\end{aligned}$$

By induction, we get

$$\|x_{n+1}-p\| \leq \max\left\{\|x_0-p\|, \frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}\right\},$$

for $n \geq 0$. Hence $\{x_n\}$ is bounded, so are $\{f(x_n)\}$ and $\{T(t_n)x_n\}$. It follows from Theorem 3.1 that there is a unique solution $x^* \in F(\mathcal{S})$ of the variational inequality (18).

Next, we show $\|x_n - T(h)x_n\| \to 0$ as $n \to \infty$. We note that

(27)

$$\begin{aligned}\|x_{n+1}-T(h)x_{n+1}\| &= \|x_{n+1}-\tfrac{1}{t_n}\int_0^{t_n}T(s)x_n ds\| + \|\tfrac{1}{t_n}\int_0^{t_n}T(s)x_n ds - T(h)\tfrac{1}{t_n}\int_0^{t_n}T(s)x_n ds\|\\ &\quad +\|T(h)\tfrac{1}{t_n}\int_0^{t_n}T(s)x_n ds - T(h)x_{n+1}\|\\ &\leq 2\|x_{n+1}-\tfrac{1}{t_n}\int_0^{t_n}T(s)x_n ds\| + \|\tfrac{1}{t_n}\int_0^{t_n}T(s)x_t ds - T(h)\tfrac{1}{t_n}\int_0^{t_n}T(s)x_n ds\|\\ &\leq 2\alpha_n\|\gamma f(x_n)-A(\tfrac{1}{t_n}\int_0^{t_n}T(s)x_n ds)\|+\beta_n\|x_n-\tfrac{1}{t_n}\int_0^{t_n}T(s)x_n ds\|\\ &\quad +\|\tfrac{1}{t_n}\int_0^{t_n}T(s)x_n ds - T(h)\tfrac{1}{t_n}\int_0^{t_n}T(s)x_n ds\|.\end{aligned}$$

Define the set $K = \{z \in C : \|z - z_0\| \leq \|x - x_0\| + \frac{\|\gamma f(x_0)-Az_0\|}{\bar{\gamma}-\gamma\alpha}\}$. Then K is a nonempty closed bounded convex subset of C which is $T(s)-$invariant for each $s \in [0, \infty]$ and contains $\{x_n\}$; it follows by Lemma 2.2 that

(28) $$\lim_{n\to\infty}\|\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds - T(h)(\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds)\| = 0,$$

for every $0 \leq h < \infty$. Since $\{x_n\}$, $\{f(x_n)\}$ and $\{T(s)x_n\}$ are bounded, by control conditions $(C1)$ and (28), into (27), we get that $\|x_{n+1} - T(h)x_{n+1}\| \to 0$ as $n \to \infty$, and hence

$$\|x_n - T(h)x_n\| \to 0 \text{ as } n \to \infty. \tag{29}$$

Let x^* be the unique solution in $F(\mathcal{S})$ of the variational inequality (18).

Now, we show that $\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle \leq 0$. We can take subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_{j\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_{n_j} - x^*)\rangle = \limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle. \tag{30}$$

Since X is uniformly convex, hence it is reflexive, and $\{x_n\}$ is bounded then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to $x \in C$ as $j \to \infty$. Again, since Banach space X has a weakly sequentially continuous duality mapping satisfying Opial's condition. By Lemma 2.4, and noting (29), we have $x \in F(\mathcal{S})$. Hence by (18), we obtain

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle = \langle \gamma f(x^*) - Ax^*, J(x - x^*)\rangle \leq 0. \tag{31}$$

Finally, we show that $x_n \to x^*$ as $n \to \infty$. For each $n \geq 0$, by Lemma 2.3 we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x^*\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x^*)\|^2 \\
&\leq \|((1-\beta_n)I - \alpha_n A)(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x^*) + \beta_n(x_n - x^*)\|^2 \\
&\quad +2\alpha_n\langle \gamma f(x_n) - Ax^*, J(x_{n+1} - x^*)\rangle \\
&\leq [(1-\beta_n - \alpha_n\bar{\gamma})\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x^*\| + \beta_n\|x_n - x^*\|]^2 \\
&\quad +2\alpha_n\langle \gamma f(x_n) - Ax^*, J(x_{n+1} - x^*)\rangle \\
&\leq (1-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x_n) - \gamma f(x^*), J(x_{n+1} - x^*)\rangle \\
&\quad +2\alpha_n\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle \\
&\leq (1-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + 2\alpha_n\|\gamma f(x_n) - \gamma f(x^*)\|\|J(x_{n+1} - x^*)\| \\
&\quad +2\alpha_n\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle \\
&= (1-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + 2\alpha_n\gamma\alpha\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad +2\alpha_n\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle \\
&\leq (1-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + \alpha_n\gamma\alpha(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
&\quad +2\alpha_n\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \left[\frac{(1-\alpha_n\bar{\gamma})^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\right]\|x_n-x^*\|^2 \\ &\quad +\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(x^*)-Ax^*,J(x_{n+1}-x^*)\rangle \\ &= \left[\frac{1-2\alpha_n\bar{\gamma}+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\right]\|x_n-x^*\|^2+\frac{\alpha_n^2\bar{\gamma}^2}{1-\alpha_n\gamma\alpha}\|x_n-x^*\|^2 \\ &\quad +\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(x^*)-Ax^*,J(x_{n+1}-x^*)\rangle \\ &= \left[1-\frac{2\alpha_n(\bar{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha}\right]\|x_n-x^*\|^2+\frac{\alpha_n^2\bar{\gamma}^2}{1-\alpha_n\gamma\alpha}\|x_n-x^*\|^2 \\ &\quad +\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(x^*)-Ax^*,J(x_{n+1}-x^*)\rangle.\end{aligned}$$

Put $\gamma_n=\frac{2\alpha_n(\bar{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha}$ and $\delta_n=\frac{\alpha_n^2\bar{\gamma}^2}{1-\alpha_n\gamma\alpha}\|x_n-x^*\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(x^*)-Ax^*,J(x_{n+1}-x^*)\rangle$. Then the above reduces to formula $\|x_{n+1}-x^*\|^2\leq(1-\gamma_n)\|x_n-x^*\|^2+\delta_n$. By control conditions $(C1)$, $(C2)$ and (31) it is easily seen that $\lim_{n\to\infty}\gamma_n=0$, $\sum_{n=0}^{\infty}\gamma_n=\infty$ and

$$\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n}=\limsup_{n\to\infty}\left[\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\gamma\alpha)}\|x_n-x^*\|^2+\frac{1}{\bar{\gamma}-\gamma\alpha}\langle\gamma f(x^*)-Ax^*,J(x_{n+1}-x^*)\rangle\right]\leq 0.$$

By Lemma 2.5, we conclude that $x_n\to x^*$ as $n\to\infty$. This completes the proof.

If X is a Hilbert space, we can get the following corollary easily.

Corollary 3.5. *Let C be a nonempty bounded closed convex subset of a Hilbert space H, $\mathcal{S}=\{T(s):0\leq s<\infty\}$ be a nonexpansive semigroup such that $F(\mathcal{S})\neq\emptyset$, $f:C\to C$ is a contraction mapping with coefficient $\alpha\in(0,1)$, A a strongly positive bounded linear operator with coefficient $\bar{\gamma}>0$ such that $0<\gamma<\frac{\bar{\gamma}}{\alpha}$ and $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $(0,1)$. Assume the following control conditions are hold:*

(C1) $\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\beta_n=0$;
(C2) $\sum_{n=0}^{\infty}\alpha_n=\infty$.

Then the sequence $\{x_n\}$ defined by (16) *converges strongly to the common fixed point x^* as $n\to\infty$, where x^* is a unique solution in $F(\mathcal{S})$ of the variational inequality* (7).

Remark 3.6. Theorem 3.4 improves and extends Theorem 3.1 of Kang et al.[11] from a Hilbert space to a Banach space.

Corollary 3.7. *Let C be a nonempty bounded closed convex subset of a Hilbert space H, $\mathcal{S}=\{T(s):0\leq s<\infty\}$ be a nonexpansive semigroup such that $F(\mathcal{S})\neq\emptyset$, $f:C\to C$ is a contraction mapping $\alpha\in(0,1)$, A a strongly positive bounded linear operator with coefficient $\bar{\gamma}>0$ such that $0<\gamma<\frac{\bar{\gamma}}{\alpha}$ and $\{\alpha_n\}$ be a sequence in $(0,1)$. Assume the following control conditions are hold:*

(C1) $\lim_{n\to\infty}\alpha_n=0$;
(C2) $\sum_{n=0}^{\infty}\alpha_n=\infty$.

Then the sequence $\{x_n\}$ defined by (14) *converges strongly to the common fixed point x^* as $n\to\infty$, where x^* is a unique solution in $F(\mathcal{S})$ of the variational inequality* (7).

Remark 3.8. Theorem 3.4 improves and extends Theorem 3.2 of Plubtieng and Wangkeeree [16] and Li et al [12] from a Hilbert space to a Banach space for a nonexpansive semigroup.

If taking $A=I$ and $\gamma=1$ in Theorem 3.4, we get the following corollary easily.

Corollary 3.9. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X which admits a weakly sequentially continuous duality mapping J from X into X^*, $\mathcal{S}=\{T(s):0\leq s<\infty\}$ be a nonexpansive semigroup such that $F(\mathcal{S})\neq\emptyset$, $f:C\to C$ is a contraction mapping with coefficient $\alpha\in(0,1)$ and $\{\alpha_n\}$ be a sequence in $(0,1)$. Assume the following control conditions are hold:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ *defined by* (12) *converges strongly to the common fixed point* x^* *as* $n \to \infty$, *where* x^* *is a unique solution in* $F(\mathcal{S})$ *of the variational inequality* (13).

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