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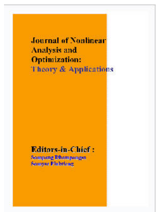


# The shrinking projection method for Generalized mixed Equilibrium Problems and Fixed Point Problems in Banach Spaces*

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**ABSTRACT**: The purpose of this paper is to introduce the iterative algorithms basing on the shrinking projection method for finding a common element of the set of common fixed points of two families of quasi-$\phi$-nonexpansive mappings and the set of solutions of the generalized mixed equilibrium problems in the framework of Banach spaces. Our results improve and extend the corresponding results announced by many others.

**KEYWORDS**: Quasi-$\phi$-nonexpansive mapping; Common fixed point; Shrinking projection method; Generalized mixed Equilibrium problems; Banach space.

## 1. Introduction

Let $E$ be a Banach space and let $E^*$ be the dual of $E$ and let $C$ be a closed convex subset of $E$. Let $J$ be the normalized duality mapping from $E$ into $2^{E^*}$ given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}, \forall x \in E,$$

where $E^*$ denoted the dual space of $E$ and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between $E$ and $E^*$. It is well known that if $E^*$ is uniformly convex, then $J$ is uniformly continuous on bounded subsets of $E$. Some properties of the duality mapping have been given in [11, 32, 39].

Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R}$ be real-valued function, and $\Psi : C \to E^*$ be a nonlinear mapping. The generalized mixed equilibrium problem is to find $u \in C$ such that

$$\Theta(u, y) + \langle \Psi u, y - u \rangle + \varphi(y) - \varphi(u) \geq 0, \ \forall y \in C. \tag{1}$$

The set of solutions to (1) is denoted by $GMEP(\Theta, \varphi, \Psi)$, i.e.,

$$GMEP(\Theta, \varphi, \Psi) = \{u \in C : \Theta(u, y) + \langle \Psi u, y - u \rangle + \varphi(y) - \varphi(u) \geq 0, \ \forall y \in C\}. \tag{2}$$

Special examples are as follows:

(I) If $\Psi = 0$, the problem (1) is equivalent to finding $u \in C$ such that

$$\Theta(u, y) + \varphi(y) - \varphi(u) \geq 0, \ \forall y \in C, \tag{3}$$

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which is called the mixed equilibrium problem (see [6]). The set of solutions to (3) is denoted by $MEP$.

(II) If $\Theta = 0$, the problem (1) is equivalent to finding $u \in C$ such that

$$\langle \Psi u, y - u \rangle + \varphi(y) - \varphi(u) \geq 0, \ \forall y \in C, \tag{4}$$

which is called the mixed variational inequality of Browder type (see [3]). The set of solutions to (4) is denoted by $VI(C, A, \varphi)$.

If $C$ is a nonempty closed convex subset of a Hilbert space $H$ and $P_C : H \to C$ is the metric projection of $H$ onto $C$, then $P_C$ is nonexpansive. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [1] recently introduced a generalized projection operator $C$ in a Banach space $E$ which is an analogue of the metric projection in Hilbert spaces.

Consider the functional $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2 \tag{5}$$

for all $x, y \in E$, where $J$ is the normalized duality mapping from $E$ to $E^*$. Observe that, in a Hilbert space $H$, (40) reduces to $\phi(y, x) = \|x - y\|^2$ for all $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = x^*$, where $x^*$ is the solution to the minimization problem:

$$\phi(x^*, x) = \inf_{y \in C} \phi(y, x). \tag{6}$$

The existence and uniqueness of the operator $\Pi_C$ follows from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping $J$ (see, for example, [1, 2, 9, 28]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function $\phi$ that

(1) $(\|y\| - \|x\|)^2 \leqslant \phi(y, x) \leqslant (\|y\| + \|x\|)^2$ for all $x, y \in E$.
(2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$ for all $x, y, z \in E$.
(3) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leqslant \|x\| \|Jx - Jy\| + \|y - x\| \|y\|$ for all $x, y \in E$.
(4) If $E$ is a reflexive, strictly convex and smooth Banach space, then, for all $x, y \in E$,

$$\phi(x, y) = 0 \text{ if and only if } x = y.$$

For more detail see [11, 32]. Let $C$ be a closed convex subset of $E$, and let $T$ be a mapping from $C$ into itself. We denote by $F(T)$ the set of fixed point of $T$. A point $p$ in $C$ is said to be an *asymptotic fixed point* of $T$ [29] if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of $T$ will be denoted by $\hat{F}(T)$. A mapping $T$ from $C$ into itself is called nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$ and relatively nonexpansive [8, 10, 12] if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leqslant \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mappings which was studied in [8, 10, 12] is of special interest in the convergence analysis of feasibility, optimization and equilibrium methods for solving the problems of image processing, rational resource allocation and optimal control. The most typical examples in this regard are the Bregman projections and the Yosida type operators which are the cornerstones of the common fixed point and optimization algorithms discussed in [9] (see also the references therein).

The mapping $T$ is said to be *$\phi$-nonexpansive* if $\phi(Tx, Ty) \leq \phi(x, y)$ for all $x, y \in C$. $T$ is said to be *quasi-$\phi$-nonexpansive* if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

**Remark 1.1.** The class of quasi-$\phi$-nonexpansive is more general than the class of relatively nonexpansive mappings [8, 10, 21, 24, 25] which requires the strong restriction $\hat{F}(T) = F(T)$.

Next, we give some examples which are closed quasi-$\phi$-nonexpansive [27].

**Example 1.2.** (1). Let $E$ be a uniformly smooth and strictly convex Banach space and $A$ be a maximal monotone mapping from $E$ to $E$ such that its zero set $A^{-1}0$ is nonempty. Then $J_r = (J + rA)^{-1}$ is a closed quasi-$\phi$-nonexpansive mapping from $E$ onto $D(A)$ and $F(J_r) = A^{-1}0$.

(2). Let $\Pi_C$ be the generalized projection from a smooth, strictly convex and reflexive Banach space $E$ onto a nonempty closed convex subset $C$ of $E$. Then $\Pi_C$ is a closed and quasi-$\phi$-nonexpansive mapping from $E$ onto $C$ with $F(\Pi_C) = C$.

On the other hand, One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping (see [4]). More precisely, let $t \in (0,1)$ and define a contraction $G_t : C \to C$ by $G_t x = tx_0 + (1-t)Tx$ for all $x \in C$, where $x_0 \in C$ is a fixed point in $C$. Applying Banach's Contraction Principle, there exists a unique fixed point $x_t$ of $G_t$ in $C$. It is unclear, in general, what is the behavior of $x_t$ as $t \to 0$ even if $T$ has a fixed point. However, in the case of $T$ having a fixed point, Browder [4] proved that the net $\{x_t\}$ defined by $x_t = tx_0 + (1-t)Tx_t$ for all $t \in (0,1)$ converges strongly to an element of $F(T)$ which is nearest to $x_0$ in a real Hilbert space. Motivated by Browder [4], Halpern [16] proposed the following innovation iteration process:

$$x_0 \in C, \ \ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n, \quad n \geqslant 0 \tag{7}$$

and proved the following theorem.

**Theorem H**. *Let $C$ be a bounded closed convex subset of a Hilbert space $H$ and let $T$ be a nonexpansive mapping on $C$. Define a real sequence $\{\alpha_n\}$ in $[0,1]$ by $\alpha_n = n^{-\theta}, 0 < \theta < 1$. Define a sequence $\{x_n\}$ by (7). Then $\{x_n\}$ converges strongly to the element of $F(T)$ nearest to $u$.*

Recently, Martinez-Yanes and Xu [20] has adapted Nakajo and Takahashi's [23] idea to modify the process (7) for a single nonexpansive mapping $T$ in a Hilbert space $H$:

$$\begin{cases} x_0 &= x \in C\text{chosen arbitrary}, \\ y_n &= \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n &= \{v \in C : \|y_n - v\|^2 \leqslant \|x_n - v\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, v\rangle)\}, \\ Q_n &= \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geqslant 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \end{cases} \tag{8}$$

where $P_C$ denotes the metric projection from $H$ onto a closed convex subset $C$ of $H$. They proved that if $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty} \alpha_n = 0$, then the sequence $\{x_n\}$ generated by (8) converges strongly to $P_{F(T)}x$.

In [25](see also [21]), Qin and Su improved the result of Martinez-Yanes and Xu [20] from Hilbert spaces to Banach spaces. To be more precise, they proved the following theorem.

**Theorem QS**. *Let $E$ be a uniformly convex and uniformly smooth Banach space, $C$ be a nonempty closed convex subset of $E$ and $T : C \to C$ be a relatively nonexpansive mapping. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Define a sequence $\{x_n\}$ in $C$ by the following algorithm:*

$$\begin{cases} x_0 &= x \in C\textit{chosen arbitrary}, \\ y_n &= J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JTx_n), \\ C_n &= \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, y_n) + (1-\alpha_n)\phi(v, x_n)\}, \\ Q_n &= \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \geqslant 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_0. \end{cases} \tag{9}$$

*where $J$ is the single-valued duality mapping on $E$. If $F(T)$ is nonempty, then $\{x_n\}$ converges to $\Pi_{F(T)}x_0$.*

Recently, Plubtieng and Ungchittrakool [24], still in the framework of Banach spaces, introduced the following hybrid projection algorithm for a pair of relatively nonexpansive mappings:

$$\begin{cases} x_0 &= x \in C\text{chosen arbitrary}, \\ y_n &= J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n), \\ z_n &= J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n), \\ H_n &= \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n) + \alpha_n(\|x_0\|^2 + 2\langle z, Jx_n - Jx\rangle)\}, \\ W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geqslant 0\}, \\ x_{n+1} &= P_{H_n \cap W_n} x, \quad n = 0, 1, 2, ..., \end{cases} \tag{10}$$

where $\{\alpha_n\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$ and $\{\beta_n^{(3)}\}$ are sequences in $[0,1]$ with $\beta_n^{(1)}+\beta_n^{(2)}+\beta_n^{(3)}=1$ for all $n\in\mathbb{N}\cup\{0\}$ and $T,S$ are relatively nonexpansive mappings and $J$ is the single-valued duality mapping on $E$. They proved that the sequence $\{x_n\}$ generated by (10) converges strongly to a common fixed point of $T$ and $S$.

Very recently, Qin, Cho, Kang and Zhou [26] introduced a new hybrid projection algorithm for two families of quasi-$\phi$-nonexpansive mappings which more general than relatively nonexpansive mappings to have strong convergence theorems in the framework of Banach spaces. To be more precise, they proved the following theorem:

**Theorem QCKZ**. Let $E$ be uniformly convex and uniformly smooth Banach space, and let $C$ be a nonempty closed convex subset of $E$. Let $\{S_i\}_{i\in I}$ and $\{T_i\}_{i\in I}$ be two families of closed quasi-$\phi$-nonexpansive mappings of $C$ into itself with $F:=\cap_{i\in I}F(T_i)\cap\cap_{i\in I}F(S_i)$ is nonempty, where $I$ is an index set. Let the sequence $\{x_n\}$ be generated by the following manner:

$$(11)\qquad \begin{cases} x_0 &= x\in C\text{chosen arbitrary},\\ z_{n,i} &= J^{-1}(\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n),\\ y_{n,i} &= J^{-1}(\alpha_{n,i}Jx_0+(1-\alpha_{n,i})Jz_{n,i}),\\ C_{n,i} &= \{u\in C:\phi(u,y_{n,i})\leqslant\phi(u,x_n)+\alpha_{n,i}(\|x_0\|^2+2\langle u,Jx_n-Jx_n\rangle)\},\\ C_n &= \cap_{i\in I}C_{n,i},\\ Q_0 &= C,\\ Q_n &= \{u\in Q_{n-1}:\langle x_n-u,Jx_0-Jx_n\rangle\geq 0\},\\ x_{n+1} &= \Pi_{C_n\cap Q_n}x_0,\quad n=0,1,2,\ldots, \end{cases}$$

where $J$ is the duality mapping on $E$, $\{\alpha_{n,i}\},\{\beta_{n,i}^{(i)}\}(i=1,2,3,\dots)$ are sequences in $(0,1)$ such that

(i) $\beta_{n,i}^{(1)}+\beta_{n,i}^{(2)}+\beta_{n,i}^{(3)}=I$ for all $i\in I$
(ii) $\lim_{n\to\infty}\alpha_{n,i}=0$ for all $i\in I$; and
(iii) $\liminf_{n\to\infty}\beta_{n,i}^{(2)}\beta_{n,i}^{(3)}>0$ and $\lim_{n\to\infty}\beta_{n,i}^{(1)}=0$ for all $i\in I$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

On the other hand, let $f: C\times C\to\mathbb{R}$ be a bifunction. The equilibrium problem for $f$ is to find $\hat{x}\in C$ such that

$$(12)\qquad f(\hat{x},y)\ \geq\ 0,\quad \forall y\in C.$$

The set of solutions of (12) is denoted by $EP(f)$.

Numerous problems in physics, optimization, and economics reduce to find a solution of the equilibrium problem. Some methods have been proposed to solve the equilibrium problem in a Hilbert space; see, for instance, Blum and Oettli [5], Combettes and Hirstoaga [7], and Moudafi [22]. On the other hand, there are some methods for approximation of fixed points of Fixed Point Theory and Applications a nonexpansive mapping. Recently, Tada and Takahashi [30, 31] and Takahashi and Takahashi [37] obtained weak and strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. In particular, Tada and Takahashi [31] established a strong convergence theorem for finding a common element of two sets by using the hybrid method introduced in Nakajo and Takahashi [23]. They also proved such a strong convergence theorem in a uniformly convex and uniformly smooth Banach space. Recently, Takahashi et al. [38] introduced a hybrid method which is different from Nakajo and Takahashis hybrid method. It is called the shrinking projection method. They obtained the strong convergence theorem in the frame work of Hilbert spaces. Based on the so-called shrinking projection method of Takahashi et al. [38], Takahashi and Zembayashi [36] introduced the following iterative scheme :

$$(13)\qquad \begin{cases} x_0 &= x \in C, \quad C_0 = C, \\ y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ u_n &\in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0, \ \forall y \in C, \\ C_{n+1} &= \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \ \forall n \geq 0, \end{cases}$$

where $J$ is the single-valued duality mapping on $E$ and $\Pi_C$ is the generalized projection from $E$ onto $C$. They proved that the sequence $\{x_n\}$ defined by (13) converges strongly to $q = \Pi_{F(T)\cap EP(f)} x_0$ under appropriate conditions imposed on the parameters.

Motivated and inspired by Iiduka and Takahashi [17], Martinez-Yanes and Xu [20], S. Matsushita and W. Takahashi [21], Plubtieng and Ungchittrakool [24], Qin and Su [25], Qin, Cho, Kang and Zhou[26], Takahashi et al. [38] and Takahashi and Zembayashi [36], we introduce a new hybrid projection algorithm basing on the shrinking projection method for two families of quasi-$\phi$-nonexpansive mappings which more general than relatively nonexpansive mappings to have strong convergence theorems for approximating the common element of the set of common fixed points of two families of quasi-$\phi$-nonexpansive mappings and the set of solutions of the equilibrium problem in the framework of Banach spaces.

## 2. **Preliminaries**

A Banach space $E$ is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in $E$ such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of $E$. Then the Banach space $E$ is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. It is well know that if $E$ is smooth, then the duality mapping $J$ is single valued. It is also known that if $E$ is uniformly smooth, then $J$ is uniformly norm-to-norm continuous on each bounded subset of $E$. Some properties of the duality mapping have been given in [14, 28, 32, 33]. A Banach space $E$ is said to have Kadec-Klee property if a sequence $\{x_n\}$ of $E$ satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if $E$ is uniformly convex, then $E$ has the Kadec-Klee property; see [14, 32, 33] for more details. Let $E$ be a smooth Banach space.

Now we collect some definitions and lemmas which will be used in the proofs for the main results in the next section. Some of them are known; others are not hard to derive.

**Lemma 2.1** (Kamimura and Takahashi [18]). *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}, \{z_n\}$ be two sequences of E such that either $\{y_n\}$ or $\{z_n\}$ is bounded. If* $\lim_{n\to\infty} \phi(y_n, z_n) = 0$, *then* $\lim_{n\to\infty} \|y_n - z_n\| = 0$.

**Lemma 2.2** (Alber [1], Alber and Reich [2], Kamimura and Takahashi [18]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \geqslant 0$ for $y \in C$.*

**Lemma 2.3** (Alber [1], Alber and Reich [2], Kamimura and Takahashi [18]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leqslant \phi(y, x)$$

*for all $y \in C$.*

**Lemma 2.4** (Qin et al. [26]). *Let $E$ be a uniformly convex and smooth Banach space, $C$ be a closed convex subset of $E$ and $T$ be a closed and quasi-$\phi$-nonexpansive mapping from $C$ into itself. Then $F(T)$ is a closed convex subset of $C$.*

Let $E$ be a reflexive strictly convex, smooth and uniformly Banach space and the duality mapping from $E$ to $E^*$. Then $J^{-1}$ is also single-valued, one to one, surjective, and it is the duality mapping from $E^*$ to $E$. We make use of the following mapping $V$ studied in Alber [1],

$$V(x,x^*) = \|x^2\| - 2\langle x, x^*\rangle + \|x\|^2 \tag{14}$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x,x^*) = \phi(x, J^{-1}(x^*))$. We know the following lemma:

**Lemma 2.5** (Kamimura and Takahashi [18]). *Let $E$ be a reflexive, strictly convex and smooth Banach space, and let $V$ be as in* (14). *Then*

$$V(x,x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*)$$

*for all $x \in E$ and $x^*, y^* \in E^*$.*

**Lemma 2.6** ([13, Lemma 1.4]). *Let $X$ be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leqslant r\}$ be a closed ball of $X$. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leqslant \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x-y\|), \tag{15}$$

*for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0,1]$ with $\lambda + \mu + \gamma = 1$.*

For solving the equilibrium problem, let us assume that a bifunction $f$ satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;
(A2) $f$ is monotone, that is, $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$,

$$\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \quad \leq \quad f(x,y); \tag{16}$$

(A4) for all $x \in C$, $f(x,\cdot)$ is convex and lower semicontinuous.

For example, let $A$ be a continuous and monotone operator of $C$ into $E^*$ and define

$$f(x,y) = \langle Ax, y - x\rangle, \forall x, y \in C.$$

Then, $f$ satisfies (A1)-(A4).

**Lemma 2.7** (Blum and Oettli [5]). *Let $C$ be a closed convex subset of a smooth, strictly convex, and reflexive Banach spaces $E$, let $f$ be a bifunction from $C \times C \to \mathbb{R}$ satisfying $(A1)-(A4)$, and let $r > 0$ and $x \in E$. Then, there exists $u \in C$ such that*

$$f(u,y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C. \tag{17}$$

**Lemma 2.8** (Takahashi and Zembayashi [35]). *Let $C$ be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space $E$, and let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$. For all $r > 0$ and $x \in E$, define a mapping*

$$T_r x = \Big\{u \in C : f(z,y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C\Big\}. \tag{18}$$

*Then, the following hold:*

(1) *$T_r$ is single-valued;*
(2) *$T_r$ is a firmly nonexpansive-type mapping* [19], *that is, for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \quad \leq \quad \langle T_r x - T_r y, Jx - Jy\rangle; \tag{19}$$

(3) *$F(T_r) = EP(f)$;*
(4) *$EP(f)$ is closed and convex.*

**Lemma 2.9** (Takahashi and Zembayashi [35]). *Let $C$ be a closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$, let $\Theta$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, and let $r > 0$. Then, for all $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \quad \leq \quad \phi(q, x).$$

**Lemma 2.10.** *Let $C$ be a closed convex subset of a smooth, strictly convex, and reflexive Banach spaces $E$, let $\Theta$ be a bifunction from $C \times C \to \mathbb{R}$ satisfying $(A1)-(A4)$. Let $\Psi : C \to E^*$ be a continuous and monotone operator and $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $r > 0$ be any given number and $x \in E$ be any given point. Then, there exists $u \in C$ such that*

$$\Theta(x,y) + \varphi(y) - \varphi(x) + \langle \Psi x, y - x\rangle + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C. \tag{20}$$

*Proof.* We define a bifunction $f : C \times C \to \mathbb{R}$ by

$$f(x,y) = \Theta(x,y) + \varphi(y) - \varphi(x) + \langle \Psi x, y - x\rangle, \quad \forall x, y \in C. \tag{21}$$

Next, we prove that the bifunction $f$ satisfies condition (A1)-(A4):
(A1) $f(x,x) = 0$ for all $x \in C$.
Since $f(x,x) = \Theta(x,x) + \varphi(x) - \varphi(x) + \langle \Psi x, x - x\rangle = 0$, for all $x \in C$.
(A2) $f$ is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$.
From the definition of $f$ we have

$$\begin{aligned} f(x,y) + f(y,x) &= \Theta(x,y) + \varphi(y) - \varphi(x) + \langle \Psi x, y - x\rangle + \Theta(y,x) + \varphi(x) - \varphi(y) + \langle \Psi x, x - y\rangle \\ &= \Theta(x,y) + \Theta(y,x) \leq 0. \end{aligned}$$

(A3) for each $x, y, z \in C$,

$$\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x,y).$$

Since

$$\begin{aligned} & \limsup_{t\downarrow 0} f(tz + (1-t)x, y) \\ = \ & \limsup_{t\downarrow 0} [\Theta(tz + (1-t)x, y) + \varphi(y) - \varphi(tz + (1-t)x) + \langle \Psi x, y - (tz + (1-t)x)\rangle] \\ \leq \ & \limsup_{t\downarrow 0} \Theta(tz + (1-t)x, y) + \varphi(y) - \liminf_{t\downarrow 0} \varphi(tz + (1-t)x) + \langle \Psi x, y\rangle - \liminf_{t\downarrow 0} \langle \Psi x, tz + (1-t)x\rangle \\ \leq \ & t\Theta(x,y) + (1-t)\Theta(x,y) + \varphi(y) - \varphi(x) + \langle \Psi x, y\rangle - \langle \Psi x, x\rangle \\ = \ & \Theta(x,y) + \varphi(y) - \varphi(x) + \langle \Psi x, y - x\rangle = f(x,y). \end{aligned}$$

(A4) for each $x \in C$, $y \mapsto \Theta(x,y)$ is a convex and lower semicontinuous.
For each $x \in C$, $\forall t \in (0,1)$ and $\forall y, z \in C$, since $G$ satisfies (A4), we have

$$\begin{aligned} f(x, ty + (1-t)z) &= \Theta(x, ty + (1-t)z) + \varphi(ty + (1-t)z) - \varphi(x) + \langle \Psi x, (ty + (1-t)z) - x\rangle \\ &\leq t\Theta(x,y) + (1-t)\Theta(x,z) + t\varphi(y) + (1-t)\varphi(z) - \varphi(x) + \langle \Psi x, (ty + (1-t)z) - x\rangle \\ &= t[\Theta(x,y) + \varphi(y) - \varphi(x) + \langle \Psi x, y - x\rangle] \\ &\quad + (1-t)[\Theta(x,z) + \varphi(z) - \varphi(x) + \langle \Psi x, z - x\rangle] \\ &= tf(x,y) + (1-t)f(x,z). \end{aligned}$$

So, $y \mapsto f(x,y)$ is convex.
Similarly, we can prove that $y \mapsto f(x,y)$ is lower semicontinuous. Hence $f$ satisfies condition (A1)-(A4). Applying Lemma 2.7, there exists $u \in C$ such that

$$f(u,y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C.$$

That is

$$\Theta(x,y) + \varphi(y) - \varphi(x) + \langle \Psi x, y - x\rangle + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C.$$

This is completes the proof. □

## 3. **Main Results**

In this section, we prove two strong convergence theorems for approximating the common element of the set of common fixed points of two families of quasi-$\phi$-nonexpansive mappings and the set of solutions of the generalized mixed equilibrium problem in the framework of a real Banach space.

**Theorem 3.1.** *Let $E$ be a uniformly convex and uniformly smooth Banach space and $C$ be a nonempty closed convex subset of $E$. Let $\Psi : C \to E^*$ be a continuous and monotone operator and $\varphi : C \to \mathbb{R}$ be a a lower semi-continuous and convex function. Let $\Theta$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, let $\{T_i\}_{i\in I}$ and $\{S_i\}_{i\in I}$ be two families of closed quasi-$\phi$-nonexpansive mappings $T_i$, $S_i : C \to C$ such that the common fixed point set $F := \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i) \cap GMEP(\Theta, \varphi, \Psi)$ is nonempty , where $I$ is an index set. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$(22)\quad \begin{cases} x_0 \in C \text{ chosen arbitrary and } C_{0,i} = C, \quad \forall i \in I, \\ z_{n,i} = J^{-1}(\beta^{(1)}_{n,i} Jx_n + \beta^{(2)}_{n,i} JT_i x_n + \beta^{(3)}_{n,i} JS_i x_n), \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_0 - (1-\alpha_{n,i}) Jz_{n,i}), \\ u_{n,i} \in C \text{ such that } \Theta(u_{n_i}, y) + \varphi(y) - \varphi(u_{n,i}) + \\ \langle \Psi u_{n,i}, y - u_{n,i}\rangle + \frac{1}{r_{n,i}}\langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i}\rangle \geq 0, \ \forall y \in C, \\ C_{n+1,i} = \{u \in C_{n,i} : \phi(u, u_{n,i}) \leq \phi(u, x_n) + \alpha_{n,i}(\|x\|^2 + 2\langle u, Jx_n - Jx_0\rangle)\}, \\ C_{n+1} = \bigcap_{i\in I} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

*where $J$ is a duality mapping on $E$, $\{\alpha_{n,i}\}$, $\{\beta^{(i)}_{n,i}\}$ $(i = 1, 2, 3)$ and $\{r_{n,i}\}$ are sequences in $(0, 1)$ satisfying*

(a) $\lim_{n\to\infty} \alpha_{n,i} = 0$ *for each* $i \in I$;
(b) $\{r_{n,i}\} \subset [a, \infty)$ *for some* $a > 0$ *and for all* $i \in I$;
(c) $\beta^{(1)}_{n,i} + \beta^{(2)}_{n,i} + \beta^{(3)}_{n,i} = 1$ *for each* $i \in I$ *and if one of the following conditions is satisfied*
 (c-1) $\liminf_{n\to\infty} \beta^{(1)}_{n,i}\beta^{(l)}_{n,i} > 0$ *for all* $l = 2, 3$ *and for all* $i \in I$ *and*
 (c-2) $\liminf_{n\to\infty} \beta^{(2)}_{n,i}\beta^{(3)}_{n,i} > 0$ *and* $\liminf_{n\to\infty} \beta^{(1)}_{n,i} = 0$ *for each* $i \in I$.

*Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.*

*Proof.* Let the bifunction $f : C \times C \to \mathbb{R}$ be defined by (21). Therefore, the mixed equilibrium problem (1) is equivalent to the following equilibrium problem: find $u \in C$ such that

$$f(u, y) \geq 0, \quad \forall y \in C,$$

and (66) can be written as:

$$(23)\quad \begin{cases} x_0 \in C \text{ chosen arbitrary and } C_{0,i} = C, \quad \forall i \in I, \\ z_{n,i} = J^{-1}(\beta^{(1)}_{n,i} Jx_n + \beta^{(2)}_{n,i} JT_i x_n + \beta^{(3)}_{n,i} JS_i x_n), \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_0 - (1-\alpha_{n,i}) Jz_{n,i}), \\ u_{n,i} \in C \text{ such that } f(u_{n_i}, y) + \frac{1}{r_{n,i}}\langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i}\rangle \geq 0, \ \forall y \in C, \\ C_{n+1,i} = \{u \in C_{n,i} : \phi(u, u_{n,i}) \leq \phi(u, x_n) + \alpha_{n,i}(\|x\|^2 + 2\langle u, Jx_n - Jx_0\rangle)\}, \\ C_{n+1} = \bigcap_{i\in I} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0. \end{cases}$$

Since the bifunction $f$ satisfies conditions (A1) - (A4), from Lemma 2.10, for given $r > 0$ and $x \in C$, we define $T_r : C \to 2^C$ by

$$T_r(x) = \{u \in C : f(u, y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C\}.$$

Moreover, $T_r$ satisfies the conclusions in Lemma 2.8. We divide the proof of Theorem 3.1 into seven steps:

**Step 1**. Show that $\Pi_F x_0$ and $\Pi_{C_{n+1}} x_0$ are well defined.

By Lemma 2.4, we know that $\bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i)$ is closed and convex. From Lemma 2.8 (4), we also have $EP(f)$ is closed and convex. Hence $F := \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i) \cap EP(f)$ is a nonempty, closed, and convex subset of $C$. Consequently, $\Pi_F x_0$ is well defined.

From the definition of $C_n$, it is obvious that $C_n$ is closed for each $n \geq 0$. We show that $C_{n+1}$ is convex for each $n \geq 0$. Notice that

$$C_{n+1,i} = \{u \in C_{n,i} : \phi(u, u_{n,i}) \leq \phi(u, x_n) + \alpha_{n,i}(\|x_0\|^2 + 2\langle u, Jx_n - Jx_0\rangle)\}$$

is equivalent to

$$C'_{n+1,i} = \{u \in C_{n,i} : 2\langle u, Jx_n - Jy_{n,i}\rangle - 2\alpha_{n,i}\langle u, Jx_n - Jx_0\rangle \leq \|x_n\|^2 - \|y_{n,i}\|^2 + \alpha_{n,i}\|x_0\|^2\}.$$

It is easy to see that $C'_{n+1,i}$ is closed and convex for all $n \geq 0$ and $i \in I$. Therefore, $C_{n+1} = \bigcap_{i\in I} C_{n+1,i} = \bigcap_{i\in I} C'_{n+1,i}$ is closed and convex for every $n \geq 0$. This shows that $\Pi_{C_{n+1}} x_0$ is well-defined.

**Step 2**. Show that $F \subset C_n$ for all $n \geq 0$.

First, we observe that $u_{n,i} = T_{r_{n,i}} y_{n,i}$ for all $n \geq 1$ and $F \subset C_0 = C$. For any $w \in F$ and all $i \in I$, one has

$$\begin{aligned}
\phi(w, z_{n,i}) &= \phi(w, J^{-1}(\beta^{(1)}_{n,i} Jx_n + \beta^{(2)}_{n,i} JT_i x_n + \beta^{(3)}_{n,i} JS_i x_n)) \\
&= \|w\|^2 - 2\langle w, \beta^{(1)}_{n,i} Jx_n + \beta^{(2)}_{n,i} JT_i x_n + \beta^{(3)}_{n,i} JS_i x_n)\rangle \\
&\quad + \|\beta^{(1)}_{n,i} Jx_n + \beta^{(2)}_{n,i} JT_i x_n + \beta^{(3)}_{n,i} JS_i x_n\|^2 \\
&\leq \|w\|^2 - 2\beta^{(1)}_{n,i}\langle w, Jx_n\rangle - 2\beta^{(2)}_{n,i}\langle w, JT_i x_n\rangle - 2\beta^{(3)}_{n,i}\langle w, JS_i x_n\rangle \\
&\quad + \beta^{(1)}_{n,i}\|x_n\|^2 + \beta^{(2)}_{n,i}\|T_i x_n\|^2 + \beta^{(3)}_{n,i}\|JS_i x_n\|^2 \\
&= \beta^{(1)}_{n,i}\phi(w, x_n) + \beta^{(2)}_{n,i}\phi(w, T_i x_n) + \beta^{(3)}_{n,i}\phi(w, S_i x_n) \\
&\leq \beta^{(1)}_{n,i}\phi(w, x_n) + \beta^{(2)}_{n,i}\phi(w, x_n) + \beta^{(3)}_{n,i}\phi(w, x_n) \\
&= \phi(w, x_n)
\end{aligned}$$

and hence

$$\begin{aligned}
\phi(w, u_{n,i}) &= \phi(w, T_{r_{n,i}} y_{n,i}) \\
&\leq \phi(w, y_{n,i}) \\
&= \phi(w, J^{-1}(\alpha_{n,i} Jx_0 - (1-\alpha_{n,i}) Jz_{n,i}) \\
&= \|w\|^2 - 2\langle w, \alpha_{n,i} Jx_0 - (1-\alpha_{n,i}) Jz_{n,i}\rangle + \|\alpha_{n,i} Jx_0 - (1-\alpha_{n,i}) Jz_{n,i}\|^2 \\
&\leq \|w\|^2 - 2\alpha_{n,i}\langle w, Jx_0\rangle - 2(1-\alpha_{n,i})\langle w, Jz_{n,i}\rangle + \alpha_{n,i}\|x_0\|^2 + (1-\alpha_{n,i})\|z_{n,i}\|^2 \\
&= \alpha_{n,i}\phi(w, x_0) + (1-\alpha_{n,i})\phi(w, z_{n,i}) \\
&\leq \alpha_{n,i}\phi(w, x_0) + (1-\alpha_{n,i})\phi(w, x_n) \\
&= \phi(w, x_n) + \alpha_{n,i}[\phi(w, x_0) - \phi(w, x_n)] \\
&\leq \phi(w, x_n) + \alpha_{n,i}(\|x_0\|^2 + 2\langle w, Jx_n - Jx_0\rangle). \qquad (24)
\end{aligned}$$

This show that $w \in C_{n+1,i}$ for each $i \in I$. That is, $w \in C_{n+1} = \bigcap_{i\in I} C_{n+1,i}$ for all $n \geq 0$. Hence $F \subset C_n$ for all $n \geq 0$.

**Step 3**. Show that $\lim_{n\to\infty} \phi(x_n, x_0)$ exists.

We note that $C_{n+1,i} \subset C_{n,i}$ for all $n \geq 0$ and for all $i \in I$. Hence

$$C_{n+1} = \cap_{i\in I} C_{n+1,i} \subset C_n = \cap_{i\in I} C_{n,i}.$$

From $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ and $x_n = \Pi_{C_n} x_0 \in C_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 1. \qquad (25)$$

This is, $\{\phi(x_n, x_0)\}$ is nondecreasing. On the other hand, from Lemma 2.3, we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(w, x_0) - \phi(w, x_n) \leq \phi(w, x_0). \tag{26}$$

for each $w \in F \subset C_n$. Combining (25) and (26), we obtain that limit $\{\phi(x_n, x_0)\}$ exists.

**Step 4**. Show that $\{x_n\}$ is a convergent sequence in $C$.

Since $x_m = \Pi_{C_m} x_0 \in C_m \subset C_n$ for $m \geq n$, by Lemma 2.3, We also have

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{Q_n} x_0) \\ &\leq \phi(x_m, x_0) - \phi(\Pi_{Q_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0). \end{aligned} \tag{27}$$

Letting $m, n \to \infty$ in (27), one has $\phi(x_m, x_n) \to 0$. It follows from Lemma 2.1 that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. Since $E$ is a Banach space and $C$ is closed and convex, one can assume that

$$x_n \to p \in C \quad (n \to \infty). \tag{28}$$

**Step 5**. Show that $p \in \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i) \cap GMEP(\Theta, \varphi, \Psi)$.

(a) We first will show that $p \in \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i)$. Taking $m = n+1$ in (27), we obtain.

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{29}$$

From Lemma 2.1, one has

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{30}$$

Noticing that $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, from the definition of $C_{n+1}$, for every $i \in I$, we obtain

$$\phi(x_{n+1}, u_{n,i}) \leq \phi(x_{n+1}, x_n) + \alpha_{n,i}(\|x_0\|^2 + 2\langle x_{n+1}, Jx_n - Jx_0\rangle).$$

It follows from (29) and $\lim_{n\to\infty} \alpha_{n,i} = 0$ that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_{n,i}) = 0, \quad \forall i \in I. \tag{31}$$

From Lemma 2.1, we have $\lim_{n\to\infty} \|x_{n+1} - u_{n,i}\| = 0$. This together with (30) implies that

$$\lim_{n\to\infty} \|x_n - u_{n,i}\| = 0, \quad \forall i \in I. \tag{32}$$

Since $J$ is uniformly norm-to-norm continuous on bounded sets, for every $i \in I$, one has

$$\lim_{n\to\infty} \|Jx_n - Ju_{n,i}\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{33}$$

It follows from $x_n \to p$ as $n \to \infty$ that

$$u_{n,i} \to p \text{ as } n \to \infty, \quad \forall i \in I. \tag{34}$$

Let $r = \sup_{n\geq 1}\{\|x_n\|, \|T_i x_n\|, \|S_i x_n\|\}$ for every $i \in I$. Therefore Lemma 2.6 implies that there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ satisfying $g(0) = 0$ and (15)

**Case I**. Assume that (c-1) holds. We observe that

$$\begin{aligned}
\phi(w, u_{n,i}) &= \phi(w, T_{r_{n,i}} y_{n,i}) \\
&\leq \phi(w, y_{n,i}) \\
&= \phi(w, J^{-1}(\beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i x_n)) \\
&= \|w\|^2 - 2\langle w, \beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i x_n \rangle \\
&\quad + \|\beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i x_n\|^2 \\
&\leq \|w\|^2 - 2\beta_{n,i}^{(1)} \langle w, Jx_n \rangle - 2\beta_{n,i}^{(2)} \langle w, JT_i x_n \rangle - 2\beta_{n,i}^{(3)} \langle w, JS_i x_n \rangle \\
&\quad + \beta_{n,i}^{(1)} \|x_n\|^2 + \beta_{n,i}^{(2)} \|T_i x_n\|^2 + \beta_{n,i}^{(3)} \|S_i x_n\|^2 \\
&\quad - \beta_{n,i}^{(1)} \beta_{n,i}^{(2)} g(\|Jx_n - JT_i x_n\|) \\
&= \beta_{n,i}^{(1)} \phi(w, x_n) + \beta_{n,i}^{(2)} \phi(w, T_i x_n) + \beta_{n,i}^{(3)} \phi(w, S_i x_n) \\
&\quad - \beta_{n,i}^{(1)} \beta_{n,i}^{(2)} g(\|Jx_n - JT_i x_n\|) \\
&\leq \beta_{n,i}^{(1)} \phi(w, x_n) + \beta_{n,i}^{(2)} \phi(w, x_n) + \beta_{n,i}^{(3)} \phi(w, x_n) - \beta_{n,i}^{(1)} \beta_{n,i}^{(2)} g(\|Jx_n - JT_i x_n\|) \\
&= \phi(w, x_n) - \beta_{n,i}^{(1)} \beta_{n,i}^{(2)} g(\|Jx_n - JT_i x_n\|).
\end{aligned}$$

This implies that

(35) $$\beta_{n,i}^{(1)} \beta_{n,i}^{(2)} g(\|Jx_n - JT_i x_n\|) \leq \phi(w, x_n) - \phi(w, u_{n,i}), \quad \forall i \in I.$$

On the other hand, for every $i \in I$, one has

$$\begin{aligned}
\phi(w, x_n) - \phi(w, u_{n,i}) &= \|x_n\|^2 - \|u_{n,i}\|^2 - 2\langle w, Jx_n - Ju_{n,i} \rangle \\
&\leq \|x_n - u_{n,i}\|(\|x_n\| + \|u_{n,i}\|) + 2\|w\|\|Jx_n - Ju_{n,i}\|.
\end{aligned}$$

It follows that (32) and (33) that

(36) $$\phi(w, x_n) - \phi(w, u_{n,i}) \to 0 \quad (n \to \infty), \quad \forall i \in I.$$

Observing that assumption $\liminf_{n\to\infty} \beta_{n,i}^{(1)} \beta_{n,i}^{(2)} > 0$, (35) and (36), one has

$$g(\|Jx_n - JT_i x_n\|) \to 0 \quad (n \to \infty), \quad \forall i \in I.$$

It follows from the property of the function $g$ that

(37) $$\|Jx_n - JT_i x_n\| \to 0 \quad (n \to \infty), \quad \forall i \in I.$$

Since $J^{-1}$ is also uniformly norm-to-norm continuous on bounded sets, for each $i \in I$, one has

(38) $$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0.$$

In a similar way, one has

(39) $$\lim_{n\to\infty} \|x_n - S_i x_n\| = 0.$$

Noticing (28), (38), (39) and the closedness of $T_i$ and $S_i$ that $p \in \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i)$.

**Case II**. Assume that (c-2) holds. We observe that

$$\begin{aligned}
\phi(w,u_{n,i}) &= \phi(w,T_{r_{n,i}}y_{n,i}) \\
&\leq \phi(w,y_{n,i}) \\
&= \phi(w,J^{-1}(\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n)) \\
&= \|w\|^2-2\langle w,\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n\rangle \\
&\quad +\|\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n\|^2 \\
&\leq \|w\|^2-2\beta_{n,i}^{(1)}\langle w,Jx_n\rangle-2\beta_{n,i}^{(2)}\langle w,JT_ix_n\rangle-2\beta_{n,i}^{(3)}\langle w,JS_ix_n\rangle \\
&\quad +\beta_{n,i}^{(1)}\|x_n\|^2+\beta_{n,i}^{(2)}\|T_ix_n\|^2+\beta_{n,i}^{(3)}\|S_ix_n\|^2 \\
&\quad -\beta_{n,i}^{(2)}\beta_{n,i}^{(3)}g(\|JS_ix_n-JT_ix_n\|) \\
&= \beta_{n,i}^{(1)}\phi(w,x_n)+\beta_{n,i}^{(2)}\phi(w,T_ix_n)+\beta_{n,i}^{(3)}\phi(w,S_ix_n) \\
&\quad -\beta_{n,i}^{(2)}\beta_{n,i}^{(3)}g(\|JT_ix_n-JS_ix_n\|) \\
&\leq \beta_{n,i}^{(1)}\phi(w,x_n)+\beta_{n,i}^{(2)}\phi(w,x_n)+\beta_{n,i}^{(3)}\phi(w,x_n)-\beta_{n,i}^{(2)}\beta_{n,i}^{(3)}g(\|JT_ix_n-JS_ix_n\|) \\
&= \phi(w,x_n)-\beta_{n,i}^{(2)}\beta_{n,i}^{(3)}g(\|JT_ix_n-JS_ix_n\|).
\end{aligned}$$

This implies that

(40) $$\beta_{n,i}^{(2)}\beta_{n,i}^{(3)}g(\|JT_ix_n-JS_ix_n\|) \leq \phi(w,x_n)-\phi(w,u_{n,i}), \quad \forall i\in I.$$

On the other hand, for every $i\in I$, one has

$$\begin{aligned}
\phi(w,x_n)-\phi(w,u_{n,i}) &= \|x_n\|^2-\|u_{n,i}\|^2-2\langle w,Jx_n-Ju_{n,i}\rangle \\
&\leq \|x_n-u_{n,i}\|(\|x_n\|+\|u_{n,i}\|)+2\|w\|\|Jx_n-Ju_{n,i}\|.
\end{aligned}$$

It follows that (32) and (33) that

(41) $$\phi(w,x_n)-\phi(w,u_{n,i})\to 0 \quad (n\to\infty), \quad \forall i\in I.$$

Observing that assumption $\liminf_{n\to\infty}\beta_{n,i}^{(2)}\beta_{n,i}^{(3)}>0$, (40) and (41), one has

$$g(\|JT_ix_n-JS_ix_n\|)\to 0 \quad (n\to\infty), \quad \forall i\in I.$$

It follows from the property of the function $g$ that

(42) $$\|JT_ix_n-JS_ix_n\|\to 0 \quad (n\to\infty), \quad \forall i\in I.$$

Since $J^{-1}$ is also uniformly norm-to-norm continuous on bounded sets, for each $i\in I$, one has

(43) $$\lim_{n\to\infty}\|T_ix_n-S_ix_n\|=0.$$

On the other hand, for each $i\in I$, one has

$$\begin{aligned}
\phi(T_ix_n,u_{n,i}) &= \phi(T_ix_n,T_{r_{n,i}}y_{n,i}) \\
&\leq \phi(T_ix_n,y_{n,i}) \\
&= \phi(T_ix_n,J^{-1}(\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n)) \\
&= \|T_ix_n\|^2-2\langle T_ix_n,\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n\rangle \\
&\quad +\|\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n\|^2 \\
&\leq \|T_ix_n\|^2-2\beta_{n,i}^{(1)}\langle T_ix_n,Jx_n\rangle-2\beta_{n,i}^{(2)}\langle T_ix_n,JT_ix_n\rangle \\
&\quad -2\beta_{n,i}^{(3)}\langle T_ix_n,JS_ix_n\rangle+\beta_{n,i}^{(1)}\|x_n\|^2+\beta_{n,i}^{(2)}\|T_ix_n\|^2+\beta_{n,i}^{(3)}\|S_ix_n\|^2 \\
&\leq \beta_{n,i}^{(1)}\phi(T_ix_n,x_n)+\beta_{n,i}^{(3)}\phi(T_ix_n,S_ix_n). \qquad (44)
\end{aligned}$$

Observe that

$$\begin{aligned}\phi(T_ix_n, S_ix_n) &= \|T_ix_n\|^2 - 2\langle T_ix_n, JS_ix_n\rangle + \|S_ix_n\|^2\\ &= \|T_ix_n\|^2 - 2\langle T_ix_n, JT_ix_n\rangle + 2\langle T_ix_n, JT_ix_n - JS_ix_n\rangle + \|S_ix_n\|^2\\ &\leq \|S_ix_n\|^2 - \|T_ix_n\|^2 + 2\|S_ix_n\|\|JT_ix_n - JS_ix_n\|\\ &\leq \|S_ix_n - T_ix_n\|(\|S_ix_n\| + \|T_ix_n\|) + 2\|S_ix_n\|\|JT_ix_n - JS_ix_n\|.\end{aligned}$$

It follows from (42) and (43) that

(45) $$\lim_{n\to\infty} \phi(T_ix_n, S_ix_n) = 0, \quad \forall i \in I.$$

Noticing that $\beta_{n,i}^{(1)} \to 0$ as $n \to \infty$, (44) and (45), one arrives at

(46) $$\lim_{n\to\infty} \phi(T_ix_n, u_{n,i}) = 0, \quad \forall i \in I.$$

From Lemma 2.1, one obtains

(47) $$\lim_{n\to\infty} \|T_ix_n - u_{n,i}\| = 0, \quad \forall i \in I.$$

Hence

(48) $$\|T_ix_n - x_n\| \leq \|T_ix_n - u_{n,i}\| + \|u_{n,i} - x_n\|, \quad \forall i \in I.$$

It follows from (32) and (47) that

(49) $$\lim_{n\to\infty} \|T_ix_n - x_n\| = 0, \quad \forall i \in I.$$

Moreover, we observe that

(50) $$\|S_ix_n - x_n\| \leq \|S_ix_n - T_ix_n\| + \|T_ix_n - x_n\|, \quad \forall i \in I.$$

Combining (43) with (49), one obtains $\lim_{n\to\infty} \|S_ix_n - x_n\| = 0$ for each $i \in I$. Noticing (28), it follows from the closedness of $T_i$ and $S_i$ and $x_n \to p$ that $p \in \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i)$.

(b) We next show that $p \in GMEP(\Theta, \varphi, \Psi)$.

From (40), we see

(51) $$\phi(u, y_{n,i}) \leq \phi(u, x_{n,i}).$$

From $u_{n,i} = T_{r_{n,i}}y_{n,i}$ and Lemma 2.8, one has

$$\begin{aligned}\phi(u_n, y_{n,i}) &= \phi(T_{r_{n,i}}y_{n,i}, y_{n,i})\\ &\leq \phi(w, y_{n,i}) - \phi(w, T_{r_{n,i}}y_{n,i})\\ &\leq \phi(w, x_{n,i}) - \phi(w, T_{r_{n,i}}y_{n,i})\\ &= \phi(w, x_{n,i}) - \phi(w, u_{n,i}). \qquad (52)\end{aligned}$$

It follows from (41) that

(53) $$\phi(u_{n,i}, y_{n,i}) \to 0 \text{ as } n \to \infty, \quad \forall i \in I.$$

Noticing Lemma 2.1, one sees

(54) $$\|u_{n,i} - y_{n,i}\| \to 0 \text{ as } n \to \infty, \quad \forall i \in I.$$

Since $J$ is uniformly norm-to-norm continuous on bounded sets, one has

(55) $$\lim_{n\to\infty} \|Ju_{n,i} - Jy_{n,i}\| = 0, \quad \forall i \in I.$$

From the assumption $r_{n,i} \geq a$, one sees

(56) $$\lim_{n\to\infty} \frac{\|Ju_{n,i} - Jy_{n,i}\|}{r_{n,i}} = 0.$$

Noticing that $u_{n,i} = T_{r_{n,i}}y_{n,i}$, one obtains

(57) $$f(u_{n,i}, y) + \frac{1}{r_{n,i}}\langle y - u_{n,i}, Ju_{n,i} - Jy\rangle \geq 0, \quad \forall y \in C.$$

From (A2), one arrives at

$$\|y - u_{n,i}\| \frac{\|Ju_{n,i} - Jy_{n,i}\|}{r_{n,i}} \geq \frac{1}{r_{n,i}} \langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i} \rangle \geq -f(u_{n,i}, y) \geq f(y, u_{n,i}), \ \forall y \in C. \tag{58}$$

By taking the limit as $n \to \infty$ in the above inequality and from $(A4)$ and (34), one has

$$f(y, p) \leq 0, \quad \forall y \in C. \tag{59}$$

For all $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)p$. Noticing that $y, p \in C$, one obtains $y_t \in C$, which yields that $f(y_t, p) \leq 0$. It follows from $(A1)$ that

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y). \tag{60}$$

That is,

$$f(y_t, y) \geq 0. \tag{61}$$

Let $t \downarrow 0$, from (A3), we obtain $f(p, y) \geq 0$, for all $y \in C$. We have $p \in EP(f)$ that is $p \in GMEP(\Theta, \varphi, \Psi)$. From (a) and (b), we conclude that $p \in F$.

**Step 6**. Show that $p = \Pi_F x_0$.

From $x_n = \Pi_{C_n} x_0$, we have

$$\langle Jx_0 - Jx_n, x_n - z \rangle \geq 0, \quad \forall z \in C_n. \tag{62}$$

Since $F \subset C_n$, we also have

$$\langle Jx_0 - Jx_n, x_n - u \rangle \geq 0, \quad \forall u \in F. \tag{63}$$

By taking limit in (63), we obtain that

$$\langle Jx_0 - Jp, p - u \rangle \geq 0, \quad \forall u \in F. \tag{64}$$

By Lemma 2.2, we can conclude that $p = \Pi_F x_0$. This completes the proof. □

If $\beta_{n,i}^{(1)} = 0$ for all $n \geq 0$ and $T_i = S_i$ for all $i \in I$ in Theorem 3.1, then we have the following.

**Corollary 3.2.** *Let $E$ be a uniformly convex and uniformly smooth Banach space and $C$ be a nonempty closed convex subset of $E$. Let $\Psi : C \to E^*$ be a continuous and monotone operator and $\varphi : C \to \mathbb{R}$ be a real-valued function. Let $\Theta$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, let $\{T_i\}_{i \in I}$ be a family of closed quasi-$\phi$-nonexpansive mappings $T_i : C \to C$ such that the common fixed point set $F := \bigcap_{i \in I} F(T_i) \cap GMEP(\Theta, \varphi, \Psi)$ is nonempty, where $I$ is an index set. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrary and } C_{0,i} = C, \quad \forall i \in I, \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_0 - (1-\alpha_{n,i}) T_i x_n), \\ u_{n,i} \in C \text{ such that } \Theta(u_{n_i}, y) + \varphi(y) - \varphi(u_{n,i}) + \\ \langle \Psi u_{n,i}, y - u_{n,i} \rangle + \frac{1}{r_{n,i}} \langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i} \rangle \geq 0, \ \forall y \in C, \\ C_{n+1,i} = \{u \in C_{n,i} : \phi(u, u_{n,i}) \leq \phi(u, x_n) + \alpha_{n,i}(\|x\|^2 + 2\langle u, Jx_n - Jx_0 \rangle)\}, \\ C_{n+1} = \bigcap_{i \in I} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{65}$$

*where $J$ is a duality mapping on $E$, $\{\alpha_{n,i}\}$, $(i = 1, 2, 3)$ and $\{r_{n,i}\}$ are sequences in $(0, 1)$ satisfying*

(a) $\lim_{n \to \infty} \alpha_{n,i} = 0$ *for each $i \in I$;*
(b) $\{r_{n,i}\} \subset [a, \infty)$ *for some $a > 0$ and for all $i \in I$;*

*Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.*

**Remark 3.3.** Corollary 3.2 improves Theorem 3.1 of Takahashi and Zembayashi [36] in the following senses:

(1) from the class of relatively nonexpansive mappings to the more general class of quasi-$\phi$-nonexpansive mappings.

(2) from one mapping to a family of mappings.

(3) from the problem of finding the solutions of the equilibrium problem to the problem of finding the solutions of the generalized mixed equilibrium problem.

**Corollary 3.4.** *Let $E$ be a uniformly convex and uniformly smooth Banach space and $C$ be a nonempty closed convex subset of $E$. Let $\{T_i\}_{i\in I}$ and $\{S_i\}_{i\in I}$ be two families of closed quasi-$\phi$-nonexpansive mappings $T_i$, $S_i : C \to C$ such that the common fixed point set $F := \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i)$ is nonempty, where $I$ is an index set. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$
(66)\quad \begin{cases} x_0 \in C \text{ chosen arbitrary and } C_{0,i} = C, \quad \forall i \in I, \\ z_{n,i} = J^{-1}(\beta^{(1)}_{n,i} Jx_n + \beta^{(2)}_{n,i} JT_i x_n + \beta^{(3)}_{n,i} JS_i x_n), \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_0 - (1-\alpha_{n,i}) Jz_{n,i}), \\ C_{n+1,i} = \{u \in C_{n,i} : \phi(u, y_{n,i}) \le \phi(u, x_n) + \alpha_{n,i}(\|x\|^2 + 2\langle u, Jx_n - Jx_0\rangle)\}, \\ C_{n+1} = \bigcap_{i\in I} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 0, \end{cases}
$$

*where $J$ is a duality mapping on $E$, $\{\alpha_{n,i}\}$ and $\{\beta^{(i)}_{n,i}\}$ $(i = 1,2,3)$ are sequences in $(0,1)$ satisfying*

(a) $\lim_{n\to\infty} \alpha_{n,i} = 0$ *for each $i \in I$;*

(b) $\beta^{(1)}_{n,i} + \beta^{(2)}_{n,i} + \beta^{(3)}_{n,i} = 1$ *for each $i \in I$ and if one of the following is satisfied.*

(b-1) $\liminf_{n\to\infty} \beta^{(1)}_{n,i}\beta^{(l)}_{n,i} > 0$ *for all $l = 2,3$ and for all $i \in I$ and*

(b-2) $\liminf_{n\to\infty} \beta^{(2)}_{n,i}\beta^{(3)}_{n,i} > 0$ *and* $\liminf_{n\to\infty} \beta^{(1)}_{n,i} = 0$ *for each $i \in I$.*

*Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.*

*Proof.* Put $f(x,y) = 0$, for all $x,y \in C$, $\Psi = \varphi = 0$ and $\{r_{n,i}\} = \{1\}, \forall i \in I$ in Theorem 3.1. Thus, we have $u_{n,i} = y_{n,i}$. Then the sequence $\{x_n\}$ generated in Corallary 3.4 converges strongly to $\Pi_F x_0$. □

**Remark 3.5.** (1) We note that the iterative method imposed in Corollary 3.4 bases on the shrinking projection method which is different from the iterative method imposed in Theorem QCKZ based on the hybrid method.

(2) We can obtain the Corollary 3.4 by using either the condition (b-1) or (b-2).

**Theorem 3.6.** *Let $E$ be a uniformly convex and uniformly smooth Banach space and $C$ be a nonempty closed convex subset of $E$. Let $\Psi : C \to E^*$ be a continuous and monotone operator and $\varphi : C \to \mathbb{R}$ be a real-valued function. Let $\Theta$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, let $\{T_i\}_{i\in I}$ and $\{S_i\}_{i\in I}$ be two families of closed quasi-$\phi$-nonexpansive mappings $T_i$, $S_i : C \to C$ such that the common fixed point set $F := \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i) \cap GMEP(\Theta, \varphi, \Psi)$ is nonempty , where $I$ is an index set. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$
(67)\quad \begin{cases} x_0 \in C \text{ chosen arbitrary}, \\ z_{n,i} = J^{-1}(\beta^{(1)}_{n,i} Jx_n + \beta^{(2)}_{n,i} JT_i x_n + \beta^{(3)}_{n,i} JS_i x_n), \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_0 - (1-\alpha_{n,i}) Jz_{n,i}), \\ u_{n,i} \in C \text{ such that } \Theta(u_{n_i}, y) + \varphi(y) - \varphi(u_{n,i}) + \\ \langle \Psi u_{n,i}, y - u_{n,i}\rangle + \frac{1}{r_{n,i}}\langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i}\rangle \ge 0, \ \forall y \in C, \\ H_{n,i} = \{u \in C : \phi(u, u_{n,i}) \le \phi(u, x_n) + \alpha_{n,i}(\|x\|^2 + 2\langle u, Jx_n - Jx_0\rangle)\}, \\ H_n = \bigcap_{i\in I} H_{n,i}, \\ W_0 = C, \\ W_n = \{u \in W_{n-1} : \langle x_n - u, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad \forall n \ge 0, \end{cases}
$$

*where $J$ is a duality mapping on $E$, $\{\alpha_{n,i}\}$, $\{\beta^{(i)}_{n,i}\}$ $(i = 1,2,3)$ are sequences in $(0,1)$ such that*

(a) $\lim_{n\to\infty} \alpha_{n,i} = 0$ *for each $i \in I$;*

(b) $\{r_{n,i}\} \subset [a, \infty)$ *for some $a > 0$ and for all $i \in I$;*

(c) $\beta_{n,i}^{(1)}+\beta_{n,i}^{(2)}+\beta_{n,i}^{(3)}=1$ *for each* $i\in I$ *and if either*
  (c-1) $\liminf_{n\to\infty}\beta_{n,i}^{(1)}\beta_{n,i}^{(l)}>0$ *for all* $l=2,3$ *and for all* $i\in I$ *or*
  (c-2) $\liminf_{n\to\infty}\beta_{n,i}^{(2)}\beta_{n,i}^{(3)}>0$ *and* $\liminf_{n\to\infty}\beta_{n,i}^{(1)}=0$ *for each* $i\in I$.

*Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$.

*Proof.* We define a bifunction $f: C\times C\to\mathbb{R}$ by

$$f(x,y)=\Theta(x,y)+\varphi(y)-\varphi(x)+\langle\Psi x,y-x\rangle,\quad \forall x,y\in C.$$

From Lemma 2.10, we have the bifunction $f$ satisfies condition (A1)-(A4). Therefore, the mixed equilibrium problem (1) is equivalent to the following equilibrium problem: find $u\in C$ such that

$$f(u,y)\geq 0,\quad \forall y\in C,$$

and (67) can be written as:

$$(68)\qquad \begin{cases} x_0\in C \text{ chosen arbitrary},\\ z_{n,i}=J^{-1}(\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n),\\ y_{n,i}=J^{-1}(\alpha_{n,i}Jx_0-(1-\alpha_{n,i})Jz_{n,i}),\\ u_{n,i}\in C \text{ such that } f(u_{n_i},y)+\frac{1}{r_{n,i}}\langle y-u_{n,i},Ju_{n,i}-Jy_{n,i}\rangle\geq 0,\ \forall y\in C,\\ H_{n,i}=\{u\in C:\phi(u,u_{n,i})\leq\phi(u,x_n)+\alpha_{n,i}(\|x\|^2+2\langle u,Jx_n-Jx_0\rangle)\},\\ H_n=\bigcap_{i\in I}H_{n,i},\\ W_0=C,\\ W_n=\{u\in W_{n-1}:\langle x_n-u,Jx_0-Jx_n\rangle\geq 0\},\\ x_{n+1}=\Pi_{H_n\cap W_n}x_0,\quad \forall n\geq 0,\end{cases}$$

It is obvious that $H_n\cap W_n$ is closed and convex. Now we show that $F\subset H_n\cap W_n$ for all $n\geq 0$. First, we show that $F\subset H_n$ for all $n\geq 0$. For $\forall w\in F$ and all $i\in I$, one has

$$\begin{aligned}\phi(w,z_{n,i}) &= \phi(w,J^{-1}(\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n))\\ &= \|w\|^2-2\langle w,\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n)\rangle\\ &\quad +\|\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n\|^2\\ &\leq \|w\|^2-2\beta_{n,i}^{(1)}\langle w,Jx_n\rangle-2\beta_{n,i}^{(2)}\langle w,JT_ix_n\rangle-2\beta_{n,i}^{(3)}\langle w,JS_ix_n\rangle\\ &\quad +\beta_{n,i}^{(1)}\|x_n\|^2+\beta_{n,i}^{(2)}\|T_ix_n\|^2+\beta_{n,i}^{(3)}\|JS_ix_n\|^2\\ &= \beta_{n,i}^{(1)}\phi(w,x_n)+\beta_{n,i}^{(2)}\phi(w,T_ix_n)+\beta_{n,i}^{(3)}\phi(w,S_ix_n)\\ &\leq \beta_{n,i}^{(1)}\phi(w,x_n)+\beta_{n,i}^{(2)}\phi(w,x_n)+\beta_{n,i}^{(3)}\phi(w,x_n)\\ &= \phi(w,x_n)\end{aligned}$$

and then

$$\begin{aligned}
\phi(w, u_{n,i}) &= \phi(w, T_{r_{n,i}} y_{n,i}) \\
&\leq \phi(w, y_{n,i}) \\
&= \phi(w, J^{-1}(\alpha_{n,i} J x_0 - (1-\alpha_{n,i}) J z_{n,i}) \\
&= \|w\|^2 - 2\langle w, \alpha_{n,i} J x_0 - (1-\alpha_{n,i}) J z_{n,i}\rangle + \|\alpha_{n,i} J x_0 - (1-\alpha_{n,i}) J z_{n,i}\|^2 \\
&\leq \|w\|^2 - 2\alpha_{n,i}\langle w, J x_0\rangle - 2(1-\alpha_{n,i})\langle w, J z_{n,i}\rangle + \alpha_{n,i}\|x_0\|^2 + (1-\alpha_{n,i})\|z_{n,i}\|^2 \\
&= \alpha_{n,i}\phi(w, x_0) + (1-\alpha_{n,i})\phi(w, z_{n,i}) \\
&\leq \alpha_{n,i}\phi(w, x_0) + (1-\alpha_{n,i})\phi(w, x_n) \\
&= \phi(w, x_n) + \alpha_{n,i}[\phi(w, x_0) - \phi(w, x_n)] \\
&\leq \phi(w, x_n) + \alpha_{n,i}(\|x_0\|^2 + 2\langle w, J x_n - J x_0\rangle). \qquad (69)
\end{aligned}$$

This show that $w \in H_{n,i}$ for each $i \in I$. That is, $w \in H_n = \bigcap_{i \in I} H_{n,i}$ for all $n \geq 0$.

Next, we show that $F \subset W_n$ for all $n \geq 0$. In fact, we prove this by induction. For $n = 0$, we have $F \subset C = W_0$. Assume that $F \subset H_{n-1}$ for some $n \geq 1$, we will show that $F \subset W_n$ for the same $n \geq 1$. Since $x_n$ is the projection of $x_0$ onto $H_{n-1} \cap W_{n-1}$, by Lemma 2.2, we have

$$\langle x_n - z, J x_0 - J x_n\rangle \geq 0, \quad \forall z \in C_{n-1} \cap Q_{n-1}. \qquad (70)$$

Since $F \subset H_{n-1} \cap W_{n-1}$ by the induction assumptions, the last inequality holds, in particular, for all $w \in F$. This together with the definition of $W_n$ implies that $F \subset W_n$. Thus we proved that $F \subset H_n \cap W_n$, $\forall n \geq 0$. This means that $\{x_n\}$ is well define.

From the definition of $W_n$, we know that

$$\langle x_n - z, J x - J x_n\rangle \geq 0, \quad \forall z \in W_n.$$

So by Lemma 2.2 we have $x_n = \Pi_{W_n} x$. If we instead $C_n$ by $W_n$ and $C_{n+1}$ by $H_n$ in the proof of Theorem 3.1, and notice that $x_{n+1} = \Pi_{H_n \cap W_n} x \in H_n \cap W_n \subset W_n$, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_n - u_{n,i}\| = 0. \qquad (71)$$

Since $J$ is uniformly norm-to-norm continuous on bounded set, we have

$$\lim_{n\to\infty} \|J x_n - J u_{n,i}\| = 0. \qquad (72)$$

Thus the proof that $\{x_n\}$ converges strongly to $\Pi_F x$ follows on the lines of Theorem 3.1. □

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