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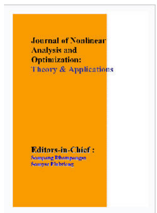

Shrinking projection methods for a family of relatively nonexpansive mappings, equilibrium problems and variational inequality problems in Banach spaces

Somyot Plubtieng, Tipphawan Thammathiwat

ABSTRACT: In this paper, we prove a strong convergence theorem by the shrinking projection method for finding a common element of the set of fixed points of a countable family of relatively nonexpansive mappings and the set of solutions of equilibrium problems and the set of solution of variational inequality problems in Banach spaces. Then, we apply our main theorem to the problem of finding a zero of a maximal monotone operator, the complementarity problems, and the convex feasibility problems.

KEYWORDS: Relatively nonexpansive mapping; Equilibrium problem; Variational inequality problem; Strong convergence.

1. **Introduction**

Let E be a real Banach space and let E^* be the dual space of E. Let C be a closed convex subset of E. Let $A : C \to E^*$ be a mapping. The classical variational inequality, denoted by $VI(A, C)$, is to find $x^* \in C$ such that

(1) $$\langle Ax^*, v - x^* \rangle \geq 0 \text{ for all} v \in C.$$

The variational inequality has emerged as a fascinating and interesting branch of mathematical and engineering sciences with a wide range of applications in industry, finance, economics, social, ecology, regional, pure and applied sciences; see, e.g. [9, 22, 33, 35, 37, 40] and the references therein. An operator A is called α-inverse-strongly monotone [7, 19] if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2$$

for all $u, v \in C$.

In 2008, Iiduka and Takahashi [13] introduce the following algorithm for finding a solution of the variational inequality for an α-inverse-strongly monotone A in a 2-uniformly convex and uniformly smooth Banach space E. For an initial point $x_1 = x \in C$, define a sequence $\{x_n\}$ by

(2) $$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n),$$

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where J is the duality mapping on E and Π_C is the generalized projection from E onto C. Then $\{x_n\}$ converges weakly to some element $z \in VI(A,C)$ where $z = \lim_{n\to\infty} \Pi_{VI(A,C)} x_n$.

Let f be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem is to find $x \in C$ such that

$$f(x,y) \geq 0 \text{ for all } y \in C. \tag{3}$$

The set of solutions of (3) is denoted by $EP(f)$. Numerous problems in physics, optimization, and economics reduce to find a solution of (3). In 1997 Combettes and Hirstoaga [12] introduced an iterative scheme of finding the best approximation to initial data when $EP(f)$ is nonempty and proved a strong convergence theorem. This equilibrium problem contains the fixed point problem, optimization problem, saddle point problem, variational inequality problem and Nash equilibrium problem as its special cases (see, e.g., Blum and Oettli [6], Combettes and Hirstoaga [12]).

A popular method is the hybrid projection method developed in Nakajo and Takahashi [22], Kamimura and Takahashi [14] and Martinez-Yanes and Xu [20]; see also Matsushita and Takahashi [21], Plubtieng and Ungchittrakool [25] and references therein. Recently Takahashi, et al. [34] introduced an alterative projection method, which is called the shrinking projection method, and they showed several strong convergence theorems for a family of nonexpansive mappings. In 2008, Takahashi and Zembayashi [36], introduced the following iterative scheme which is called the shrinking projection method:

$$\begin{cases} x_0 = x \in C, \ \ C_1 = C, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \ \forall n \geq 1, \end{cases} \tag{4}$$

where J is the duality mapping on E and Π_C is the generalized projection from E onto C. They proved that the sequence $\{x_n\}$ converges strongly to $q \in \Pi_{F(T)\cap EP(f)} x_0$. Recently, Cholamjiak [10] introduced a new hybrid projection algorithm and proved a strong convergence theorem for finding a common element of the set of solutions of the equilibrium problem and the set of the variational inequality for an inverse-strongly monotone operator and the set of fixed points of relatively quasinonexpansive mappings in a Banach space.

On the other hand, Aoyama, et al. [1] introduced a Halpern type iterative sequence for finding a common fixed point of a countable family of nonexpansive mappings. Let $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1-\alpha_n) T_n x_n \tag{5}$$

for all $n \in \mathbb{N}$, where C is a nonempty closed convex subset of a Banach space, $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings with some condition. They proved that $\{x_n\}$ defined by (5) converges strongly to a common fixed point of $\{T_n\}$. Recently, Nakajo et al. [23] introduced the more general condition so-called the NST*-condition. A sequence $\{T_n\}$ is said to satisfy the NST*-condition if for every bounded sequence $\{z_n\}$ in C,

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = \lim_{n\to\infty} \|z_n - z_{n+1}\| = 0 \text{ implies } \omega_w(z_n) \subset F.$$

where F is the set of common fixed point of $\{T_n\}$ and $\omega_w(z_n) = \{z : \exists z_{n_r} \rightharpoonup z\}$ denotes the weak ω-limit set of $\{z_n\}$. They also prove strong convergence theorems by the hybrid method for families of mappings in a uniformly convex Banach space E whose norm is G$\hat{a}$teaux differentiable.

Motivated and inspired by Takahashi and Zembayashi [36] and Cholamjiak [10], this paper is organized as follows. In section 2, we present some basic concepts and useful lemmas for proving the convergence result of this paper. In section 3, we introduce an iterative processes (15) below for finding a common element of the set of fixed points of a countable family of relatively nonexpansive mappings and the set of solutions of equilibrium problems and the set of solution of variational inequality problem. Then, we prove a strong convergence theorem. Moreover, we obtain corollary which extend the result of Takahashi and Zembayashi [36]. In section 4, we apply our main theorems to the problem of finding a zero of a maximal monotone operator, the complementarity problems, and the convex feasibility problems.

2. **Preliminaries**

Let E be a real Banach space and let $S = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is said to be *strictly convex* if for any $x, y \in S$,

$$x \neq y \text{ implies } \|\frac{x+y}{2}\| < 1. \tag{6}$$

It is also said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in S$,

$$\|x - y\| \geq \varepsilon \text{ implies } \|\frac{x+y}{2}\| < 1 - \delta. \tag{7}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. We define a function $\delta : [0, 2] \to [0, 1]$, is called the *modulus of convexity of E*, as follows:

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\}. \tag{8}$$

Then E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0, 2]$; see [4, 5, 32] for more details. A Banach space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{9}$$

exists for all $x, y \in S$. It is also said to be *uniformly smooth* if the limit (9) is attained uniformly for $x, y \in S$. One should note that no Banach space is p-uniformly convex for $1 < p < 2$; see [32]. It is well known that a Hilbert space is 2-uniformly convex, uniformly smooth. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\} \tag{10}$$

for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. See [30, 31] for more details.

Lemma 2.1. [5] *Let p be a given real number with $p \geq 2$ and E a p-uniformly convex Banach space. Then, for all $x, y \in E$, $j_x \in J_p(x)$ and $j_y \in J_p(y)$,*

$$\langle x - y, j_x - j_y \rangle \geq \frac{c^p}{2^{p-2}p} \|x - y\|^p, \tag{11}$$

where J_p is the generalized duality mapping of E and $\frac{1}{c}$ is the p-uniformly convexity constant of E.

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \tag{12}$$

for all $x, y \in E$. In a Hilbert space H, we have $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$.

It is obvious from the definition of the function ϕ that

(1) $(\|y\| - \|x\|)^2 \leqslant \phi(y, x) \leqslant (\|y\| + \|x\|)^2$,
(2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$,
(3) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leqslant \|x\| \|Jx - Jy\| + \|y - x\| \|y\|$,

for all $x, y, z \in E$. Let E be a strictly convex, smooth, and reflexive Banach space, and let J be the duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one, and surjective, and it is the duality mapping from E^* into E. We make use of the following mapping V studied in Alber [2]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{13}$$

for all $x \in E$ and $x^* \in E^*$. In other words, $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. For each $x \in E$, the mapping $V(x, \cdot) : E^* \to \mathbb{R}$ is a continuous and convex function from E^* into $\mathbb{R}$.

Lemma 2.2. [14] *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Lemma 2.3. [2, 14] *Let E be a a smooth, strictly convex, and reflexive Banach space and let V be as in (13). Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, for any $x \in E$, there exists a point $x_0 \in C$ such that $\phi(x_0, x) = \min_{y \in C} \phi(y, x)$. The mapping $\Pi_C : E \to C$ defined by $\Pi_C x = x_0$ is called the *generalized projection* [2, 14]. The following are well-known results. For example, see [2, 14].

Lemma 2.4. [2, 14] *Let C be a nonempty closed convex subset of a smooth Banach space E, let $x \in E$, and let $x_0 \in C$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \geqslant 0$ for all $y \in C$.*

Lemma 2.5. [2, 14] *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then $\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leqslant \phi(y, x)$ for all $y \in C$.*

Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let T be a mapping from C into itself, and let $F(T)$ be the set of all fixed points of T. Then a point $p \in C$ is said to be an *asymptotic fixed point* of T (see Reich [27]) if there exists a sequence $\{x_n\}$ in C converging weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$ and we say that T is a *relatively nonexpansive mapping* if the following conditions are satisfied:

(R1) $F(T)$ is nonempty;
(R2) $\phi(u, Tx) \leqslant \phi(u, x)$ for all $u \in F(T)$ and $x \in C$;
(R3) $\hat{F}(T) = F(T)$.

Lemma 2.6. [21] *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E , and let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.7. [39] *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leqslant r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + \|(1-t)y\|^2 - t(1-t)g(\|x - y\|), \tag{14}$$

for all $x, y \in B_r(0)$ and $t \in [0, 1]$.

Lemma 2.8. [26] *Let C be a closed convex subset of a smooth Banach space E and let $x, y \in E$. Then the set $K := \{v \in C : \phi(v, y) \leqslant \phi(v, x)\}$ is closed and convex.*

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbf{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t\to 0} f(tz + (1-t)x, y) \leq f(x, y)$;
(A4) for each $x \in C, y \mapsto f(x, y)$ is convex and lower semicontinuous.

Lemma 2.9. [6] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \text{ for all } y \in C.$$

Lemma 2.10. [36] *et C be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

(1) *T_r is single- valued;*
(2) *T_r is firmly nonexpansive-type mapping, i.e., for any $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle$$

(3) *$F(T_r) = EP(f)$;*
(4) *$EP(f)$ is closed and convex.*

Lemma 2.11. [36] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4) and let $r > 0$. Then, for all $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x)$$

An operator A of C into E^* is said to be *hemicontinuous* if for all $x, y \in C$, the mapping F of $[0, 1]$ into E^* defined by $F(t) = A(tx + (1 - t)y)$ is continuous with respect to the weak* topology of E^*. We define the *normal cone for C at a point $v \in C$*, $N_C(v)$ by

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^*\rangle \geq 0, \ \forall y \in C\}.$$

Lemma 2.12. [28] *Let C be a closed convex subset of a Banach space E, and let A be a monotone, hemicontinuous operator of C into E^*. Let $T_e \subset E \times E^*$ be an operator defined as follows:*

$$T_e v = \begin{cases} Av + N_C v, & v \in C; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then T_e is maximal monotone and $T_e^{-1}0 = VI(A, C)$.

Throughout the paper, we will use the notations:

(1) $\to$ for strong convergence.
(2) $\omega_w(x_n) = \{x : \exists x_{n_r} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

3. **Main result**

In this section, we prove the strong convergence theorems for finding a common element of the set of solutions of equilibrium problem, the set of the solutions of the variational inequality problem and the set of fixed point of a countable family of relatively nonexpansive mappings in Banach spaces by using the hybrid method in mathematical programming.

Theorem 3.1. *Let C be a closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)-(A4) and let A be an α-inverse strongly monotone of E into E^* such that $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A, C)$. Let $\{T_n\}$ be a family of relatively nonexpansive mappings of C into itself such that satisfies the NST^*-condition* *and* $F := \cap_{n=1}^{\infty} F(T_n) \cap EP(f) \cap VI(A,C) \neq \emptyset$. *For an initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1} x_0$ *and* $C_1 = C$, *define a sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JT_n z_n), \\ u_n \in C \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \ \forall n \geq 1 \end{cases} \tag{15}$$

where J *is the duality mapping on* E. *Assume that* $\{\alpha_n\} \subset [0,1], \{\lambda_n\} \subset (0,\infty)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfying the retrictions*

(C1) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$,
(C2) $\{r_n\} \subset [s,\infty)$ *for some* $s > 0$,
(C3) $\{\lambda_n\} \subset [a,b]$ *for some* a, b *with* $0 < a < b < \frac{c^2\alpha}{2}$, *where* $\frac{1}{c}$ *is the 2-uniformly convexity of* E.

Then the sequence $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $q = \Pi_F x_0$.

Proof. ***Step 1.*** Show that $\Pi_F x_0$ and $\Pi_{C_{n+1}} x_0$ are well-defined.

It is obvious that $VI(A,C)$ is closed convex subset of C. Thus, it follows form Lemma 2.6 and Lemma 2.10 that $\emptyset \neq F = \cap_{n=1}^{\infty} F(T_n) \cap EP(f) \cap VI(A,C)$ is closed and convex. This implies that $\Pi_F x_0$ is well-defined. Now, we claim that $F \subset C_n$ and C_n is closed convex for all $n \in \mathbb{N}$ .Obvious that $F \subset C = C_1$ is closed and convex. So $x_1 = \Pi_{C_1} x_0$ is well-defined. Next, suppose that $F \subset C_k$ and C_k is closed convex for some $k \in \mathbb{N}$. Thus $x_k = \Pi_{C_k} x_0$ is well-defined. We note from Lemma 2.8 that C_{k+1} is closed and convex. Consequently, C_n is closed and convex for all $n \in \mathbb{N}$. Set $v_n = J^{-1}(Jx_n - \lambda_n Ax_n)$ and $u_n = T_{r_n} y_n$. For $u \in F \subset C_k$, we know from Lemma 2.3 and Lemma 2.5 that

$$\begin{aligned} \phi(u, z_k) &= \phi(u, \Pi_C v_k) \\ &\leq \phi(u, v_k) \\ &= \phi(u, J^{-1}(Jx_k - \lambda_k Ax_k)) \\ &= V(u, Jx_k - \lambda_k Ax_k) \\ &\leq V(u, (Jx_k - \lambda_k Ax_k) + \lambda_k Ax_k) - 2\langle J^{-1}(Jx_k - \lambda_k Ax_k) - u, \lambda_k Ax_k\rangle \\ &= V(u, Jx_k) - 2\lambda_k \langle v_k - u, Ax_k\rangle \\ &= \phi(u, x_k) - 2\lambda_k\langle x_k - u, Ax_k\rangle + 2\langle v_k - x_k, -\lambda_k Ax_k\rangle. \end{aligned} \tag{16}$$

Since $u \in VI(A,C)$ and A is an α-inverse strongly monotone, we have

$$\begin{aligned} -2\lambda_k\langle x_k - u, Ax_k\rangle &= -2\lambda_k\langle x_k - u, Ax_k - Au\rangle - 2\lambda_k\langle x_k - u, Au\rangle \\ &\leq -2\alpha\lambda_k\|Ax_k - Au\|^2. \end{aligned} \tag{17}$$

Using Lemma 2.1 and $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A,C)$, we obtain

$$\begin{aligned} 2\langle v_k - x_k, -\lambda_k Ax_k\rangle &= 2\langle J^{-1}(Jx_k - \lambda_k Ax_k) - J^{-1}(Jx_k), -\lambda_k Ax_k\rangle \\ &\leq 2\|J^{-1}(Jx_k - \lambda_k Ax_k) - J^{-1}(Jx_k)\|\|\lambda_k Ax_k\| \\ &\leq \frac{4}{c^2}\|JJ^{-1}(Jx_k - \lambda_k Ax_k) - JJ^{-1}(Jx_k)\|\|\lambda_k Ax_k\| \\ &\leq \frac{4}{c^2}\lambda_k^2\|Ax_k\|^2 \\ &\leq \frac{4}{c^2}\lambda_k^2\|Ax_k - Au\|^2. \end{aligned} \tag{18}$$

Replacing (17) and (18) into (16), we have

$$\phi(u, z_k) \leq \phi(u, x_k) + 2\lambda_k(\frac{2}{c^2}\lambda_k - \alpha)\|Ax_k - Au\|^2 \leq \phi(u, x_k). \tag{19}$$

By convexity of $\|\cdot\|^2$ and (19) , for each $u \in F \subset C_k$, we obtain

$$\begin{aligned}
\phi(u,u_k) &= \phi(u,T_{r_k}y_k)\\
&\le \phi(u,y_k)\\
&= \phi(u,J^{-1}(\alpha_k Jx_k + (1-\alpha_k)JT_k z_k))\\
&= \|u\|^2 - 2\alpha_k\langle u, Jx_k\rangle - 2(1-\alpha_k)\langle u, JT_k z_k\rangle + \|\alpha_k Jx_k + (1-\alpha_k)JT_k z_k\|^2\\
&\le \|u\|^2 - 2\alpha_k\langle u, Jx_k\rangle - 2(1-\alpha_k)\langle u, JT_k z_k\rangle + \alpha_k\|Jx_k\|^2 + (1-\alpha_k)\|JT_k z_k\|^2\\
&= \alpha_k\phi(u,x_k) + (1-\alpha_k)\phi(u,T_k z_k)\\
&\le \alpha_k\phi(u,x_k) + (1-\alpha_k)\phi(u,z_k)\\
&\le \alpha_k\phi(u,x_k) + (1-\alpha_k)\phi(u,x_k)\\
&= \phi(u,x_k). \qquad (20)
\end{aligned}$$

This show that $u \in C_{k+1}$ and so $F \subset C_{k+1}$. Consequently, $\emptyset \neq F \subset C_n$and C_n closed convex for all $n \geq 1$. This implies that $\Pi_{C_n}x_0$ for all $n \geq 1$ is well-defined.

Step 2. Show that $\lim_{n\to\infty}\phi(x_n,x_0)$ exists and $\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}\|x_n-u_n\| = 0$. Since $x_n = \Pi_{C_n}x_0$, it follows from Lemma 2.5 that

$$\phi(x_n,x_0) \le \phi(u,x_0) - \phi(u,x_n) \le \phi(u,x_0) \text{ for all } u \in C_n. \qquad (21)$$

From step 2 and (21), we get

$$\phi(x_n,x_0) \le \phi(u,x_0) \text{ for all } u \in \mathbb{F} \text{ and for all } n \in \mathbb{N}. \qquad (22)$$

Therefore $\{\phi(x_n,x_0)\}$ is bounded and hence $\{x_n\}$ is bounded by (1). From $x_n = \Pi_{C_n}x_0$ and $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\phi(x_n,x_0) = min_{y\in C_n}\phi(y,x_0) \le \phi(x_{n+1},x_0) \text{ for all } n \in \mathbb{N}. \qquad (23)$$

Hence $\{\phi(x_n,x_0)\}$ is bounded and nondecreasing. This implies that there exists the limit of $\{\phi(x_n,x_0)\}$. It follows form Lemma 2.5 that

$$\phi(x_{n+1},x_n) = \phi(x_{n+1},\Pi_{C_n}x_0) \le \phi(x_{n+1},x_0) - \phi(\Pi_{C_n}x_0,x_0) = \phi(x_{n+1},x_0) - \phi(x_n,x_0), \qquad (24)$$

for all $n \in \mathbb{N}$. Thus, we have

$$\lim_{n\to\infty}\phi(x_{n+1},x_n) = 0. \qquad (25)$$

Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1}$, it follows from the definition of C_{n+1} that

$$\phi(x_{n+1},u_n) \le \phi(x_{n+1},x_n) \to 0. \qquad (26)$$

By Lemma 2.2, 25 and 26, we note that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}\|x_{n+1}-u_n\| = \lim_{n\to\infty}\|x_n-u_n\| = 0. \qquad (27)$$

Since J is uniformly norm-to-norm continuous on bounded subset, we also obtain

$$\lim_{n\to\infty}\|Jx_{n+1}-Jx_n\| = \lim_{n\to\infty}\|Jx_n-Ju_n\| = 0. \qquad (28)$$

Step 3. Show that $\{x_n\}$ is a Cauchy sequence.
Since $x_m = \Pi_{C_m}x_0 \in C_m \subset C_n$ for $m > n$, it follows from Lemma 2.5 that

$$\phi(x_m,x_n) = \phi(x_m,\Pi_{C_n}x_0) \le \phi(x_m,x_0) - \phi(\Pi_{C_n}x_0,x_0) = \phi(x_m,x_0) - \phi(x_n,x_0). \qquad (29)$$

Taking $m,n \to \infty$, we obtain that $\phi(x_m,x_n) \to 0$. From Lemma 2.2, implies that $\lim_{m,n\to\infty}\|x_m - x_n\| = 0$. Hence $\{x_n\}$ is a Cauchy sequence and so by the completeness of E and the closedness of C, we can assume that $x_n \to q \in C$ as $n \to \infty$.

Step 4. Show that $q \in \cap_{n=1}^{\infty}F(T_n)$.
Since $\{x_n\}$ and $\{T_n\}$ are bounded, there exists $r > 0$ such that $r = \sup_{n\geq 1}\{\|x_n\|, \|T_n\|\}$. Then, it

follows from (19), (20) and Lemma 2.7 that there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\begin{aligned}
\phi(u,u_n) &\leq \|u\|^2 - 2\alpha_n\langle u, Jx_n\rangle - 2(1-\alpha_n)\langle u, JT_nz_n\rangle + \|\alpha_n Jx_n + (1-\alpha_n)JT_nz_n\|^2 \\
&\leq \|u\|^2 - 2\alpha_n\langle u, Jx_n\rangle - 2(1-\alpha_n)\langle u, JT_nz_n\rangle + \alpha_n\|Jx_n\|^2 + (1-\alpha_n)\|JT_nz_n\|^2 \\
&\quad -\alpha_n(1-\alpha_n)g(\|Jx_n - JT_nz_n\|) \\
&= \alpha_n\phi(u,x_n) + (1-\alpha_n)\phi(u,T_nz_n) - \alpha_n(1-\alpha_n)g(\|Jx_n - JT_nz_n\|) \\
&\leq \alpha_n\phi(u,x_n) + (1-\alpha_n)\phi(u,z_n) - \alpha_n(1-\alpha_n)g(\|Jx_n - JT_nz_n\|) \\
&\leq \alpha_n\phi(u,x_n) + (1-\alpha_n)[\phi(u,x_n) + 2\lambda_n(\frac{2}{c^2}\lambda_n - \alpha)\|Ax_k - Au\|^2] \\
&\quad -\alpha_n(1-\alpha_n)g(\|Jx_n - JT_nz_n\|) \\
&= \phi(u,x_n) + 2(1-\alpha_n)\lambda_n(\frac{2}{c^2}\lambda_n - \alpha)\|Ax_k - Au\|^2 - \alpha_n(1-\alpha_n)g(\|Jx_n - JT_nz_n\|) \\
&\leq \phi(u,x_n) - \alpha_n(1-\alpha_n)g(\|Jx_n - JT_nz_n\|). \qquad (30)
\end{aligned}$$

This implies that

$$\begin{aligned}
\alpha_n(1-\alpha_n)g(\|Jx_n - JT_nz_n\|) &\leq \phi(u,x_n) - \phi(u,u_n) \\
&= \|x_n\|^2 - \|u_n\|^2 - 2\langle u, Jx_n - Ju_n\rangle \\
&\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|Jx_n - Ju_n\|. \qquad (31)
\end{aligned}$$

Using (27), (28), (31) and (C1), we get $\lim_{n\to\infty} g(\|Jx_n - JT_nz_n\|) = 0$. By the property of g, we have $\lim_{n\to\infty}\|Jx_n - JT_nz_n\| = 0$. Since J and J^{-1} are uniformly norm-to-norm continuous on bounded subset, it follows that

$$\lim_{n\to\infty}\|x_n - T_nz_n\| = \lim_{n\to\infty}\|J^{-1}(Jx_n) - J^{-1}(JT_nz_n)\| = 0 \qquad (32)$$

Again by 30, we have

$$\begin{aligned}
2a(\alpha - \frac{2}{c^2}b)\|Ax_k - Au\|^2 &\leq \frac{1}{1-\alpha_n}\phi(u,x_n) - \phi(u,u_n) \\
&\leq \frac{1}{1-\alpha_n}[\|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|Jx_n - Ju_n\|]. \qquad (33)
\end{aligned}$$

It follows from (27), (28), (33) and (C1), we get that

$$\lim_{n\to\infty}\|Ax_k - Au\| = 0. \qquad (34)$$

From Lemma 2.3, Lemma 2.5 and (18), we have

$$\begin{aligned}
\phi(x_n,z_n) &= \phi(x_n,\Pi_C v_n) \\
&\leq \phi(x_n,v_n) \\
&= \phi(x_n, J^{-1}(Jx_n - \lambda_nAx_n)) \\
&= V(x_n, Jx_n - \lambda_nAx_n) \\
&\leq V(x_n,(Jx_n - \lambda_nAx_n) + \lambda_nAx_n) - 2\langle J^{-1}(Jx_n - \lambda_nAx_n) - x_n, \lambda_nAx_n\rangle \\
&= V(x_n,Jx_n) - 2\lambda_n\langle v_n - x_n, Ax_n\rangle \\
&= \phi(x_n,x_n) + 2\langle v_n - x_n, -\lambda_nAx_n\rangle \\
&\leq \frac{4}{c^2}\lambda_n^2\|Ax_n - Au\|^2. \qquad (35)
\end{aligned}$$

By Lemma 2.2, (35) and J is uniformly norm-to-norm continuous on bounded subset, we note that

$$\lim_{n\to\infty}\|x_n - z_n\| = \lim_{n\to\infty}\|Jx_n - Jz_n\| = 0. \qquad (36)$$

Since $x_n \to q$ as $n\to\infty$, $z_n \to q$ as $n\to\infty$. Combining (27), (32) and (36), we also obtain

$$\|T_nz_n - z_n\| \leq \|x_n - z_n\| + \|T_nz_n - x_n\| \to 0,$$

and hence

$$\|z_{n+1} - z_n\| \leq \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_n\| \to 0.$$

Since $\{T_n\}$ satisfies the NST*-condition, we have $q \in \cap_{n=1}^{\infty} F(T_n)$.

Step 5. Show that $q \in EP(f)$.

From (20) and Lemma 2.11, we have

$$\begin{aligned} \phi(u_n, y_n) &= \phi(T_{r_n} y_n, y_n) \\ &\leq \phi(u, y_n) - \phi(u, T_{r_n} y_n) \\ &\leq \phi(u, x_n) - \phi(u, T_{r_n} y_n) \\ &= \phi(u, x_n) - \phi(u, u_n). \end{aligned} \tag{37}$$

It follows from (31) that $\lim_{n\to\infty} \phi(u_n, y_n) = 0$. Hence, by Lemma 2.2, we note that $\lim_{n\to\infty} \|u_n - y_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded subset and (C2), we get

$$\lim_{n\to\infty} \|\frac{Ju_n - Jy_n}{r_n}\| = 0. \tag{38}$$

Using (A2), we note that, for each $y \in C$,

$$\begin{aligned} \|y - u_n\|\|\frac{Ju_n - Jy_n}{r_n}\| &\geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \\ &\geq -f(u_n, y) \\ &\geq f(y, u_n). \end{aligned} \tag{39}$$

It follows from (A4) and $u_n \to q$ that $f(y, q) \leq 0$ for all $y \in C$. For each $0 < t < 1$ and $y \in C$, we define $y_t = ty + (1-t)q$. Hence $y_t \in C$ and therefore $f(y_t, q) \leq 0$. From (A1), we obtain $0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, q) \leq tf(y_t, y)$. Thus, $f(y_t, y) \geq 0$ and so from (A3) we get $f(q, y) \geq 0$. Since y is arbitrary element in C, we get $q \in EP(f)$.

Step 6. Show that $q \in VI(A, C)$.

Define $T_e \subset E \times E^*$ be as in Lemma 2.12 and let $(v, w) \in G(T_e)$. Then $w - Av \in N_C(v)$. Since $z_n \in C$ and by definition of $N_C(v)$, it follows that

$$\langle v - z_n, w - Av\rangle \geq 0. \tag{40}$$

On the other hand, by Lemma 2.5, we obtain

$$\langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} - Ax_n\rangle \leq 0. \tag{41}$$

Combining (40) and (41), we have

$$\begin{aligned} \langle v - z_n, w\rangle &\geq \langle v - z_n, Av\rangle \\ &\geq \langle v - z_n, Av\rangle + \langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} - Ax_n\rangle \\ &= \langle v - z_n, Av - Ax_n\rangle + \langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n}\rangle \\ &= \langle v - z_n, Av - Az_n\rangle + \langle v - z_n, Az_n - Ax_n\rangle \\ &\quad + \langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n}\rangle \\ &\geq -\|v - z_n\|\frac{\|z_n - x_n\|}{\alpha} - \|v - z_n\|\frac{\|Jx_n - Jz_n\|}{a} \\ &\geq -M(\frac{\|z_n - x_n\|}{\alpha} + \frac{\|Jx_n - Jz_n\|}{a}), \end{aligned} \tag{42}$$

where $M = \sup_{n\geq 1}\{\|v - z_n\|\}$. By taking the limit as $n \to \infty$ and from (36), we note that $\langle v - q, w\rangle \geq 0$. Since T_e is maximal monotone and $T_e^{-1}0 = VI(A, C)$, we obtain $q \in VI(A, C)$.

Step 7. Show that $q = \Pi_F x_0$.

Since $x_n = \Pi_{C_n} x_0$ and $F \subset C_n$ for all $n \in \mathbb{N}$, we obtain that

$$\langle Jx_0 - Jx_n, x_n - u\rangle \geq 0 \ \ \forall u \in F. \tag{43}$$

Taking the limit as $n \to \infty$ in (43), we get

$$\langle Jx_0 - Jq, q - u\rangle \geq 0 \;\; \forall u \in F. \tag{44}$$

By Lemma 2.4, we can conclude that $q = \Pi_F x_0$. This completes the proof. □

Setting $T_n = T$ for all $n \in \mathbb{N}$ and $A \equiv 0$ in Theorem 3.1, we have following result.

Corollary 3.2. [36] *Let C be a closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)-(A4) and let T be a relatively nonexpansive mappings of C into itself $F := F(T) \cap EP(f) \neq \emptyset$. Assume that $\{\alpha_n\} \subset [0,1]$ satisfy $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [s,\infty)$ for some $s > 0$. Then the sequence $\{x_n\}$ generated by (4) converge strongly to $q = \Pi_F x_0$.*

Proof. Since $\hat{F}(T) = F(T)$, it follows that T satisfies the NST*-condition which is the desired result. □

Remark 3.3. It would be interesting to investigate convergent sequence when the countable family of relatively nonexpansive mappings of C into C is a semgroup, which is abelian or amenable. See: [17, 18].

4. Applications

4.1. Complementarity problems

Let K be a nonempty, closed convex cone in E, A an operator of K into E^*. We define its *polar* in E^* to be the set

$$K^* = \{y^* \in E^* : \langle x, y^*\rangle \geq 0, \forall x \in K\}. \tag{45}$$

Then the element $u \in K$ is called a solution of the *complementarity problem* if

$$Au \in K^*, \;\; \langle u, Au\rangle = 0. \tag{46}$$

The set of solutions of the complementarity problem is denoted by $C(K, A)$.

Theorem 4.1. *Let K be a closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $K \times K \to \mathbb{R}$ satisfying (A1)-(A4) and let A be an α-inverse strongly monotone of E into E^* such that $\|Ay\| \leq \|Ay - Au\|$ for all $y \in K$ and $u \in C(K, A)$. Let $\{T_n\}$ be a family of relatively nonexpansive mappings of K into itself such that satisfies the NST*-condition and $F := \cap_{n=1}^{\infty} F(T_n) \cap EP(f) \cap C(K, A) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{K_1} x_0$ and $K_1 = K$, define a sequence $\{x_n\}$ as follows:*

$$\begin{cases} z_n = \Pi_K J^{-1}(Jx_n - \lambda_n A x_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JT_n z_n), \\ u_n \in C \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \;\forall y \in K, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \;\; \forall n \geq 1 \end{cases} \tag{47}$$

where J is the duality mapping on E. Assume that $\{\alpha_n\} \subset [0,1], \{\lambda_n\} \subset (0,\infty)$ and $\{r_n\} \subset (0,\infty)$ satisfying the condition (C1)-(C3) of Theorem 3.1. Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = \Pi_F x_0$.

Proof. As in the proof of Takahashi [Lemma 7.11, [30]], we note that $VI(K, A) = C(K, A)$. Hence, we obtain the desired result. □

4.2. **Approximation of a zero of a maximal monotone operator**

Let B be a multivalued operator from E to E^* with domain $D(B) = \{z \in E : Az \neq \varnothing\}$ and range $R(B) = \cup\{Bz : z \in D(B)\}$. An operator B is said to be monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geqslant 0$ for each $x_i \in D(B)$ and $y_i \in Ax_i$, $i = 1, 2$. A monotone operator B is said to be maximal if its graph $G(B) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. We know that if B is a maximal monotone operator, then $B^{-1}(0)$ is closed and convex. Let E be a reflexive, strictly convex and smooth Banach space, and let B be a monotone operator from E to E^*, we known from Rockafellar [28] that B is maximal if and only if $R(J + rB) = E^*$ for all $r > 0$. Let $J_r : E \to D(B)$ defined by $J_r = (J + rB)^{-1}J$ and such a J_r is called the resolvent of B. We know that J_r is a relatively nonexpansive; see [21] and $B^{-1}(0) = F(J_r)$ for all $r > 0$; see [30, 31] for more details.

Theorem 4.2. *Let C be a closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)-(A4) and let A be an α-inverse strongly monotone of E into E^* such that $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A, C)$. Let B be a maximal monotone operator of E into E^* such that $F := B^{-1}(0) \cap EP(f) \cap VI(A, C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}x_0$ and $C_1 = C$, define a sequence $\{x_n\}$ as follows:*

$$(48)\qquad \begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JJ_{t_n}z_n), \\ u_n \in C \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \ \ \forall n \geq 1 \end{cases}$$

where J is the duality mapping on E and J_{t_n} is the resolvent of B. Assume that $\{\alpha_n\} \subset [0, 1]$, $\{\lambda_n\} \subset (0, \infty)$ and $\{r_n\}, \{t_n\} \subset (0, \infty)$ satisfying the retrictions

(C1) $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$,

(C2) $\{r_n\}, \{t_n\} \subset [s, \infty)$ *for some* $s > 0$,

(C3) $\{\lambda_n\} \subset [a, b]$ *for some* a, b *with* $0 < a < b < \frac{c^2\alpha}{2}$, *where* $\frac{1}{c}$ *is the 2-uniformly convexity of E.*

Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = \Pi_F x_0$.

Proof. As in the proof of Nakajo et. al. [Theorem 4.2, [24]], we get that $\{J_{t_n}\}$ satisfies the NST*-condition. Hence, we obtain the desired result. □

4.3. **Convex feasibility problems**

Let I be a countable set and C_i be a nonempty closed convex subset of a Banach space E such that $C := \cap_{i\in I}C_i \neq \emptyset$. Then we are concerned with the *convex feasibility problem* (CFP)

$$\text{finding an } x \in C.$$

This problem is a frequently appearing problem in diverse areas of mathematical and physical sciences. There is a considerable investigation on (CFP) in the framework of Hilbert spaces which captures applications in various disciplines such as image restoration [11, 15, 16, 38], computer tomography [29], and radiation theraphy treatment planning [8]. In computer tomography with limited data, in which an unknown image has to be reconstructed from a *priori* knowledge and from measured results, each piece of information gives a constraint which in turn, gives rise to a convex set C_i to which the unknown image should belong (see [3]). It follows from Lemma 2.5 that the *generalized projection* Π_C is a relatively nonexpansive mapping. Then we get the following result by Theorem 3.1.

Theorem 4.3. *Let C be a closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E and let $\{\Omega_i\}_{i\in I}$ be a family of nonempty closed convex subset of C. Let f be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)-(A4) and let A be an α-inverse strongly monotone of E into E^* such that $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A, C)$. Let $\Omega = \cap_{i\in I}\Omega_i \neq \emptyset$ and*

$F := \Omega \cap EP(f) \cap VI(A,C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define a sequence $\{x_n\}$ as follows:

$$(49) \quad \begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J \Pi_{\Omega_{i(n)}} z_n), \\ u_n \in C \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \forall n \geq 1 \end{cases}$$

where J is the duality mapping on E and $\Pi_{\Omega_{i(n)}}$ is the generalized projection from E into $\Omega_{i(n)}$ and the index mapping $i : \mathbb{N} \cup \{0\} \to I$ satisfies for each $i \in I$, there exists $M_i > 0$, for all $n \in \mathbb{N} \cup \{0\}, i \in \{i(n), ..., i(n + M_i - 1)\}$. Assume that $\{\alpha_n\} \subset [0,1], \{\lambda_n\} \subset (0,\infty)$ and $\{r_n\} \subset (0,\infty)$ satisfying the condition (C1)-(C3) of Theorem 3.1. Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = \Pi_F x_0$.

Proof. We shall show that $\{\Pi_{\Omega_{i(n)}}\}$ satisfies the NST*-condition. Let $T_n = \Pi_{\Omega_{i(n)}}$ and $\{z_n\}$ be bounded sequence in E such that $\lim_{n\to\infty} \|z_{n+1} - z_n\| = \lim_{n\to\infty} \|z_n - T_n z_n\| = 0$. Suppose that $z_{n_k} \rightharpoonup z$. Fixed $i \in I$. There exists a strictly increasing sequence $\{p_k\} \subset \mathbb{N} \cup \{0\}$ such that

$$n_k \leq p_k \leq n_k + M_i + 1 \text{ and } i(p_k), (\forall\ k \in \mathbb{N} \cup \{0\}).$$

Then we have,

$$\|z_{p_k} - z_{n_k}\| \leq \sum_{l=n_k}^{n_k+M_i-1} \|z_{l+1} - z_l\|.$$

for all $k \in \mathbb{N} \cup \{0\}$ which implies that $z_{p_k} \rightharpoonup z$. From $\|z_{p_k} - T_{p_k} z_{p_k}\| \to 0$, we get $T_{p_k} z_{p_k} = \Pi_{\Omega_i} z_{p_k} \rightharpoonup z$. So $z \in \Omega_i, \ \forall i$ and hence $z \in \Omega$. This implies that $\omega_w(z_n) \subset \Omega$. By Theorem 3.1, we obtain the result. □

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