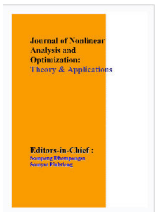
Journal of Nonlinear Analysis and Optimization
Vol. 1 No. 1 (2010), 71-91
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



An extragradient type method for a system of equilibrium problems, variational inequality problems and fixed points of finitely many nonexpansive mappings

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ABSTRACT: The purpose of this paper is to investigate the problem of finding the common element of the set of common fixed points of a finite family of nonexpansive mappings, the set of solutions of a system of equilibrium problems and the set of solutions of the variational inequality problem for a monotone and k-Lipschitz continuous mapping in Hilbert spaces. Consequently, we obtain the strong convergence theorem of the proposed iterative algorithm to the unique solutions of variational inequality, which is the optimality condition for a minimization problem. The results presented in this paper generalize, improve and extend some well-known results in the literature.

KEYWORDS: Nonexpansive mapping; Monotone mapping, Variational inequality; Fixed points, System of equilibrium problems, Extragradient approximation method.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $EP(F)$, that is,

$$EP(F) = \{\, x \in C : F(x,y) \geq 0, \quad \forall y \in C \,\}. \tag{2}$$

Given a mapping $B : C \to H$, let $F(x,y) = \langle Bx, y - x\rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Bz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality problems. Numerous problems in physics, optimization, saddle point problems, complementarity problems, mechanics and economics reduce to find a solution of (1). In 1997, Combettes and Hirstoaga [4] introduced an iterative scheme of finding the best approximation to initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. Some methods have

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Manuscript received May 11, 2010. Accepted June 14, 2010.

been proposed to solve the problem (1); see, for instance, [10, 16, 21, 22, 23, 24, 29, 33, 38, 39] and the references therein.

Let $\Im = \{F_k\}_{k\in\Lambda}$ be a family of bifunctions from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The system of equilibrium problems for $\Im = \{F_k\}_{k\in\Lambda}$ is to determine common equilibrium points for $\Im = \{F_k\}_{k\in\Lambda}$ such that

$$F_k(x,y) \geq 0, \quad \forall k \in \Lambda, \quad \forall y \in C. \tag{3}$$

where Λ is an arbitrary index set. The set of solutions of (3) is denoted by $SEP(\Im)$, that is,

$$SEP(\Im) = \left\{ x \in C : F_k(x,y) \geq 0, \quad \forall k \in \Lambda, \quad \forall y \in C \right\}. \tag{4}$$

If Λ is a singleton, then the problem (3) is reduced to the problem (1). The problem (3) is very general in the sense that it includes, as special case, some optimization , variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games, economics and others (see, for instance, [2, 4, 5]). The classical variational inequality problem is to find $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0, \quad \forall y \in C. \tag{5}$$

The set of solutions of (5) is denoted by $VI(C,B)$, that is,

$$VI(C,B) = \left\{ x \in C : \langle Bx, y - x\rangle \geq 0, \quad \forall y \in C \right\}. \tag{6}$$

The variational inequality has been extensively studied in the literature; see, for instance [6, 7, 9, 12, 16, 29, 39]. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Recall the following definitions:

(1) A mapping B of C into H is called *monotone* if

$$\langle Bx - By, x - y\rangle \geq 0, \ \ \forall x,y \in C.$$

(2) B is called *β-strongly monotone* (see [3, 18]) if there exists a constant $\beta > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \beta\|x-y\|^2, \quad \forall x,y \in C.$$

(3) B is called *k-Lipschitz continuous* if there exists a positive real number k such that

$$\|Bx - By\| \leq k\|x-y\|, \quad \forall x,y \in C.$$

(4) B is called *β-inverse-strongly monotone* (see [3, 18]) if there exists a constant $\beta > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \beta\|Bx-By\|^2, \ \ \forall x,y \in C.$$

Remark 1.1. It is obvious that any β-inverse-strongly monotone mapping B is monotone and $\frac{1}{\beta}$-Lipschitz continuous.

(5) A mapping T of C into itself is called *nonexpansive* (see [30]) if

$$\|Tx - Ty\| \leq \|x-y\|, \quad \forall x,y \in C.$$

We denote $F(T) = \{x \in C : Tx = x\}$ be the set of fixed points of T.

(6) Let $f : C \to C$ is said to be a *α-contraction* if there exists a coefficient α $(0 < \alpha < 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x-y\|, \quad \forall x,y \in C.$$

(7) An operator A is *strongly positive linear bounded operator* on H if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \ \ \forall x \in H.$$

(8) A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x,y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$.

(9) A monotone mapping $T : H \to 2^H$ is *maximal* if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let B be a monotone mapping of C into H and let $N_C v$ be the *normal cone* to C at $v \in C$, that is,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$$

and define

$$Tv = \begin{cases} Bv + N_C v, & v \in C; \\ \emptyset, & v \notin C. \end{cases}$$

Then T is the maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, B)$; see [27].

In 1976, Korpelevich [17] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda B x_n), \\ x_{n+1} = P_C(x_n - \lambda B y_n), \end{cases} \tag{7}$$

for all $n \geq 0$, where $\lambda \in (0, \frac{1}{k}), C$ is a closed convex subset of $\mathbb{R}^n$ and B is a monotone and k-Lipschitz continuous mapping of C into $\mathbb{R}^n$. He proved that if $VI(C, B)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$, generated by (7), converge to the same point $z \in VI(C, B)$. For finding a common element of the set of fixed points of a nonexpansive mapping and the set of solution of variational inequalities for an β-inverse-strongly monotone, Takahashi and Toyoda [31] introduced the following iterative scheme:

$$\begin{cases} x_0 \in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n B x_n), \ \ \forall n \geq 0, \end{cases} \tag{8}$$

where B is β-inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in (0,1) and $\{\lambda_n\}$ is a sequence in $(0, 2\beta)$. They showed that if $F(S) \cap VI(C, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (8) converges weakly to some $z \in F(S) \cap VI(C, B)$. Recently, Iiduka and Takahashi [15] proposed a new iterative scheme as following

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) S P_C(x_n - \lambda_n B x_n), \ \ \forall n \geq 0, \end{cases} \tag{9}$$

where B is β-inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in (0, 1), and $\{\lambda_n\}$ is a sequence in $(0, 2\beta)$. They showed that if $F(S) \cap VI(C, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (9) converges strongly to some $z \in F(S) \cap VI(C, B)$.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see e.g., [13, 35, 36, 37] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{10}$$

where A is a linear bounded operator, C is the fixed point set of a nonexpansive mapping S on H and b is a given point in H. Moreover, it is shown in [19] that the sequence $\{x_n\}$ defined by the scheme

$$x_{n+1} = \epsilon_n \gamma f(x_n) + (1 - \epsilon_n A) S x_n \tag{11}$$

converges strongly to $z = P_{F(S)}(I - A + \gamma f)(z)$. Recently, Plubtieng and Punpaeng [24] proposed the following iterative algorithm:

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A) S u_n. \end{cases} \tag{12}$$

They proved that if the sequence $\{\epsilon_n\}$ and $\{r_n\}$ of parameters satisfy appropriate condition, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge to the unique solution z of the variational inequality

$$\Big\langle (A - \gamma f)z, x - z \Big\rangle \geq 0, \ \forall x \in F(S) \cap EP(F), \tag{13}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S) \cap EP(F)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{14}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

In 2009, Peng and Yao [20] introduced an iterative scheme for finding a common element of the set of solutions of the system equilibrium problems (3), the set of solutions to the variational inequality for a monotone and Lipschitz continuous mapping and the set of common fixed points of a countable family of nonexpansive mappings in a Hilbert spaces and proved a strong convergence theorem.

Definition 1.2. [16]. For a finite family of nonexpansive mappings of $T_1, T_2, ..., T_N$ and sequence $\{\mu_{n,i}\}_{i=1}^N$ in $[0, 1]$, we define the mapping W_n of C into itself as follows:

$$\begin{aligned} U_{n,0} &= I, \\ U_{n,1} &= \mu_{n,1} T_1 U_{n,0} + (1 - \mu_{n,1}) U_{n,0}, \\ U_{n,2} &= \mu_{n,2} T_2 U_{n,1} + (1 - \mu_{n,2}) U_{n,1}, \\ &\vdots \\ U_{n,N-1} &= \mu_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \mu_{n,N-1}) U_{n,N-2}, \\ W_n = U_{n,N} &= \mu_{n,N} T_N U_{n,N-1} + (1 - \mu_{n,N}) U_{n,N-1}. \end{aligned} \tag{15}$$

On the other hand, Colao et al. [10] introduced and considered an iterative scheme for finding a common element of the set of solutions of the equilibrium problem (1) and the set of common fixed points of a finite family of nonexpansive mappings on C. Starting with an arbitrary initial $x_0 \in C$ and defining a sequence $\{x_n\}$ recursively by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + \big((1 - \beta)I - \epsilon_n A\big) W_n u_n, \end{cases} \tag{16}$$

where $\{\epsilon_n\}$ be a sequences in $(0, 1)$. It is proved [10] that under certain appropriate conditions imposed on $\{\epsilon_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (16) converges strongly to $z \in \cap_{n=1}^\infty F(T_n) \cap EP(F)$, where z is an equilibrium point for F and the unique solution of the variational inequality (13), i.e., $z = P_{\cap_{n=1}^\infty F(T_n) \cap EP(F)}(I - (A - \gamma f))z$.

In 2009, Colao et al. [11] introduced and considered an implicit iterative scheme for finding a common element of the set of solutions of the system equilibrium problems (3) and the set of common fixed points of an infinite family of nonexpansive mappings on C. Starting with an arbitrary initial $x_0 \in C$ and defining a sequence $\{z_n\}$ recursively by

$$z_n = \epsilon_n \gamma f(z_n) + (1 - \epsilon_n A) W_n J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} J_{r_{M-2,n}}^{F_{M-2}} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} z_n, \tag{17}$$

where $\{\epsilon_n\}$ be a sequences in $(0, 1)$. It is proved [11] that under certain appropriate conditions imposed on $\{\epsilon_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (17) converges strongly to $z \in \cap_{n=1}^\infty F(T_n) \cap \big(\cap_{k=1}^M SEP(F_k)\big)$, where z is the unique solution of the variational inequality (13) and which is the optimality condition for the minimization problem (14). In 2010, He et al. [14] introduced an explicit iterative scheme for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of an infinite family of nonexpansive mappings.

In this paper, motivated by Colao et al. [10, 11], He et al. [14], and Peng and Yao [20, 21], we introduce a new iterative scheme in a Hilbert space H which is mixed the iterative schemes of (16) and (17). We prove that the sequence converges strongly to a common element of the set of solutions of the system equilibrium problems (3), the set of common fixed points of a finite family of nonexpansive mappings and the set of solutions of variational inequality (5) for be a monotone and k-Lipschitz continuous mapping in Hilbert spaces by using the extragradient approximation method. The results presented in this paper generalize, improve and extend some well-known results in the literature.

2. **Preliminaries**

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. When $\{x_n\}$ is a sequence in H, we denote strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. In a real Hilbert space H, it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

Lemma 2.1. [26] *Let $(C, \langle .,. \rangle)$ be an inner product space. Then for all $x, y, z \in C$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2.$$

Recall that the metric (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point in $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y\in C} \|x-y\|.$$

The following characterizes the projection P_C.
In order to prove our main results, we need the following lemmas.

Lemma 2.2. *For a given $z \in H$, $u \in C$,*

$$u = P_C z \Leftrightarrow \langle u-z, v-u\rangle \geq 0, \quad \forall v \in C.$$

It is well known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x-y\rangle, \quad \forall x, y \in H. \tag{18}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and for all $x \in H, y \in C$,

$$\langle x - P_C x, y - P_C x\rangle \leq 0. \tag{19}$$

It is easy to see that (19) is equivalent to the following inequality:

$$\|x-y\|^2 \geq \|x-P_C x\|^2 + \|y - P_C x\|^2. \tag{20}$$

Using Lemma 2.2, one can see that the variational inequality (5) is equivalent to a fixed point problem.

It is easy to see that the following is true:

$$u \in VI(C,B) \Leftrightarrow u = P_C(u - \lambda Bu), \quad \lambda > 0. \tag{21}$$

Lemma 2.3. [25]. *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

hold for each $y \in H$ with $y \neq x$.

Lemma 2.4. [28] . *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.5. [32]. *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - l_n)a_n + \sigma_n, \ n \geq 0,$$

where $\{l_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} l_n = \infty$,
(2) $\limsup_{n\to\infty} \frac{\sigma_n}{l_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. . *Let H be a real Hilbert space. Then for all $x, y \in H$,*

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$,
(2) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$.

Lemma 2.7. [19]. *Let C be a nonempty closed convex subset of H and let f be a contraction of H into itself with $\alpha \in (0,1)$, and A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$. Then , for $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$,*

$$\Big\langle x - y, (A - \gamma f)x - (A - \gamma f)y \Big\rangle \geq (\bar{\gamma} - \alpha\gamma)\|x - y\|^2, \quad x, y \in H.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\bar{\gamma} - \gamma\alpha$.

Lemma 2.8. [19]. *Assume A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;
(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [2].

Lemma 2.9. [2]. *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

The following lemma was also given in [5].

Lemma 2.10. [5]. *Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $J_r^F : H \to C$ as follows:*

$$J_r^F(x) = \Big\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \Big\}$$

for all $z \in H$. Then, the following hold:

(1) J_r^F *is single- valued;*
(2) J_r^F *is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|J_r^F x - J_r^F y\|^2 \leq \Big\langle J_r^F x - J_r^F y, x - y \Big\rangle;$$

(3) $F(J_r^F) = EP(F)$; *and*
(4) $EP(F)$ *is closed and convex.*

3. **Main Results**

In this section, we deal with the strong convergence of extragradient approximation method (23) for finding a common element of the set of solutions of the system equilibrium problems (3), the set of common fixed points of a finite family of nonexpansive mappings and the set of solutions of variational inequality (5) for be a monotone and k-Lipschitz continuous mapping in Hilbert spaces.

First, let $T_i : C \to C$, where $i = 1, 2, \dots, N$, be a family of finitely nonexpansive mappings. Let the mapping W_n be defined by

$$(22) \qquad \begin{cases} U_{n,0} = I, \\ U_{n,1} = \lambda_{n,1} T_1 U_{n,0} + (1 - \lambda_{n,1}) U_{n,0}, \\ U_{n,2} = \lambda_{n,2} T_2 U_{n,1} + (1 - \lambda_{n,2}) U_{n,1}, \\ \vdots \\ U_{n,N-1} = \lambda_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \lambda_{n,N-1}) U_{N-2}, \\ W_n = U_{n,N} = \lambda_{n,N} T_N U_{n,N-1} + (1 - \lambda_{n,N}) U_{N-1}, \end{cases}$$

where $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}, \dots, \{\lambda_{n,N}\} \in (0, 1]$. Such a mapping W_n is called the *W-mapping* generated by $T_1, T_2, \dots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$. Nonexpansivity of each T_i ensures the nonexpansivity of W_n. Moreover, in ([1], Lemma 3.1), it is shown that $F(W_n) = \cap_{i=1}^{N} F(T_i)$.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_k, k \in \{1, 2, 3, \dots, M\}$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $T_1, T_2, \dots, T_N$ be a family of finitely nonexpansive mappings of C into itself and let W_n be the W-mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$. Let B be a monotone and k-Lipschitz continuous mapping of C into H such that*

$$\Omega := \cap_{n=1}^{N} F(T_n) \cap \left(\cap_{k=1}^{M} SEP(F_k)\right) \cap VI(C, B) \neq \emptyset.$$

Let f be a contraction of H into itself with $\alpha \in (0, 1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$(23) \qquad \begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ u_n = J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} J_{r_{M-2,n}}^{F_{M-2}} \dots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} x_n, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) W_n P_C(u_n - \lambda_n B y_n), \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $(0, 1)$, $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{k})$ and $\{r_{k,n}\}, k \in \{1, 2, 3, \dots, M\}$ are a real sequence in $(0, \infty)$ satisfy the following conditions:

- (C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
- (C3) $\liminf_{n\to\infty} r_{k,n} > 0$ *and* $\lim_{n\to\infty} |r_{k,n+1} - r_{k,n}| = 0$ *for each* $k \in \{1, 2, 3, \dots, M\}$,
- (C4) $\lim_{n\to\infty} \lambda_n = 0$,
- (C5) $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_{n-1,i}| = 0$ *for all* $i = 1, 2, \dots, N$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Omega$ which is the unique solution of the variational inequality

$$(24) \qquad \Big\langle (A - \gamma f) z, x - z \Big\rangle \geq 0, \quad \forall x \in \Omega.$$

Equivalently, we have $z = P_\Omega(I - A + \gamma f)(z)$, which is the optimality condition for the minimization problem

$$(25) \qquad \min_{x \in \Omega} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Proof. Note that from the condition (C1), we may assume, without loss of generality, that $\alpha_n \leq (1-\beta_n)\|A\|^{-1}$ for all $n \in \mathbb{N}$. From Lemma 2.8, we know that if $0 \leq \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$. We will assume that $\|I - A\| \leq 1 - \bar{\gamma}$. Since A is a strongly positive bounded linear operator on H, we have

$$\|A\| = \sup\Big\{\big|\langle Ax, x\rangle\big| : x \in H, \|x\| = 1\Big\}.$$

Observe that

$$\begin{aligned}\Big\langle \big((1-\beta_n)I - \alpha_n A\big)x, x\Big\rangle &= 1 - \beta_n - \alpha_n\langle Ax, x\rangle \\ &\geq 1 - \beta_n - \alpha_n\|A\| \\ &\geq 0,\end{aligned}$$

this show that $(1-\beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I - \alpha_n A\| &= \sup\Bigg\{\Big|\Big\langle \big((1-\beta_n)I - \alpha_n A\big)x, x\Big\rangle\Big| : x \in H, \|x\| = 1\Bigg\} \\ &= \sup\Big\{1 - \beta_n - \alpha_n\langle Ax, x\rangle : x \in H, \|x\| = 1\Big\} \\ &\leq 1 - \beta_n - \alpha_n\bar{\gamma}.\end{aligned}$$

We will divide the proof of Theorem 3.1 into several steps.

Step 1. We claim that $\{x_n\}$ is bounded.

Indeed, pick any $x^* \in \Omega$. By taking $\Im_n^k = J_{r_{k,n}}^{F_k} J_{r_{k-1,n}}^{F_{k-1}} J_{r_{k-2,n}}^{F_{k-2}} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1}$ for $k \in \{1, 2, 3, \ldots, M\}$ and $\Im_n^0 = I$ for all n. The nonexpansivity of $J_{r_{k,n}}^{F_k}$ for each $k = 1, 2, 3, \ldots, M$ implies that $\Im_n^k$ is nonexpansive. Let $x^* = \Im_n^k x^*$, we note that $u_n = \Im_n^M x_n$, it follows that

$$\|u_n - x^*\| = \|\Im_n^M x_n - \Im_n^M x^*\| \leq \|x_n - x^*\|.$$

Put $v_n = P_C(u_n - \lambda_n B y_n)$. Then, from (20) and the monotonicity of B, we compute

$$\begin{aligned}\|v_n - x^*\|^2 &\leq \|u_n - \lambda_n B y_n - x^*\|^2 - \|u_n - \lambda_n B y_n - v_n\|^2 \\ &= \|u_n - x^*\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle By_n, x^* - v_n\rangle \\ &= \|u_n - x^*\|^2 - \|u_n - v_n\|^2 \\ &\quad + 2\lambda_n\big(\langle By_n - Bx^*, x^* - y_n\rangle + \langle Bx^*, x^* - y_n\rangle + \langle By_n, y_n - v_n\rangle\big) \\ &\leq \|u_n - x^*\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle By_n, y_n - v_n\rangle \\ &= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - v_n\rangle - \|y_n - v_n\|^2 \\ &\quad + 2\lambda_n\langle By_n, y_n - v_n\rangle \\ &= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\ &\quad + 2\langle u_n - \lambda_n B y_n - y_n, v_n - y_n\rangle.\end{aligned}$$

Moreover, since $y_n = P_C(u_n - \lambda_n B u_n)$ and (19), we have

$$\langle u_n - \lambda_n B u_n - y_n, v_n - y_n\rangle \leq 0. \tag{26}$$

Since B is k-Lipschitz continuous and (26), we obtain

$$\begin{aligned}&\langle u_n - \lambda_n B y_n - y_n, v_n - y_n\rangle \\ &\quad = \langle u_n - \lambda_n B u_n - y_n, v_n - y_n\rangle + \langle \lambda_n B u_n - \lambda_n B y_n, v_n - y_n\rangle \\ &\quad \leq \langle \lambda_n B u_n - \lambda_n B y_n, v_n - y_n\rangle \\ &\quad \leq \lambda_n\|Bu_n - By_n\|\|v_n - y_n\| \\ &\quad \leq \lambda_n k\|u_n - y_n\|\|v_n - y_n\|.\end{aligned}$$

Since $\lambda_n \in (0, \frac{1}{k})$, we have

$$\begin{aligned}\|v_n - x^*\|^2 &\le \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\lambda_n k\|u_n - y_n\|\|v_n - y_n\| \\ &\le \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + \lambda_n^2 k^2\|u_n - y_n\|^2 + \|v_n - y_n\|^2 \\ &= \|u_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \\ &\le \|u_n - x^*\|^2,\end{aligned} \tag{27}$$

and hence

$$\|v_n - x^*\| \le \|u_n - x^*\| \le \|x_n - x^*\|. \tag{28}$$

Thus, we can note that

$$\begin{aligned}\|x_{n+1} - x^*\| &= \Big\|\alpha_n\big(\gamma f(x_n) - Ax^*\big) + \beta_n(x_n - x^*) + \big((1-\beta_n)I - \alpha_n A\big)(W_n v_n - x^*)\Big\| \\ &\le (1 - \beta_n - \alpha_n\bar{\gamma})\|v_n - x^*\| + \beta_n\|x_n - x^*\| + \alpha_n\|\gamma f(x_n) - Ax^*\| \\ &\le (1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - x^*\| + \beta_n\|x_n - x^*\| + \alpha_n\|\gamma f(x_n) - Ax^*\| \\ &= (1 - \alpha_n\bar{\gamma})\|x_n - x^*\| + \alpha_n\gamma\|f(x_n) - f(x^*)\| + \alpha_n\|\gamma f(x^*) - Ax^*\| \\ &\le (1 - \alpha_n\bar{\gamma})\|x_n - x^*\| + \alpha_n\gamma\alpha\|x_n - x^*\| + \alpha_n\|\gamma f(x^*) - Ax^*\| \\ &= (1 - (\bar{\gamma} - \gamma\alpha)\alpha_n)\|x_n - x^*\| + (\bar{\gamma} - \gamma\alpha)\alpha_n\frac{\|\gamma f(x^*) - Ax^*\|}{\bar{\gamma} - \gamma\alpha}.\end{aligned} \tag{29}$$

By induction that

$$\|x_n - x^*\| \le \max\left\{\|x_1 - x^*\|, \frac{\|\gamma f(x^*) - Ax^*\|}{\bar{\gamma} - \gamma\alpha}\right\}, \quad n \in \mathbb{N}. \tag{30}$$

Hence, $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{v_n\}$, $\{Bu_n\}$, $\{Bv_n\}$, $\{W_n v_n\}$ and $\{f(x_n)\}$.

Step 2. We claim that

$$\lim_{n\to\infty}\|\Im_n^k x_n - \Im_{n+1}^k x_n\| = 0 \tag{31}$$

for every $k \in \{1, 2, 3, \ldots, M\}$. From Step 2 of the proof in [10, Theorem 3.1], we have for $k \in \{1, 2, 3, \ldots, M\}$,

$$\lim_{n\to\infty}\|J^{F_k}_{r_{k,n+1}} x_n - J^{F_k}_{r_{k,n}} x_n\| = 0. \tag{32}$$

Note that for every $k \in \{1, 2, 3, \ldots, M\}$, we obtain

$$\Im_n^k = J^{F_k}_{r_{k,n}} J^{F_{k-1}}_{r_{k-1,n}} J^{F_{k-2}}_{r_{k-2,n}} \ldots J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} = J^{F_k}_{r_{k,n}} \Im_n^{k-1}.$$

So, we have

$$\begin{aligned}&\|\Im_n^k x_n - \Im_{n+1}^k x_n\| \\ =\ &\|J^{F_k}_{r_{k,n}}\Im_n^{k-1}x_n - J^{F_k}_{r_{k,n+1}}\Im_{n+1}^{k-1}x_n\| \\ \le\ &\|J^{F_k}_{r_{k,n}}\Im_n^{k-1}x_n - J^{F_k}_{r_{k,n+1}}\Im_n^{k-1}x_n\| + \|J^{F_k}_{r_{k,n+1}}\Im_n^{k-1}x_n - J^{F_k}_{r_{k,n+1}}\Im_{n+1}^{k-1}x_n\| \\ \le\ &\|J^{F_k}_{r_{k,n}}\Im_n^{k-1}x_n - J^{F_k}_{r_{k,n+1}}\Im_n^{k-1}x_n\| + \|\Im_n^{k-1}x_n - \Im_{n+1}^{k-1}x_n\| \\ \le\ &\|J^{F_k}_{r_{k,n}}\Im_n^{k-1}x_n - J^{F_k}_{r_{k,n+1}}\Im_n^{k-1}x_n\| + \|J^{F_{k-1}}_{r_{k-1,n}}\Im_n^{k-2}x_n - J^{F_{k-1}}_{r_{k-1,n+1}}\Im_n^{k-2}x_n\| \\ &+\|\Im_n^{k-2}x_n - \Im_{n+1}^{k-2}x_n\| \\ \le\ &\|J^{F_k}_{r_{k,n}}\Im_n^{k-1}x_n - J^{F_k}_{r_{k,n+1}}\Im_n^{k-1}x_n\| + \|J^{F_{k-1}}_{r_{k-1,n}}\Im_n^{k-2}x_n - J^{F_{k-1}}_{r_{k-1,n+1}}\Im_n^{k-2}x_n\| \\ &+ \ldots + \|J^{F_2}_{r_{2,n}}\Im_n^1 x_n - J^{F_2}_{r_{2,n+1}}\Im_n^1 x_n\| + \|J^{F_1}_{r_{1,n}}x_n - J^{F_1}_{r_{1,n+1}}x_n\|.\end{aligned} \tag{33}$$

Now, apply (32) to (33), we conclude (31).

Step 3. We claim that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

On the other hand, from $u_n = \Im_n^M x_n$ and $u_{n+1} = \Im_{n+1}^M x_{n+1}$, by the triangular inequality, we have

$$\begin{aligned} \|u_{n+1}-u_n\| &= \|\Im^M_{n+1}x_{n+1}-\Im^M_n x_n\| \\ &= \|\Im^M_{n+1}x_{n+1}-\Im^M_{n+1}x_n\|+\|\Im^M_{n+1}x_n-\Im^M_n x_n\| \\ &\leq \|x_{n+1}-x_n\|+\|\Im^M_{n+1}x_n-\Im^M_n x_n\|. \end{aligned} \tag{34}$$

We note that

$$\begin{aligned} \|v_{n+1}-v_n\| &= \|P_C(u_{n+1}-\lambda_{n+1}By_{n+1})-P_C(u_n-\lambda_n By_n)\| \\ &\leq \|u_{n+1}-\lambda_{n+1}By_{n+1}-(u_n-\lambda_n By_n)\| \\ &= \|(u_{n+1}-\lambda_{n+1}Bu_{n+1})-(u_n-\lambda_{n+1}Bu_n) \\ &\quad +\lambda_{n+1}(Bu_{n+1}-By_{n+1}-Bu_n)+\lambda_n By_n\| \\ &\leq \|(u_{n+1}-\lambda_{n+1}Bu_{n+1})-(u_n-\lambda_{n+1}Bu_n)\| \\ &\quad +\lambda_{n+1}(\|Bu_{n+1}\|+\|By_{n+1}\|+\|Bu_n\|)+\lambda_n\|By_n\| \\ &\leq (1+\lambda_{n+1}k)\|u_{n+1}-u_n\|+\lambda_{n+1}\big(\|Bu_{n+1}\|+\|By_{n+1}\|+\|Bu_n\|\big) \\ &\quad +\lambda_n\|By_n\|. \end{aligned} \tag{35}$$

Substituting (34) into (35), we have

$$\begin{aligned} \|v_{n+1}-v_n\| &\leq (1+\lambda_{n+1}k)\|u_{n+1}-u_n\|+\lambda_{n+1}\big(\|Bu_{n+1}\|+\|By_{n+1}\|+\|Bu_n\|\big)+\lambda_n\|By_n\| \\ &\leq (1+\lambda_{n+1}k)\|x_{n+1}-x_n\|+(1+\lambda_{n+1}k)\|\Im^M_{n+1}x_n-\Im^M_n x_n\| \\ &\quad +\lambda_{n+1}\big(\|Bu_{n+1}\|+\|By_{n+1}\|+\|Bu_n\|\big)+\lambda_n\|By_n\|. \end{aligned} \tag{36}$$

Putting $z_n=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}=\frac{\alpha_n\gamma f(x_n)+\big((1-\beta_n)I-\alpha_n A\big)W_n v_n}{1-\beta_n}$ then, we get $x_{n+1}=(1-\beta_n)z_n+\beta_n x_n,\ n\geq 1$. It follows that

$$\begin{aligned} z_{n+1}-z_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1})+\big((1-\beta_{n+1})I-\alpha_{n+1}A\big)W_{n+1}v_{n+1}}{1-\beta_{n+1}} \\ &\quad -\frac{\alpha_n\gamma f(x_n)+\big((1-\beta_n)I-\alpha_n A\big)W_n v_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1})-\frac{\alpha_n}{1-\beta_n}\gamma f(x_n)+W_{n+1}v_{n+1}-W_n v_n \\ &\quad -\frac{\alpha_{n+1}}{1-\beta_{n+1}}AW_{n+1}v_{n+1}+\frac{\alpha_n}{1-\beta_n}AW_n v_n \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(\gamma f(x_{n+1})-AW_{n+1}v_{n+1}\big)+\frac{\alpha_n}{1-\beta_n}\big(AW_n v_n-\gamma f(x_n)\big) \\ &\quad +W_{n+1}v_{n+1}-W_{n+1}v_n+W_{n+1}v_n-W_n v_n. \end{aligned} \tag{37}$$

It follows from (36) and (37) that

$$\begin{aligned}\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(x_{n+1})\|+\|AW_{n+1}v_{n+1}\|\big)\\ &\quad+\frac{\alpha_n}{1-\beta_n}\big(\|AW_nv_n\|+\|\gamma f(x_n)\|\big)+\|W_{n+1}v_{n+1}-W_{n+1}v_n\|\\ &\quad+\|W_{n+1}v_n-W_nv_n\|-\|x_{n+1}-x_n\|\\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(x_{n+1})\|+\|AW_{n+1}v_{n+1}\|\big)\\ &\quad+\frac{\alpha_n}{1-\beta_n}\big(\|AW_nv_n\|+\|\gamma f(x_n)\|\big)+\|v_{n+1}-v_n\|\\ &\quad+\|W_{n+1}v_n-W_nv_n\|-\|x_{n+1}-x_n\|\\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(x_{n+1})\|+\|AW_{n+1}v_{n+1}\|\big)\\ &\quad+\frac{\alpha_n}{1-\beta_n}\big(\|AW_nv_n\|+\|\gamma f(x_n)\|\big)+\lambda_{n+1}k\|x_{n+1}-x_n\|\\ &\quad+(1+\lambda_{n+1}k)\|\Im^M_{n+1}x_n-\Im^M_nx_n\|\\ &\quad+\lambda_{n+1}\big(\|Bu_{n+1}\|+\|By_{n+1}\|+\|Bu_n\|\big)\\ &\quad+\lambda_n\|By_n\|+\|W_{n+1}v_n-W_nv_n\|. \end{aligned}\tag{38}$$

By the definition of W_n that

$$\begin{aligned}\|W_{n+1}v_n-W_nv_n\| &= \|\lambda_{n+1,N}T_NU_{n+1,N-1}v_n+(1-\lambda_{n+1,N})v_n-\lambda_{n,N}T_NU_{n,N-1}v_n-(1-\lambda_{n,N})v_n\|\\ &\leq |\lambda_{n+1,N}-\lambda_{n,N}|\|v_n\|+\|\lambda_{n+1,N}T_NU_{n+1,N-1}v_n-\lambda_{n,N}T_NU_{n,N-1}v_n\|\\ &\leq |\lambda_{n+1,N}-\lambda_{n,N}|\|v_n\|+\|\lambda_{n+1,N}(T_NU_{n+1,N-1}v_n-T_NU_{n,N-1}v_n)\|\\ &\quad+|\lambda_{n+1,N}-\lambda_{n,N}|\|T_NU_{n,N-1}v_n\|\\ &\leq 2M|\lambda_{n+1,N}-\lambda_{n,N}|+\lambda_{n+1,N}\|U_{n+1,N-1}v_n-U_{n,N-1}v_n\|,\end{aligned}\tag{39}$$

where M is an approximate constant such that $M\geq\max\{\sup_{n\geq1}\{\|v_n\|\},\sup_{n\geq1}\{\|T_mU_{n,m-1}v_n\|\}\mid m=1,2,\dots,N\}$.

Since $0<\lambda_{n_i}\leq1$ for all $n\geq1$ and $i=1,2,\dots,N$, we compute

$$\begin{aligned}&\|U_{n+1,N-1}v_n-U_{n,N-1}v_n\|\\ =\;&\|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}v_n+(1-\lambda_{n+1,N-1})v_n-\lambda_{n,N-1}T_{N-1}U_{n,N-2}v_n-(1-\lambda_{n,N-1})v_n\|\\ \leq\;&|\lambda_{n+1,N-1}-\lambda_{n,N-1}|\|v_n\|+\|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}v_n-\lambda_{n,N-1}T_{N-1}U_{n,N-2}v_n\|\\ \leq\;&|\lambda_{n+1,N-1}-\lambda_{n,N-1}|\|v_n\|+\|\lambda_{n+1,N-1}(T_{N-1}U_{n+1,N-2}v_n-T_{N-1}U_{n,N-2}v_n)\|\\ &+|\lambda_{n+1,N-1}-\lambda_{n,N-1}|\|T_{N-1}U_{n,N-2}v_n\|\\ \leq\;&2M|\lambda_{n+1,N-1}-\lambda_{n,N-1}|+\|U_{n+1,N-2}v_n-U_{n,N-2}v_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|U_{n+1,N-1}v_n-U_{n,N-1}v_n\| &\leq 2M|\lambda_{n+1,N-1}-\lambda_{n,N-1}|+2M|\lambda_{n+1,N-2}-\lambda_{n,N-2}|\\ &\quad+\|U_{n+1,N-3}v_n-U_{n,N-3}v_n\|\\ &\leq 2M\sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|+\|U_{n+1,1}v_n-U_{n,1}v_n\|\\ &= 2M\sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|\\ &\quad+\|\lambda_{n+1,1}T_1v_n+(1-\lambda_{n+1,1})v_n-\lambda_{n,1}T_1v_n-(1-\lambda_{n,1})v_n\|\\ &\leq 2M\sum_{i=1}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|.\end{aligned}\tag{40}$$

Substituting (40) into (39) yields that

$$\begin{aligned}\|W_{n+1}v_n - W_nv_n\| &\leq 2M|\lambda_{n+1,N} - \lambda_{n,N}| + 2\lambda_{n+1,N}M\sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| \\ &\leq 2M\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|, \qquad (41)\end{aligned}$$

Applying (41) in (38), we get

$$\begin{aligned}\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(x_{n+1})\| + \|AW_{n+1}v_{n+1}\|\big) \\ &\quad + \frac{\alpha_n}{1-\beta_n}\big(\|AW_nv_n\| + \|\gamma f(x_n)\|\big) + \lambda_{n+1}k\|x_{n+1} - x_n\| \\ &\quad + (1+\lambda_{n+1}k)\|\Im^M_{n+1}x_n - \Im^M_n x_n\| \\ &\quad + \lambda_{n+1}\big(\|Bu_{n+1}\| + \|By_{n+1}\| + \|Bu_n\|\big) \\ &\quad + \lambda_n\|By_n\| + 2M\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|.\end{aligned}$$

By (31) and (C1)-(C5), imply that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, by Lemma 2.4, we obtain

$$\lim_{n\to\infty}\|z_n - x_n\| = 0.$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \qquad (42)$$

Applying (31), (42) and (C4) to (34) and (35), we obtain that

$$\lim_{n\to\infty}\|u_{n+1} - u_n\| = \lim_{n\to\infty}\|v_{n+1} - v_n\| = 0. \qquad (43)$$

Step 4. We claim that $\lim_{n\to\infty}\|x_n - W_nv_n\| = 0$.
Since $x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_nv_n$, we have

$$\begin{aligned}\|x_n - W_nv_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - W_nv_n\| \\ &= \|x_n - x_{n+1}\| + \Big\|\alpha_n\gamma f(x_n) + \beta_n x_n + \big((1-\beta_n)I - \alpha_n A\big)W_nv_n - W_nv_n\Big\| \\ &= \|x_n - x_{n+1}\| + \Big\|\alpha_n\big(\gamma f(x_n) - AW_nv_n\big) + \beta_n(x_n - W_nv_n)\Big\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\big(\|\gamma f(x_n)\| + \|AW_nv_n\|\big) + \beta_n\|x_n - W_nv_n\|,\end{aligned}$$

it follows that

$$\|x_n - W_nv_n\| \leq \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\big(\|\gamma f(x_n)\| + \|AW_nv_n\|\big).$$

By (C1), (C2) and (42), we obtain

$$\lim_{n\to\infty}\|W_nv_n - x_n\| = 0. \qquad (44)$$

Step 5. We claim that the following statements hold:

(i) $\lim_{n\to\infty}\|u_n - y_n\| = 0$;
(ii) $\lim_{n\to\infty}\|u_n - v_n\| = 0$;
(iii) $\lim_{n\to\infty}\|W_ny_n - y_n\| = 0$.

For any $x^* \in \Omega$ and (23), we compute

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \Big\|\big((1-\beta_n)I - \alpha_n A\big)(W_n v_n - x^*) + \beta_n(x_n - x^*) + \alpha_n\big(\gamma f(x_n) - Ax^*\big)\Big\|^2 \\
&= \|\big((1-\beta_n)I - \alpha_n A\big)(W_n v_n - x^*) + \beta_n(x_n - x^*)\|^2 + \alpha_n^2\|\gamma f(x_n) - Ax^*\|^2 \\
&\quad + 2\beta_n\alpha_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\alpha_n\Big\langle \big((1-\beta_n)I - \alpha_n A\big)(W_n v_n - x^*), \gamma f(x_n) - Ax^*\Big\rangle \\
&\le \big[(1-\beta_n-\alpha_n\bar{\gamma})\|W_n v_n - x^*\| + \beta_n\|x_n - x^*\|\big]^2 + \alpha_n^2\|\gamma f(x_n) - Ax^*\|^2 \\
&\quad + 2\beta_n\alpha_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\alpha_n\Big\langle \big((1-\beta_n)I - \alpha_n A\big)(W_n v_n - x^*), \gamma f(x_n) - Ax^*\Big\rangle \\
&\le \big[(1-\beta_n-\alpha_n\bar{\gamma})\|v_n - x^*\| + \beta_n\|x_n - x^*\|\big]^2 + c_n \\
&\le (1-\beta_n-\alpha_n\bar{\gamma})^2\|v_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 \\
&\quad + 2(1-\beta_n-\alpha_n\bar{\gamma})\beta_n\|v_n - x^*\|\|x_n - x^*\| + c_n \\
&\le (1-\beta_n-\alpha_n\bar{\gamma})^2\|v_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 \\
&\quad + (1-\beta_n-\alpha_n\bar{\gamma})\beta_n\big(\|v_n - x^*\|^2 + \|x_n - x^*\|^2\big) + c_n \\
&= \big[(1-\alpha_n\bar{\gamma})^2 - 2(1-\alpha_n\bar{\gamma})\beta_n + \beta_n^2\big]\|v_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 \\
&\quad + \big((1-\alpha_n\bar{\gamma})\beta_n - \beta_n^2\big)\big(\|v_n - x^*\|^2 + \|x_n - x^*\|^2\big) + c_n \\
&= (1-\alpha_n\bar{\gamma})^2\|v_n - x^*\|^2 - (1-\alpha_n\bar{\gamma})\beta_n\|v_n - x^*\|^2 + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - x^*\|^2 + c_n \\
&= (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\|v_n - x^*\|^2 + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - x^*\|^2 + c_n, \qquad (45)
\end{aligned}$$

where

$$\begin{aligned}
c_n &= \alpha_n^2\|\gamma f(x_n) - Ax^*\|^2 + 2\beta_n\alpha_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\alpha_n\Big\langle \big((1-\beta_n)I - \alpha_n A\big)(W_n v_n - x^*), \gamma f(x_n) - Ax^*\Big\rangle.
\end{aligned}$$

It follows from the condition (C1) that

$$\lim_{n\to\infty} c_n = 0. \qquad (46)$$

Substituting (27) into (45), and (C4), we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\|v_n - x^*\|^2 + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - x^*\|^2 + c_n \\
&\le (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\Big\{\|u_n - x^*\|^2 + (\lambda_n^2k^2 - 1)\|u_n - y_n\|^2\Big\} \\
&\quad + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - x^*\|^2 + c_n \\
&\le (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\Big\{\|x_n - x^*\|^2 + (\lambda_n^2k^2 - 1)\|u_n - y_n\|^2\Big\} \\
&\quad + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - x^*\|^2 + c_n \\
&= (1-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})(\lambda_n^2k^2 - 1)\|u_n - y_n\|^2 + c_n \\
&\le \|x_n - x^*\|^2 + (\lambda_n^2k^2 - 1)\|u_n - y_n\|^2 + c_n.
\end{aligned}$$

Hence

$$\begin{aligned}
(1-\lambda_n^2k^2)\|u_n - y_n\|^2 &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + c_n \\
&= (\|x_n - x^*\| - \|x_{n+1} - x^*\|)(\|x_n - x^*\| + \|x_{n+1} - x^*\|) + c_n \\
&\le \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) + c_n.
\end{aligned}$$

Since (46) and (42), we obtain

$$\lim_{n\to\infty}\|u_n - y_n\| = 0. \qquad (47)$$

By the same argument as in (27), we also have

$$\begin{aligned}
\|v_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\lambda_n k\|u_n - y_n\|\|v_n - y_n\| \\
&\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + \|u_n - y_n\|^2 + \lambda_n^2 k^2\|v_n - y_n\|^2 \\
&= \|u_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - v_n\|^2 \\
&\leq \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - v_n\|^2.
\end{aligned} \tag{48}$$

Substituting (48) into (45), and (C4), we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\Big\{\|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - v_n\|^2\Big\} \\
&\quad + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - x^*\|^2 + c_n \\
&= (1-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})(\lambda_n^2 k^2 - 1)\|y_n - v_n\|^2 + c_n \\
&\leq \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - v_n\|^2 + c_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
(1-\lambda_n^2 k^2)\|y_n - v_n\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + c_n \\
&\leq \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) + c_n.
\end{aligned}$$

Again from (46) and (42), we have

$$\lim_{n\to\infty}\|y_n - v_n\| = 0. \tag{49}$$

On the other hand, we note that

$$\|u_n - v_n\| \leq \|u_n - y_n\| + \|y_n - v_n\|.$$

Applying (47)and (49), we get

$$\lim_{n\to\infty}\|u_n - v_n\| = 0. \tag{50}$$

For any $x^* \in \Omega$, note that $J_{r_{k,n}}^{F_k}$ is firmly nonexpansive (Lemma 2.10) for $k \in \{1,2,3,\dots,M\}$, then we get

$$\begin{aligned}
\|\Im_n^k x_n - x^*\|^2 &= \|J_{r_{k,n}}^{F_k}\Im_n^{k-1}x_n - J_{r_{k,n}}^{F_k}x^*\|^2 \\
&\leq \Big\langle J_{r_{k,n}}^{F_k}\Im_n^{k-1}x_n - J_{r_{k,n}}^{F_k}x^*, \Im_n^{k-1}x_n - x^*\Big\rangle \\
&= \Big\langle \Im_n^k x_n - x^*, \Im_n^{k-1}x_n - x^*\Big\rangle \\
&= \frac{1}{2}\Big(\|\Im_n^k x_n - x^*\|^2 + \|\Im_n^{k-1}x_n - x^*\|^2 - \|\Im_n^k x_n - \Im_n^{k-1}x_n\|^2\Big)
\end{aligned}$$

and hence

$$\|\Im_n^k x_n - x^*\|^2 \leq \|\Im_n^{k-1}x_n - x^*\|^2 - \|\Im_n^k x_n - \Im_n^{k-1}x_n\|^2, \quad k = 1,2,3,\dots,M$$

which implies that for each $k \in \{1,2,3,\dots,M\}$,

$$\begin{aligned}
\|\Im_n^k x_n - x^*\|^2 &\leq \|\Im_n^0 x_n - x^*\|^2 - \|\Im_n^k x_n - \Im_n^{k-1}x_n\|^2 \\
&\quad - \|\Im_n^{k-1}x_n - \Im_n^{k-2}x_n\|^2 - \ldots - \|\Im_n^2 x_n - \Im_n^1 x_n\|^2 - \|\Im_n^1 x_n - \Im_n^0 x_n\|^2 \\
&\leq \|x_n - x^*\|^2 - \|\Im_n^k x_n - \Im_n^{k-1}x_n\|^2.
\end{aligned}$$

Together with (27) and (48), we compute

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
\leq\ & (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\|v_n-x^*\|^2+(1-\alpha_n\bar{\gamma})\beta_n\|x_n-x^*\|^2+c_n\\
\leq\ & (1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\Big\{\|u_n-x^*\|^2+(\lambda_n^2k^2-1)\|u_n-y_n\|^2\Big\}\\
&+(1-\alpha_n\bar{\gamma})\beta_n\|x_n-x^*\|^2+c_n\\
=\ & (1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\|u_n-x^*\|^2+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2\\
&+(1-\alpha_n\bar{\gamma})\beta_n\|x_n-x^*\|^2+c_n\\
=\ & (1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-x^*\|^2+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2\\
&+(1-\alpha_n\bar{\gamma})\beta_n\|x_n-x^*\|^2+c_n\\
\leq\ & (1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\Big\{\|x_n-x^*\|^2-\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\Big\}\\
&+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2+(1-\alpha_n\bar{\gamma})\beta_n\|x_n-x^*\|^2+c_n\\
=\ & (1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\|x_n-x^*\|^2-(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\\
&+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2+(1-\alpha_n\bar{\gamma})\beta_n\|x_n-x^*\|^2+c_n\\
=\ & (1-\alpha_n\bar{\gamma})^2\|x_n-x^*\|^2-(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\\
&+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2+c_n\\
=\ & \big[1-2\alpha_n\bar{\gamma}+(\alpha_n\bar{\gamma})^2\big]\|x_n-x^*\|^2-(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\\
&+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2+c_n\\
\leq\ & \|x_n-x^*\|^2+(\alpha_n\bar{\gamma})^2\|x_n-x^*\|^2-(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\\
&+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2+c_n.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\\
\leq\ & \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+(\alpha_n\bar{\gamma})^2\|x_n-x^*\|^2\\
&+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2+c_n\\
\leq\ & \|x_n-x_{n+1}\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)+(\alpha_n\bar{\gamma})^2\|x_n-x^*\|^2\\
&+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2+c_n.
\end{aligned}$$

Using (C1), (42) , (46) and (47), we get

(51) $$\lim_{n\to\infty}\|\Im_n^k x_n-\Im_n^{k-1}x_n\|=0, \forall k=1,2,...,M.$$

Observe that

$$\begin{aligned}
\|W_ny_n-y_n\| \leq\ & \|W_ny_n-W_nv_n\|+\|W_nv_n-x_n\|+\|x_n-u_n\|+\|u_n-y_n\|\\
\leq\ & \|y_n-v_n\|+\|W_nv_n-x_n\|+\|x_n-\Im_n^M x_n\|+\|u_n-y_n\|\\
\leq\ & \|y_n-v_n\|+\|W_nv_n-x_n\|+\|\Im_n^0 x_n-\Im_n^1 x_n\|+\|\Im_n^1 x_n-\Im_n^2 x_n\|\\
&+\ \dots\ +\|\Im_n^{M-1}x_n-\Im_n^M x_n\|+\|u_n-y_n\|.
\end{aligned}$$

Applying (44), (47), (49) and (51) to the last inequality, we have

(52) $$\lim_{n\to\infty}\|W_ny_n-y_n\|=0.$$

We also have

$$\begin{aligned}
\|W_nu_n-u_n\| \leq\ & \|W_nu_n-W_nv_n\|+\|W_nv_n-x_n\|+\|x_n-u_n\|\\
\leq\ & \|W_nu_n-W_nv_n\|+\|W_nv_n-x_n\|+\|x_n-\Im_n^M x_n\|\\
\leq\ & \|u_n-v_n\|+\|W_nv_n-x_n\|+\|\Im_n^0 x_n-\Im_n^1 x_n\|+\|\Im_n^1 x_n-\Im_n^2 x_n\|\\
&+\ \dots\ +\|\Im_n^{M-1}x_n-\Im_n^M x_n\|.
\end{aligned}$$

Applying (44), (50) and (51) to the last inequality, we have

$$\lim_{n\to\infty}\|W_n u_n - u_n\| = 0. \tag{53}$$

Step 6. We claim that $\limsup_{n\to\infty}\big\langle (A-\gamma f)z, z-x_n\big\rangle \le 0$, which z is the unique solution of the variational inequality $\big\langle (A-\gamma f)z, x-z\big\rangle \ge 0,\ \forall x\in\Omega$.

Observe that $P_\Omega(I-A+\gamma f)$ is a contraction of H into itself. Indeed, for all $x,y\in H$, we have

$$\begin{aligned}
&\|P_\Omega(I-A+\gamma f)(x) - P_\Omega(I-A+\gamma f)(y)\|\\
&\quad\le \|(I-A+\gamma f)(x)-(I-A+\gamma f)(y)\|\\
&\quad\le \|I-A\|\|x-y\| + \gamma\|f(x)-f(y)\|\\
&\quad\le (1-\bar\gamma)\|x-y\| + \gamma\alpha\|x-y\|\\
&\quad= (1-(\bar\gamma-\gamma\alpha))\|x-y\|,
\end{aligned}$$

since H is complete, there exists a unique fixed point $z\in H$ such that $z = P_\Omega(I-A+\gamma f)(z)$.

Since $z = P_\Omega(I-A+\gamma f)(z)$ is a unique solution of the variational inequality (24). To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}\big\langle (A-\gamma f)z, z-x_{n_i}\big\rangle = \limsup_{n\to\infty}\big\langle (A-\gamma f)z, z-x_n\big\rangle.$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to $w\in C$. Without loss of generality, we can assume that $u_{n_i}\rightharpoonup w$. Since $\lim_{n\to\infty}\|\Im_n^k x_n - \Im_n^{k-1}x_n\| = 0$ for each $k=1,2,...,M$, we have $\Im_{n_i}^k x_{n_i}\rightharpoonup w$ for each $k=1,2,...,M$. From $\|u_n-y_n\|\to 0$ and $\|u_n-v_n\|\to 0$, we also obtain $y_{n_i}\rightharpoonup w$ and $v_{n_i}\rightharpoonup w$. Since $\{u_{n_i}\}\subset C$ and C is closed and convex, we have $w\in C$.

Next, we show that $w\in\Omega$, where $\Omega := \cap_{n=1}^N F(T_n)\cap\big(\cap_{k=1}^M SEP(F_k)\big)\cap VI(C,B)$.

First, we show that $w\in\cap_{k=1}^M SEP(F_k)$. Since $u_n = \Im_n^k x_n$ for $k=1,2,3,\dots,M$, we also have

$$F_k(\Im_n^k x_n, y) + \frac{1}{r_n}\langle y-\Im_n^k x_n, \Im_n^k x_n - \Im_n^{k-1}x_n\rangle \ge 0,\quad \forall y\in C.$$

If follows from (A2) that,

$$\frac{1}{r_n}\langle y-\Im_n^k x_n, \Im_n^k x_n - \Im_n^{k-1}x_n\rangle \ge -F_k(\Im_n^k x_n, y) \ge F_k(y,\Im_n^k x_n)$$

and hence

$$\left\langle y-\Im_{n_i}^k x_{n_i}, \frac{\Im_{n_i}^k x_{n_i} - \Im_{n_i}^{k-1}x_{n_i}}{r_{n_i}}\right\rangle \ge F_k(y,\Im_{n_i}^k x_{n_i}).$$

Since $\frac{\Im_{n_i}^k x_{n_i} - \Im_{n_i}^{k-1}x_{n_i}}{r_{n_i}}\to 0$ and $u_{n_i} = \Im_{n_i}^k x_{n_i}\rightharpoonup w$, it follows by (A4) that

$$F_k(y,w)\le 0\quad \forall y\in C,$$

for each $k=1,2,3,\dots,M$.

For t with $0<t\le 1$ and $y\in H$, let $y_t = ty+(1-t)w$. Since $y\in C$ and $w\in C$, we have $y_t\in C$ and hence $F_k(y_t,w)\le 0$. So, from (A1) and (A4) we have

$$0 = F_k(y_t,y_t) \le tF_k(y_t,y) + (1-t)F_k(y_t,w) \le tF_k(y_t,y)$$

and hence $F_k(y_t,y)\ge 0$. From (A3), we have $F_k(w,y)\ge 0$ for all $y\in C$ and hence $w\in EP(F_k)$ for $k=1,2,3,\dots,M$, that is, $w\in\cap_{k=1}^M SEP(F_k)$.

Next, we show that $w\in\cap_{i=1}^N F(T_i)$. By [1, Lemma 3.1], we have $F(W_n) = \cap_{i=1}^N F(T_n)$. Assume $w\notin F(W_n)$. Since $y_{n_i}\rightharpoonup w$, $\|W_n y_{n_i} - y_{n_i}\|\to 0$ and $w\ne W_n w$, it follows by the Opial's condition that

$$\begin{aligned}\liminf_{i\to\infty}\|y_{n_i}-w\| &< \liminf_{i\to\infty}\|y_{n_i}-W_nw\|\\ &\le \liminf_{i\to\infty}\{\|y_{n_i}-W_ny_{n_i}\|+\|W_ny_{n_i}-W_nw\|\}\\ &\le \liminf_{i\to\infty}\|y_{n_i}-w\|\end{aligned}$$

which derives a contradiction. Thus, we have $w\in F(W_n)=\cap_{i=1}^N F(T_i)$.

Finally, we show that $w\in VI(C,B)$. Define

$$Tv=\begin{cases} Bv+N_Cv, & v\in C,\\ \emptyset, & v\notin C.\end{cases}$$

Then, T is maximal monotone. Let $(v,w_1)\in G(T)$. Since $w_1-Bv\in N_Cv$ and $v_n\in C$, we have $\langle v-v_n,w_1-Bv\rangle\ge 0$. On the other hand, $v_n=P_C(u_n-\lambda_nBy_n)$, we have

$$\langle v-v_n,v_n-(u_n-\lambda_nBy_n)\rangle\ge 0,$$

and hence

$$\left\langle v-v_n,\frac{v_n-u_n}{\lambda_n}+By_n\right\rangle\ge 0.$$

Therefore, we have

$$\begin{aligned}\langle v-v_{n_i},w\rangle &\ge \langle v-v_{n_i},Bv\rangle\\ &\ge \langle v-v_{n_i},Bv\rangle-\left\langle v-v_{n_i},\frac{v_{n_i}-u_{n_i}}{\lambda_{n_i}}+By_{n_i}\right\rangle\\ &= \left\langle v-v_{n_i},Bv-By_{n_i}-\frac{v_{n_i}-u_{n_i}}{\lambda_{n_i}}\right\rangle\\ &= \langle v-v_{n_i},Bv-Bv_{n_i}\rangle+\langle v-v_{n_i},Bv_{n_i}-By_{n_i}\rangle\\ &\quad -\left\langle v-v_{n_i},\frac{v_{n_i}-u_{n_i}}{\lambda_{n_i}}\right\rangle\\ &\ge \langle v-v_{n_i},Bv_{n_i}-By_{n_i}\rangle-\left\langle v-v_{n_i},\frac{v_{n_i}-u_{n_i}}{\lambda_{n_i}}\right\rangle.\end{aligned}$$

Since $\lim_{n\to\infty}\|v_n-u_n\|=\lim_{n\to\infty}\|v_n-y_n\|=0$, $u_{n_i}\rightharpoonup w$ and B is Lipschitz continuous, we obtain that $\lim_{n\to\infty}\|Bv_n-By_n\|=0$ and $v_{n_i}\rightharpoonup w$. From $\liminf_{n\to\infty}\lambda_n>0$ and $\lim_{n\to\infty}\|v_n-u_n\|=0$, we obtain

$$\langle v-w,w_1\rangle\ge 0.$$

Since T is maximal monotone, we have $w\in T^{-1}0$ and hence $w\in VI(C,B)$. Hence $w\in\Omega$, where $\Omega:=\cap_{i=1}^N F(T_i)\cap\left(\cap_{k=1}^M SEP(F_k)\right)\cap VI(C,B)$.

Since $z=P_\Omega(I-A+\gamma f)(z)$, it follows that

$$\begin{aligned}\limsup_{n\to\infty}\big\langle (A-\gamma f)z,z-x_n\big\rangle &= \limsup_{n\to\infty}\big\langle (A-\gamma f)z,z-x_n\big\rangle\\ &= \lim_{i\to\infty}\big\langle (A-\gamma f)z,z-x_{n_i}\big\rangle\\ &= \big\langle (A-\gamma f)z,z-w\big\rangle\le 0.\end{aligned}\tag{54}$$

It follows from the last inequality and (44) that

$$\limsup_{n\to\infty}\big\langle \gamma f(z)-Az,W_nv_n-z\big\rangle\le 0.\tag{55}$$

Step 7. Finally, we claim that $\{x_n\}$ converges strongly to $z=P_\Omega(I-A+\gamma f)(z)$.

Indeed, from (23) , we have

$$
\begin{aligned}
(56)\qquad & \|x_{n+1}-z\|^2 \\
= {} & \big\|\alpha_n\gamma f(x_n)+\beta_n x_n+\big((1-\beta_n)I-\alpha_n A\big)W_n v_n - z\big\|^2 \\
= {} & \|\big((1-\beta_n)I-\alpha_n A\big)(W_n v_n - z)+\beta_n(x_n-z)+\alpha_n\big(\gamma f(x_n)-Az\big)\|^2 \\
= {} & \|\big((1-\beta_n)I-\alpha_n A\big)(W_n v_n - z)+\beta_n(x_n-z)\|^2+\alpha_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\alpha_n\langle x_n-z,\gamma f(x_n)-Az\rangle \\
& +2\alpha_n\Big\langle\big((1-\beta_n)I-\alpha_n A\big)(W_n v_n - z),\gamma f(x_n)-Az\Big\rangle \\
\leq {} & \Big[(1-\beta_n-\alpha_n\bar{\gamma})\|W_n v_n - z\|+\beta_n\|x_n-z\|\Big]^2+\alpha_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\alpha_n\gamma\langle x_n-z,f(x_n)-f(z)\rangle+2\beta_n\alpha_n\langle x_n-z,\gamma f(z)-Az\rangle \\
& +2(1-\beta_n)\gamma\alpha_n\langle W_n v_n - z,f(x_n)-f(z)\rangle+2(1-\beta_n)\alpha_n\langle W_n v_n - z,\gamma f(z)-Az\rangle \\
& -2\alpha_n^2\langle A(W_n v_n - z),\gamma f(z)-Az\rangle \\
\leq {} & \Big[(1-\beta_n-\alpha_n\bar{\gamma})\|W_n v_n - z\|+\beta_n\|x_n-z\|\Big]^2+\alpha_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\alpha_n\gamma\|x_n-z\|\|f(x_n)-f(z)\|+2\beta_n\alpha_n\langle x_n-z,\gamma f(z)-Az\rangle \\
& +2(1-\beta_n)\gamma\alpha_n\|W_n v_n - z\|\|f(x_n)-f(z)\|+2(1-\beta_n)\alpha_n\langle W_n v_n - z,\gamma f(z)-Az\rangle \\
& -2\alpha_n^2\langle A(W_n v_n - z),\gamma f(z)-Az\rangle \\
\leq {} & \Big[(1-\beta_n-\alpha_n\bar{\gamma})\|x_n-z\|+\beta_n\|x_n-z\|\Big]^2+\alpha_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\alpha_n\gamma\alpha\|x_n-z\|^2+2\beta_n\alpha_n\langle x_n-z,\gamma f(z)-Az\rangle \\
& +2(1-\beta_n)\gamma\alpha_n\alpha\|x_n-z\|^2+2(1-\beta_n)\alpha_n\langle W_n v_n - z,\gamma f(z)-Az\rangle \\
& -2\alpha_n^2\langle A(W_n v_n - z),\gamma f(z)-Az\rangle \\
= {} & \Big[(1-\alpha_n\bar{\gamma})^2+2\beta_n\alpha_n\gamma\alpha+2(1-\beta_n)\gamma\alpha_n\alpha\Big]\|x_n-z\|^2+\alpha_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\alpha_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n)\alpha_n\langle W_n v_n - z,\gamma f(z)-Az\rangle \\
& -2\alpha_n^2\langle A(W_n v_n - z),\gamma f(z)-Az\rangle \\
\leq {} & \big[1-2(\bar{\gamma}-\alpha\gamma)\alpha_n\big]\|x_n-z\|^2+\bar{\gamma}^2\alpha_n^2\|x_n-z\|^2+\alpha_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\alpha_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n)\alpha_n\langle W_n v_n - z,\gamma f(z)-Az\rangle \\
& +2\alpha_n^2\|A(W_n v_n - z)\|\|\gamma f(z)-Az\| \\
= {} & \big[1-2(\bar{\gamma}-\alpha\gamma)\alpha_n\big]\|x_n-z\|^2+\alpha_n\Big\{\alpha_n\Big[\bar{\gamma}^2\|x_n-z\|^2+\|\gamma f(x_n)-Az\|^2 \\
& +2\|A(W_n v_n - z)\|\|\gamma f(z)-Az\|\Big]+2\beta_n\langle x_n-z,\gamma f(z)-Az\rangle \\
& +2(1-\beta_n)\langle W_n v_n - z,\gamma f(z)-Az\rangle\Big\}
\end{aligned}
$$

Since $\{x_n\}$, $\{f(x_n)\}$ and $\{W_n v_n\}$ are bounded, we can take a constant $K>0$ such that

$$\bar{\gamma}^2\|x_n-z\|^2+\|\gamma f(x_n)-Az\|^2+2\|A(W_n v_n - z)\|\|\gamma f(z)-Az\|\leq K,$$

for all $n\geq 0$. It then follows that

$$\|x_{n+1}-z\|^2\leq\big[1-2(\bar{\gamma}-\alpha\gamma)\alpha_n\big]\|x_n-z\|^2+\alpha_n\sigma_n, \tag{57}$$

where

$$\sigma_n=2\beta_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n)\langle W_n v_n - z,\gamma f(z)-Az\rangle+\alpha_n K.$$

Using (C1), (54) and (55), we get $\limsup_{n\to\infty}\sigma_n \le 0$. Applying Lemma 2.5 to (57), we conclude that $x_n \to z$ in norm. This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_k, k \in \{1,2,3,\ldots,M\}$ be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let B be a monotone and k-Lipschitz continuous mapping of C into itself such that*

$$\Omega := \big(\cap_{k=1}^{M} SEP(F_k)\big) \cap VI(C,B) \neq \emptyset.$$

Let f be a contraction of H into itself with $\alpha \in (0,1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ u_n = J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} J_{r_{M-2,n}}^{F_{M-2}} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} x_n, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + \big((1-\beta_n)I - \alpha_n A\big) P_C(u_n - \lambda_n B y_n), \quad \forall n \ge 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $(0,1)$, $\{\lambda_n\} \subset [a,b] \subset (0,\frac{1}{k})$ and $\{r_{k,n}\}, k \in \{1,2,3,\ldots,M\}$ are real sequence in $(0,\infty)$ satisfy the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,
(C2) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,
(C3) $\liminf_{n\to\infty} r_{k,n} > 0$ *and* $\lim_{n\to\infty}|r_{k,n+1} - r_{k,n}| = 0$,
(C4) $\lim_{n\to\infty}\lambda_n = 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Omega$ which is the unique solution of the variational inequality

$$\Big\langle (A-\gamma f)z, x - z \Big\rangle \ge 0, \quad \forall x \in \Omega.$$

Equivalently, we have $z = P_\Omega(I - A + \gamma f)(z)$.

Proof. Put $T_n = I$ for all $n \in \mathbb{N}$ and for all $x \in C$. Then $W_n = I$ for all $x \in C$. The conclusion follows from Theorem 3.1. This completes the proof. □

If $A = I, \gamma \equiv 1$ and $\gamma_n = 1 - \alpha_n - \beta_n$ in Theorem 3.1, then we can obtain the following result immediately.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_k, k \in \{1,2,3,\ldots,M\}$ be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4), let $\{T_n\}$ be a family of finitely nonexpansive mappings of C into itself and let B be a monotone and k-Lipschitz continuous mapping of C into H such that*

$$\Omega := \cap_{n=1}^{N} F(T_n) \cap \big(\cap_{k=1}^{M} SEP(F_k)\big) \cap VI(C,B) \neq \emptyset.$$

Let f be a contraction of H into itself with $\alpha \in (0,1)$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ u_n = J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} J_{r_{M-2,n}}^{F_{M-2}} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} x_n, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n P_C(u_n - \lambda_n B y_n), \quad \forall n \ge 1, \end{cases}$$

where $\{W_n\}$ is the sequence generated by (15)and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $(0,1)$, $\{\lambda_n\} \subset [a,b] \subset (0,\frac{1}{k})$ and $\{r_{k,n}\}, k \in \{1,2,3,\ldots,M\}$ are a real sequence in $(0,\infty)$ satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$,
(C2) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,
(C3) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,
(C4) $\liminf_{n\to\infty} r_{k,n} > 0$ *and* $\lim_{n\to\infty}|r_{k,n+1} - r_{k,n}| = 0$ *for each* $k \in \{1,2,3,\ldots,M\}$,
(C5) $\lim_{n\to\infty}\lambda_n = 0$.

(C6) $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_{n-1,i}| = 0$ *for all* $i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to a point* $z \in \Omega$ *which is the unique solution of the variational inequality*

$$\left\langle z - f(z), x - z \right\rangle \geq 0, \quad \forall x \in \Omega.$$

Equivalently, we have $z = P_\Omega f(z)$.

Acknowledgement(s) : The authors would like to thanks the "National Centre of Excellence in Mathematics", under the Commission on Higher Education (CHE), Ministry of Education, Thailand. Moreover, we would like to thank Professor Dr. Somyot Plubiteng for providing valuable suggestions.

References

[1] S. Atsushiba and W. Takahashi, Strong convergence theorems for a finite family of nonexpansive mappings and applications, *Indian J. Math.*, 41 (1999), pp. 435-453.
[2] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, The Mathematics Student. vol. 63 no. 1–4, pp. 123–145, 1994.
[3] F.E. Browder and W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, Journal of Mathematical Analysis and Applications. vol. 20, pp. 197–228, 1967.
[4] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming using proximal-like algorithms, Mathematics Programming. vol.78, pp. 29–41, 1997.
[5] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. vol.6, pp. 29–41, 2005.
[6] L. C. Ceng and J. C. Yao, An Extragradient-Like Approximation Method for Variational Inequality Problems and Fixed Point Problems, Applied Mathematics and Computation. 190, pp. 205–215, 2007.
[7] L. C. Ceng and J. C. Yao, Relaxed Viscosity Approximation Methods for Fixed Point Problems and Variational Inequality Problems, Nonlinear Analysis Series A: Theory, Methods & Applications, 69, pp. 3299–3309, 2008.
[8] L. C. Ceng, C. Lee and J. C. Yao, Strong Weak Convergence Theorems of Implicit Hybrid Steepest-Descent Methods for Variational Inequalities, Taiwanese Journal of Mathematics, 12 (2008), 227-244.
[9] L. C. Ceng, A. Petruşel and J. C. Yao, Weak convergence theorem by a modified extragradient method for nonexpansive mappings and monotone mappings. Fixed Point Theory, 9, pp. 73–88, 2008.
[10] V. Colao, G. Marino and H.K. Xu, An iterative method for finding common solutions of equilibrium and fixed point problems, Journal of Mathematical Analysis and Applications, vol.344, pp. 340–352, 2008.
[11] V. Colao, G.L. Acedo and G. Marino, An implicit method for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of infinite family of nonexpansive mappings, Nonlinear Analysis, vol.71, pp. 2708–2715, 2009.
[12] L. C. Ceng, P. Cubiotti and J. C. Yao, An Implicit Iterative Scheme for Monotone Variational Inequalities and Fixed Point Problems, Nonlinear Analysis Series A: Theory, Methods & Applications, 69, pp. 2445–2457, 2008.
[13] F. Deutsch, I. Yamada, Minimizing certain convex functions over the intersection of the fixed point set of nonexpansive mappings, Numerical Functional Analysis and Optimization, vol.19, pp. 33–56, 1998
[14] H. He, S. Liu and H. Zhou, An explicit method for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of an infinite family of nonexpansive mappings, Nonlinear Analysis:Theory, Methods and Applications, vol.72 (2010) 341–350.
[15] H. Iiduka, W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, Nonlinear Analysis:Theory, Methods and Applications, vol.61 (2005) 341–350.
[16] A. Kangtunyakarn and S. Suantai, A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mapping, Nonlinear Analysis, vol.71, (2009) 4448–4460.
[17] G.M. Korpelevich, The extragradient method for finding saddle points and other problems, E'konomika i Matematicheskie Metody, vol.12, no.4, (1976) 747–756.
[18] F. Liu and M.Z. Nashed, Regularization of nonlinear Ill-posed variational inequalities and convergence rates, Set-Valued Analysis, vol.6, pp. 313–344, 1998.
[19] G. Marino, H.K. Xu, A general iterative method for nonexpansive mapping in Hilbert spaces, Journal of Mathematical Analysis and Applications. vol.318, pp. 43–52, 2006.
[20] J.W. Peng and J.C. Yao, A viscosity approximation scheme for system of equilibrium problems, nonexpansive mappings and monotone mappings, Nonlinear Analysis. 71 (2009), 6001-6010.
[21] J.-W. Peng and J.-C. Yao, Strong convergence theorems of iterative scheme based on the extragradient method for mixed equilibrium problems and fixed point problems, *Math. and Comp. Model.*, 49 (2009), 1816-1828.

[22] J. W. Peng and J. C. Yao, A new hybrid-extragradient method for generalized mixed equilibrium problems and fixed point problems and variational inequality problems, Taiwanese Journal of Mathematics, 12 (2008), 1401-1433.
[23] J. W. Peng and J. C. Yao, Some new iterative algorithms for generalized mixed equilibrium problems with strict pseudo-contractions and monotone mappings, Taiwanese Journal of Mathematics, 2009, to appear.
[24] S. Plubtieng and R. Punpaenga, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, Journal of Mathematical Analysis and Applications, vol.336, no.1, pp. 455–469, 2007.
[25] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol.73, pp. 595–597, 1967.
[26] M.O. Osilike, D.I. Igbokwe, Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations, Comp. Math. Appl. 40:559–567(2000).
[27] R.T. Rockafellar, "On the maximality of sums of nonlinear monotone operators, Transacrions of the American Mathematical Society, vol.149, pp. 75–88, 1970.
[28] T. Suzuki, Strong convergence of krasnoselskii and manns type sequences for one-parameter nonexpansive semigroups without bochner integrals, Journal of Mathematical Analysis and Applications, vol.305, pp. 227–239, 2005.
[29] M. Shang, Y. Su and X. Qin, A General Iterative Method for Equilibrium Problems and Fixed Point Problems in Hilbert Spaces, Fixed Point Theory and Applications, vol.2007, Article ID 95412, 9 pages.
[30] W. Takahashi, Nonlinear functional analysis, Yokohama Publishers, Yokohama, 2000.
[31] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, Journal of Optimization Theory and Applications, vol. 118, pp. 417–428, 2003.
[32] H.K. Xu, Viscosity approximation methods for nonexpansive mappings, Journal of Mathematical Analysis and Applications, vol. 298, pp. 279–291, 2004.
[33] Y. Yao, Y.C. Liou, J.C. Yao, Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings, Fixed Point Theory and Applications, vol.2007, Article ID 64363, 12 pages.
[34] K. Shimoji, W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, Taiwanese Journal of Mathematics, vol.5, pp. 387–404, 2001.
[35] H.K. Xu, Iterative algorithms for nonlinear operators, Journal of the London Mathematical Society, vol.66, pp. 240–256, 2002.
[36] H.K. Xu, An iterative approach to quadratic optimization, Journal of Optimization Theory and Applications, vol.116 no. 2, pp. 659–678, 2003.
[37] I. Yamada, " The hybrid steepest descent method for the variational inequality problem of the intersection of fixed point sets of nonexpansive mappings," in: D. Butnariu, Y. Censor, S. Reich (Eds.), Inherently Parallel Algorithm for Feasibility and Optimization, Elsevier, pp. 473–504, 2001.
[38] Y. Yao, M.A. Noor and Y.-C. Liou, On iterative methods for equilibrium problems, Nonlinear Analysis 70 (2009) 497509.
[39] L. C. Zeng and J. C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, Taiwan. J. Math., 10 (2006), 1293-1303.

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