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Mean nonexpansive mappings and Suzuki-generalized nonexpansive mappings*

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ABSTRACT: We give an example of a mapping that is mean nonexpansive but not Suzuki-generalized nonexpansive, and vice versa. But in case of increasing mappings, we show that mean nonexpansiveness implies Suzuki-generalized nonexpansiveness.

KEYWORDS: Mean nonexpansive mapping; Suzuki-generalized nonexpansive mapping.

1. **Introduction**

Let C be a subset of a Banach space X. For nonnegative real numbers a and b such that $a + b \leq 1$, a mapping $T : C \to C$ is said to be (a, b)-*mean nonexpansive* if

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Ty\| \quad \text{for all } x, y \in C.$$

We also say that T is *mean nonexpansive* if T is (a, b)-mean nonexpansive for some nonnegative real numbers a and b such that $a + b \leq 1$. This type of mappings is introduced in [4] and extensively studied in [2] and [3].

In [1], T. Suzuki introduced a weaker condition of nonpansiveness which is now known as Suzuki-generalized nonexpansive. We say that T is Suzuki-generalized nonexpansive if $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$, then $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

Incidentally, examples of mean nonexpansive mappings and Suzuki-generalized nonexpansive mappings in the known literature are essentially the same. So the question naturally arises whether there exists a subset relation between the class of mean nonexpansive mappings and the class of Suzuki-generalized nonexpansive mappings.

We find the answer negative by an example of a mapping that is mean nonexpansive but not Suzuki-generalized nonexpansive, and vice versa. However we prove that in case of increasing mappings, mean nonexpansiveness implies Suzuki-generalized nonexpansiveness.

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2. A mean nonexpansive mapping that is not Suzuki-generalized nonexpansive

Lemma 2.1. *Suppose that $T : [0,5] \to [0,2]$ is a mapping defined by*

$$Tx = \begin{cases} 2 & \text{if } x \in [0,4], \\ 1 & \text{if } x \in (4,5), \\ 0 & \text{if } x = 5. \end{cases}$$

Then T is mean nonexpansive but not Suzuki-generalized nonexpansive.

Proof. Let $x = 4$ and $y = 5$. We have

$$\frac{1}{2}\|x - Tx\| = 1 = \|x - y\|.$$

But $\|Tx - Ty\| = 2 > \|x - y\|$. Thus T is not Suzuki-generalized nonexpansive.

Next we show that for each $0 \le x, y \le 5$,

$$\|Tx - Ty\| \le \frac{1}{2}\|x - y\| + \frac{1}{2}\|x - Ty\|.$$

Case 1: $x \in [0,5], y \in (4,5)$. Then

$$\begin{aligned} \|Tx - Ty\| &< \tfrac{3}{2} \\ &< \tfrac{1}{2}\|y - x + x - Ty\| \\ &= \tfrac{1}{2}\|x - y\| + \tfrac{1}{2}\|x - Ty\|. \end{aligned}$$

Case 2: $x \in [0,5], y = 5$. Then

$$\begin{aligned} \|Tx - Ty\| &< \tfrac{5}{2} \\ &= \tfrac{1}{2}\|y - x + x - Ty\| \\ &= \tfrac{1}{2}\|x - y\| + \tfrac{1}{2}\|x - Ty\|. \end{aligned}$$

Case 3: $x \in (4,5), y \in [0,4]$. Then

$$\begin{aligned} \|Tx - Ty\| &< \tfrac{3}{2} \\ &< \tfrac{1}{2}\|x - y + x - Ty\| \\ &\le \tfrac{1}{2}\|x - y\| + \tfrac{1}{2}\|x - Ty\|. \end{aligned}$$

Case 4: $x = 5, y \in [0,4]$. Then

$$\begin{aligned} \|Tx - Ty\| &< \tfrac{5}{2} \\ &\le \tfrac{1}{2}\|x - y + x - Ty\| \\ &\le \tfrac{1}{2}\|x - y\| + \tfrac{1}{2}\|x - Ty\|. \end{aligned}$$

We have $\|Tx - Ty\| = 0$ in a remaining case, so T is mean nonexpansive. □

3. A Suzuki-generalized nonexpansive mapping that is not mean nonexpansive

Lemma 3.1. *Suppose that $T : [0,11] \to [0,1]$ is a mapping defined by*

$$Tx = \begin{cases} 1 - x & \text{if } x \in [0,1], \\ 0 & \text{if } x \in (1,11) \\ 1 & \text{if } x = 11. \end{cases}$$

Then T is Suzuki-generalized nonexpansive but not mean nonexpansive.

Proof. Suppose T is mean nonexpansive. So there are nonnegative real numbers a and b such that $a + b \le 1$ and

$$\|Tx - Ty\| \le a\|x - y\| + b\|x - Ty\| \quad \text{for all } x, y \in [0,11].$$

But if $x = 0$ and $y = 1$, then

$$\begin{aligned} \|Tx - Ty\| &= 1 \\ &\leq a\|x-y\| + b\|x - Ty\| \\ &= a. \end{aligned}$$

So $a = 1$ and $b = 0$, i.e., T is nonexpansive. But this contradicts to the fact that T is not continuous. So T is not mean nonexpansive.

Next we show that T is Suzuki-generalized nonexpansive by contradiction. Suppose there are x and y such that

(1) $$\|Tx - Ty\| > \|x - y\|$$

but

(2) $$\frac{1}{2}\|x - Tx\| \leq \|x - y\|$$

or

(3) $$\frac{1}{2}\|y - Ty\| \leq \|x - y\|.$$

We may assume that $x = 11$ because T is nonexpansive on $[0, 11)$. Combining with $\|Tx - Ty\| \leq 1$ and (1), we have $y > 10$. But then

$$\begin{aligned} \tfrac{1}{2}\|x - Tx\| &= 5 \\ &> 1 \\ &\geq \|x - y\| \end{aligned}$$

and

$$\begin{aligned} \tfrac{1}{2}\|y - Ty\| &\geq \tfrac{9}{2} \\ &> 1 \\ &\geq \|x - y\| \end{aligned}$$

which contradict to (2) and (3). Thus T is Suzuki-generalized nonexpansive □

4. Increasing mean nonexpansive mapping is Suzuki-generalized nonexpansive

Lemma 4.1. *If T is an increasing mean nonexpansive mapping, then T is Suzuki-generalized nonexpansive.*

Proof. Let T be an increasing mean nonexpansive mapping.
Let $y < x$. We show that $\|Tx - Ty\| \leq \|x - y\|$ if

(4) $$\frac{1}{2}\|x - Tx\| \leq \|x - y\|$$

or

(5) $$\frac{1}{2}\|y - Ty\| \leq \|x - y\|.$$

We may assume

(6) $$\|Tx - Ty\| < \min\{\|Tx - y\|, \|x - Ty\|\}.$$

Otherwise $\|Tx - Ty\| \leq \|x - y\|$ by mean nonexpansive condition of T.

Case 1: $Ty \leq Tx \leq y \leq x$.

Suppose x, y satisfy (4), we have

$$\|x - y\| + \|y - Tx\| = \|x - Tx\| \leq 2\|x - y\|.$$

So

$$\|y - Tx\| \leq \|x - y\|.$$

From (6), we have
$$\|Tx - Ty\| < \|Tx - y\|.$$
Thus
$$\|Tx - Ty\| < \|x - y\|.$$
Suppose x, y satisfy (5). Then
$$\|y - Tx\| + \|Tx - Ty\| = \|y - Ty\| \leq 2\|x - y\|.$$
So
$$\|Tx - Ty\| \leq \|x - y\| \quad \text{or} \quad \|y - Tx\| \leq \|x - y\|.$$
Thus $\|Tx - Ty\| \leq \|x - y\|$ immediately or by (6).

Case 2: $Ty \leq y \leq Tx \leq x$.
This case does not satisfy $\|Tx - Ty\| < \|Tx - y\|$ in (6). Thus the case is impossible.

Case 3: $y \leq Ty \leq Tx \leq x$.
We have $\|Tx - Ty\| \leq \|x - y\|$ immediately.

Case 4:$y \leq Ty \leq x \leq Tx$.
This case does not satisfy $\|Tx - Ty\| < \|x - Ty\|$ in (6). Thus the case is impossible.

Case 5:$y \leq x \leq Ty \leq Tx$.

Suppose x, y satisfy (4). We have $\|Tx - Ty\| < \|x - Ty\|$ by (6), then
$$\begin{aligned} \|x - y\| &\geq \tfrac{1}{2}\|Tx - x\| \\ &= \tfrac{1}{2}\|Tx - Ty\| + \tfrac{1}{2}\|Ty - x\| \\ &> \tfrac{1}{2}\|Tx - Ty\| + \tfrac{1}{2}\|Tx - Ty\| \\ &= \|Tx - Ty\|. \end{aligned}$$
Suppose x, y satisfy (5). Then
$$\|Ty - x\| + \|x - y\| = \|Ty - y\| \leq 2\|x - y\|.$$
So
$$\|Ty - x\| \leq \|x - y\|.$$
From (6), we have
$$\|Tx - Ty\| < \|Ty - x\|.$$
Thus $\|Tx - Ty\| < \|x - y\|$. This completes the proof.

□

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