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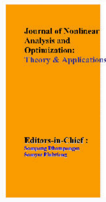

A new hybrid method for solving generalized equilibrium problem and common fixed points of asymptotically quasi nonexpansive mappings in Banach spaces*

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ABSTRACT: In this paper, we introduce a new hybrid projection iterative scheme based on the shrinking projection method for two asymptotically quasi-ϕ-nonexpansive mappings, for finding a common element of the set of solutions of the generalized mixed equilibrium problems and the set of common fixed points of two asymptotically quasi-ϕ-nonexpansive mappings in Banach spaces. The results obtained in this paper improve and extend the recent ones announced by Matsushita and Takahashi [S. Matsushita, W. Takahashi, Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces, Fixed Point Theory Appl. (2004), 2004, 37-47], Qin et al. [X. Qin, S.Y. Cho, S.M Kang, On hybrid projection methods for asymptotically quasi-ϕ-nonexpansive mappings, Applied Mathematics and Computation 215 (2010) 38743883], and Chang, Lee and Chan [S.-s. Chang, H.W. Joseph Lee, C.K. Chan, A new hybrid method for solving generalized equilibrium problem variational inequality and common fixed point in banach spaces with applications, Nonlinear Analysis (2010), doi:10.1016/j.na.2010.06.006] and many others.

KEYWORDS: Generalized mixed equilibrium problem, Asymptotically quasi-ϕ-nonexpansive mapping, Strong convergence theorem, and Banach space.

1. **Introduction**

Let E be a real Banach space, and E^* the dual space of E. Let C be a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R}$ be a real-valued function, and $A : C \to E^*$ be a nonlinear mapping. The generalized mixed equilibrium problem, is to find $x \in C$ such that

(1) $$f(x,y) + \langle Ax, y-x \rangle + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C.$$

The set of solutions to (1) is denoted by EP, i.e.,

(2) $$f(x,y) + \langle Ax, y-x \rangle + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C.$$

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If $\varphi = 0$, the problem (1) reduces to the generalized equilibrium problem for f, denoted by $GEP(f)$, which is to find $x \in C$ such that

$$f(x,y) + \langle Ax, y - x\rangle \geq 0, \ \forall y \in C. \tag{3}$$

If $A = 0$, the problem (1) reduces to the mixed equilibrium problem for f, denoted by $MEP(f, \varphi)$, which is to find $x \in C$ such that

$$f(x,y) + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C. \tag{4}$$

If $f \equiv 0$, the problem (1) reduces to the mixed variational inequality of Browder type, denoted by $VI(C, A, \varphi)$, which is to find $x \in C$ such that

$$\langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C. \tag{5}$$

If $A = 0$ and $\varphi = 0$, the problem (1) reduces to the equilibrium problem for f, denoted by $EP(f)$, which is to find $x \in C$ such that

$$f(x,y) \geq 0, \ \forall y \in C. \tag{6}$$

Let $f(x,y) = \langle Ax, y - x\rangle$ for all $x, y \in C$. Then $p \in EP(f)$ if and only if $\langle Ap, y - p\rangle \geq 0$ for all $y \in C$, i.e., p is a solution of the variational inequality; there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an EP. In other words, the EP is a unifying model for several problems arising in physical sciences. In the last two decades, many papers have appeared in the literature on the existence of solutions of the EP; see, for example [5, 17, 19, 20] and references therein. Some solution methods have been proposed to solve the EP; see, for example [9, 10, 15, 16, 22, 24, 30, 32, 34] and references therein.

Recall that a mappings $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C.$$

T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\|, \text{ for all } x \in C, y \in F(T).$$

T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \text{ for all } x, y \in C.$$

T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - y\| \leq k_n \|x - y\|, \text{ for all } x \in C, y \in F(T).$$

T is called uniformly L-Lipschitzian continuous if there exists a $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \text{ for all } x, y \in C.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [18] in 1972. Since 1972, a host of authors have studied the weak and strong convergence of iterative processes for such a class of mappings.

If C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is a nonexpansive mapping. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [3] recently introduced a generalized projection operator C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Consider the functional $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 \tag{7}$$

for all $x, y \in E$, where J is the normalized duality mapping from E to E^*. Observe that, in a Hilbert space H, (7) reduces to $\phi(y, x) = \|x - y\|^2$ for all $x, y \in H$. The generalized projection

$\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = x^*$, where x^* is the solution to the minimization problem:

$$\phi(x^*, x) = \inf_{y \in C} \phi(y, x). \tag{8}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J (see, for example, [1, 2, 9, 28]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

(1) $(\|y\| - \|x\|)^2 \leqslant \phi(y, x) \leqslant (\|y\| + \|x\|)^2$ for all $x, y \in E$.
(2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$ for all $x, y, z \in E$.
(3) $\phi(x, y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leqslant \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$ for all $x, y \in E$.
(4) If E is a reflexive, strictly convex and smooth Banach space, then, for all $x, y \in E$,

$$\phi(x, y) = 0 \text{ if and only if } x = y.$$

For more detail see [14, 31]. Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed point of T. A point p in C is said to be an *asymptotic fixed point* of T [29] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. Recall that the following :

(i) A mapping $T : C \to C$ is called *relatively nonexpansive* [7, 8, 11] if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leqslant \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.
The asymptotic behavior of relatively nonexpansive mappings were studied in [7, 8].
(ii) $T : C \to C$ is said to be *relatively asymptotically nonexpansive* [1, 28] if $\hat{F}(T) = F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p, T^n x) \leqslant k_n \phi(p, x)$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$.
(iii) $T : C \to C$ is said to be *ϕ-nonexpansive* [26, 36] if $\phi(Tx, Ty) \leq \phi(x, y)$ for all $x, y \in C$.
(iv) $T : C \to C$ is said to be *quasi-ϕ-nonexpansive* [26, 36] if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.
(v) $T : C \to C$ is said to be *asymptotically ϕ-nonexpansive* [36] if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(T^n x, T^n y) \leq k_n \phi(x, y)$ for all $x, y \in C$.
(vi) $T : C \to C$ is said to be *asymptotically quasi-ϕ-nonexpansive* [36] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p, T^n x) \leq k_n \phi(p, x)$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$.
(vii) $T : C \to C$ is said to be *asymptotically regular* on C if, for any bounded subset D of C, there holds the following equality :

$$\lim_{n\to\infty} \sup_{x \in D} \|T^{n+1}x - T^n x\| = 0.$$

(viii) $T : C \to C$ is said to be *closed* if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$.

Remark 1.1. The class of (asymptotically) quasi-ϕ-nonexpansive mappings is more general than the class of relatively (asymptotically) nonexpansive mappings which requires the strong restriction $\hat{F}(T) = F(T)$.

Remark 1.2. In real Hilbert spaces, the class of (asymptotically) quasi-ϕ-nonexpansive mappings is reduced to the class of (asymptotically) quasi-nonexpansive mappings.

We give some examples which are closed and asymptotically quasi-ϕ-nonexpansive.

Example 1.3. (1). Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ be a maximal monotone mapping such that its zero set $A^{-1}0$ is nonempty. Then $J_r = (J + rA)^{-1}J$ is a closed and asymptotically quasi-ϕ-nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

(2). Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed and convex subset C of E. Then Π_C is a closed and asymptotically quasi-ϕ-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

Recently, Matsushita and Takahashi [25] obtained the following results in a Banach space.
Theorem MT. Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty} < 1$. Suppose that $\{x_n\}$ is given by

$$(9) \qquad \begin{cases} x_0 = x \in C \text{ chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \ n = 0, 1, 2, \ldots, \end{cases}$$

where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the generalized projection from C onto $F(T)$.
Recently, Qin et al. [27] further extended Theorem MT by considering a pair of asymptotically quasi-ϕ-nonexpansive mappings. To be more precise, they proved the following results.

Theorem QCK. Let E be a uniformly smooth and uniformly convex Banach space and C a nonempty closed and convex subset of E. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(t)}\} \subset [1, \infty)$ such that $k_n^{(t)} \to 1$ as $n \to \infty$ and $S : C \to C$ a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_s^{(t)}\} \subset [1, \infty)$ such that $k_n^{(s)} \to 1$ as $n \to \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be real number sequences in $[0, 1]$. Assume that T and S are uniformly asymptotically regular on C and $\Omega = F(T) \cap F(S)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:

$$(10) \qquad \begin{cases} x_0 \in E \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ z_n = J^{-1}(\beta_n Jx_n + \gamma_n J(T^n x_n) + \delta_n J(S^n x_n)), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n), \\ C_{n+1} = \{w \in C_n : \phi(w, y_n) \leq \phi(w, x_n) + (k_n - 1)M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $k_n = \max\{k_n^{(t)}, k_n^{(s)}\}$ for each $n \geq 1$, J is the duality mapping on E, $M_n = \sup\{\phi(z, x_n) : z \in \Omega\}$ for each $n \geq 1$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ satisfy the following restrictions :

(a) $\beta_n + \gamma_n + \delta_n = 1, \ \forall n \geq 1$;
(b) $\liminf_{n\to\infty} \gamma_n \delta_n, \ \lim_{n\to\infty} \beta_n = 0$;
(c) $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$.

On the other hand, very recently, Chang, Lee and Chan [12] proved a strong convergence theorem for finding a common element of the set of solutions for a generalized equilibrium problem (3) and the set of common fixed points for a pair of relatively nonexpansive mappings in Banach spaces. They proved the following results.

Theorem CLC. Let E be a uniformly smooth and uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a α-inverse-strongly monotone mapping and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A1) - (A4)$. Let $S, T : C \to C$ be two relatively nonexpansive mappings such that $\Omega := F(T) \cap F(S) \cap GEP(f)$.

Let $\{x_n\}$ be the sequence generated by

$$(11)\qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JSx_n), \\ u_n \in C \textit{ such that} \\ \quad f(u_n, y) + \langle Au_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, Ju_n - Jy_n\rangle \geq 0,\ \forall y \in C, \\ H_n = \{v \in C : \phi(v,u_n) \leq \beta_n\phi(v,x_n) + (1-\beta_n)\phi(v,x_n)\}; \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}; \\ x_{n+1} = \Pi_{H_n\cap W_n}x_0,\ \forall n \geq 0, \end{cases}$$

where $J : E \to E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and $\{r_n\} \subset [a,1)$ for some $a > 0$. If the following conditions are satisfied:

(a) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(b) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;

Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is the generalized projection of E onto Ω.

In this paper, motivated and inspired by the work of Matsushita and Takahashi [25], Qin et al. [27], and Chang, Lee and Chan [12], we introduce a new hybrid projection iterative scheme based on the shrinking projection method for two asymptotically quasi-ϕ-nonexpansive mappings, for finding a common element of the set of solutions of the generalized mixed equilibrium problems and the set of common fixed points of two asymptotically quasi-ϕ-nonexpansive mappings in Banach spaces. The results obtained in this paper improve and extend the recent ones announced by Matsushita and Takahashi [25], Qin et al. [27], and Chang, Lee and Chan [12] and many others.

2. **Preliminaries**

For the sake of convenience, we first recall some definitions and conclusions which will be needed in proving our main results. The mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\},\ x \in E.$$

is called the normalized duality mapping. By the Hahn-Banach theorem, $J(x) \neq \emptyset$ for each $x \in E$.

In the sequel, we denote the strong convergence, weak convergence and weak* convergence of a sequence $\{x_n\}$ by $x_n \to x$, $x_n \rightharpoonup x$ and $x_n \rightharpoonup^* x$, respectively.

A Banach space E is said to be strictly convex, if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. It is said to be uniformly convex, if for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} \leq 1-\delta$ for all $x, y \in U$ with $\|x-y\| \geq \epsilon$.

It is well-known that a uniformly convex Banach space has the Kadec-Klee property, i.e., if $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

The space E is said to be smooth, if the limit

$$(12)\qquad \lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t}$$

exists for all $x, y \in U$. And E is said to be uniformly smooth, if the limit (12) exists uniformly in $x, y \in U$.

Remark 2.1. It is wellknown that if E is a smooth, strictly convex and reflexive Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is single-valued, one-to-one and onto (see [14]).

Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Throughout this paper the Lyapunov functional $\phi : E \times E \to \mathbb{R}^+$ is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,\ \forall x,y \in E. \tag{13}$$

Following Alber [4], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = argmin_{y\in C}\phi(y,x),\ \forall x \in E. \tag{14}$$

If E is a real Hilbert space H, then $\phi(x,y) = \|x-y\|^2$ and Π_C is the metric projection of H onto C.

In order to our main results, we need the following concepts and lemmas.

Let E be a real Banach space, C a nonempty subset of E and $T : C \to C$ a nonlinear mapping. The mapping T is said to be uniformly asymptotically regular on C if

$$\lim_{n\to\infty}\left(\sup_{x\in C}\|T^{n+1}x - T^n x\|\right) = 0.$$

The mapping T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$.

Lemma 2.2. ([2, 4, 23]) *Let E be a reflexive, strictly convex and smooth Banach space, C a nonempty closed convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y,x),\ \forall y \in C.$$

Lemma 2.3. ([4, 23]) *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset. Then the following conclusion hold:*

(1) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x,y);\ \forall x \in C,\ y \in E;$
(2) *Let $x \in E$ and $z \in C$, then*

$$z = \Pi_C x \Leftrightarrow \langle z - y, Jx - Jz\rangle,\ \forall y \in C.$$

Lemma 2.4. ([13]) *Let E be a uniformly convex Banach space and $B_r(0)$ a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \le \|\alpha x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|x-y\|)$$

for all $x, y \in B_r(0)$ and $\alpha \in [0,1]$.

Lemma 2.5. ([27]) *Let E be a uniformly convex and smooth Banach space, C a nonempty closed convex subset of E and $T : C \to C$ a closed asymptotically quasi-ϕ-nonexpansive mapping. Then $F(T)$ is a closed convex subset of C.*

Lemma 2.6. ([23]) *Let E be a smooth and uniformly convex Banach space. Let x_n and y_n be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $lim_{n\to\infty}\phi(x_n, y_n) = 0$, then $lim_{n\to\infty}\|x_n - y_n\| = 0$.*

For solving the generalized equilibrium problem, let us assume that the nonlinear mapping $A : C \to E^*$ is α-inverse strongly monotone and the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x,x) = 0\ \forall x \in C$;
(A2) f is monotone, i.e., $f(x,y) + f(y,x) \le 0,\ \forall x,y \in C$;
(A3) $\limsup_{t\downarrow 0} f(x + t(z-x), y) \le f(x,y),\ \forall x,y,z \in C$;
(A4) the function $y \mapsto f(x,y)$ is convex and lower semicontinuous.

Lemma 2.7. ([5]) *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A1)-(A4)$. Let $r > 0$ and $x \in E$, then there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0,\ \forall y \in C. \tag{15}$$

Lemma 2.8. ([33]) *Let C be a closed convex subset of a uniform smooth, strictly convex and reflexive Banach space E and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$. For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0,\ \forall y \in C \right\},$$

for all $x \in C$. Then, the following conclusions holds:

(1) *T_r is single-valued ;*
(2) *T_r is a firmly nonexpansive-type mapping, i.e.;*

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle,\ \forall x, y \in E;$$

(A3) *$F(T_r) = EP(f)$;*
(A4) *$EP(f)$ is a closed convex.*

Lemma 2.9. ([34]) *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$ and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r(x), x) \leq \phi(q, x).$$

Lemma 2.10. ([35]) *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $A : C \to E^*$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function, and f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$. For $r > 0$ and $x \in E$, then there exists $u \in C$ such that*

$$f(u,y) + \langle Au, y - u \rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx \rangle,\ \forall y \in C.$$

Define a mapping $K_r : C \to C$ as follows:

$$K_r(x) = \left\{ u \in C : f(u,y) + \langle Au, y - u \rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx \rangle \geq 0,\ \forall y \in C \right\} \tag{16}$$

for all $x \in C$. Then, the following conclusions holds:

(a) K_r is single-valued ;
(b) K_r is a firmly nonexpansive-type mapping, *i.e.;*

$$\langle K_r x - K_r y, JK_r x - JK_r y \rangle \leq \langle K_r x - K_r y, Jx - Jy \rangle,\ \forall x, y \in E;$$

(c) $F(K_r) = \hat{F}(K_r) = EP$;
(d) EP is a closed convex,
(e) $\phi(p, K_r z) + \phi(K_r z, z) \leq \phi(p, x),\ \forall p \in F(K_r),\ z \in E.$

Remark 2.11. ([35]) It follows from Lemma 2.10 that the mapping $K_r : C \to C$ defined by (16) is a relatively nonexpansive mapping. Thus, it is quasi-ϕ-nonexpansive.

3. Main Results

In this section, we shall prove a strong convergence theorem for finding a common element of the set of solutions for a generalized mixed equilibrium problem (1) and the set of common fixed points for a pair of asymptotically quasi-ϕ-nonexpansive mapping mappings in Banach spaces.

Theorem 3.1. *Let E be a uniformly smooth and uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a continuous and monotone mapping and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A1)$ - $(A4)$, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(t)}\} \subset [1, \infty)$ such that $k_n^{(t)} \to 1$ as $n \to \infty$ and $S : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(s)}\} \subset [1, \infty)$ such that $k_n^{(s)} \to 1$ as $n \to \infty$.*

Assume that T and S are uniformly asymptotically regular on C and $\Omega = F(T) \cap F(S) \cap EP \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by

$$(17) \quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n x_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JS^n z_n), \\ f(u_n, y) + \langle Au_n, y-u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y-u_n, Ju_n - Jy_n\rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n\phi(v, x_n) + (1-\beta_n)k_n\phi(v, z_n) \leq \phi(v, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ \forall n \geq 1, \end{cases}$$

where $\theta_n = (1-\beta_n)(k_n^2 - 1)M_n \to 0$ as $n \to \infty$, $k_n = \max\{k_n^{(t)}, k_n^{(s)}\}$ for each $n \geq 1$, $M_n = \sup\{\phi(v, x_n) : v \in \Omega\}$ for each $n \geq 1$, $J : E \to E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Suppose that the following conditions are satisfied:

(i) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(ii) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega x_1$, where Π_Ω is generalized projection of E onto Ω.

Proof. First, we define two bifunctions $H : C \times C \to \mathbb{R}$ and $K_r : C \to C$ by

$$(18) \qquad H(x, y) = f(x, y) + \langle Ax, y-x\rangle + \varphi(y) - \varphi(x), \quad \forall x, y \in C,$$

and

$$(19) \qquad K_r(x) = \{u \in C : H(u, y) + \frac{1}{r}\langle y-u, Ju - Jx\rangle \geq 0, \ \forall y \in C\}.$$

By Lemma 2.10, we know that the function H satisfies conditions (A1) - (A4) and K_r has the properties (a)-(e). Therefore, (17) is equivalent to

$$(20) \quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n x_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JS^n z_n), \\ H(u_n, y) + \frac{1}{r_n}\langle y-u_n, Ju_n - Jy_n\rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n\phi(v, x_n) + (1-\beta_n)k_n\phi(v, z_n) \leq \phi(v, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ \forall n \geq 1, \end{cases}$$

(I) We show first that the sequence $\{x_n\}$ is well defined. By the same argument as in the proof of [36, Lemma 2.4], one can show that $F(T) \cap F(S)$ is closed and convex. Hence $\Omega := F(S) \cap F(T) \cap EP$ is a nonempty, closed and convex subset of C. Consequently, Π_Ω is well defined. Next, we prove that C_n is closed and convex for each $n \geq 1$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_h is closed and convex for some positive integer h. Next, we prove that C_{h+1} is closed and convex. For $w \in C_{h+1}$, we see that

$$\phi(w, u_h) \leq \beta_h\phi(w, x_n) + (1-\beta_h)k_h\phi(w, z_h)$$

is equivalent to

$$\begin{aligned} 2\langle w, (1-\beta_h)Jz_h + \beta_h Jx_h - Ju_h\rangle \quad \leq \quad & (1-\beta_h)k_h\|z_h\|^2 - \|u_h\|^2 + \beta_h\|x_h\|^2 \\ & + (\beta_h + k_h - \beta_h k_h - 1)\|w\|^2, \end{aligned}$$

and

$$\beta_h\phi(w, x_h) + (1-\beta_h)k_h\phi(w, z_h) \leq \phi(w, x_h) + \theta_n$$

is equivalent to

$$\phi(w, z_h) \leq \phi(w, x_h) + (k_h^2 - 1)M_h.$$

It is easy to see that C_{h+1} is closed and convex. Then, for each $n \geq 1$, we see C_n is closed and convex.

(II) Next we prove that $\Omega \subset C_n$ for each $n \geq 1$.

If $n = 1$, $\Omega \subset C_1 = C$ is obvious. Suppose that $\Omega \subset C_h$ for some positive integer h. Next, we claim that $\Omega \subset C_{h+1}$ for the same h. For every $w \in \Omega$, we obtain from the assumption that $w \in C_h$: On the other hand, we have

$$\begin{aligned}
\phi(w, z_h) &= \phi(w, J^{-1}(\alpha_h J x_h + (1-\alpha_h) J T^h x_h)) \\
&= \|w\|^2 - 2\langle w, \alpha_h J x_h + (1-\alpha_h) J T^h x_h\rangle + \|\alpha_h J x_h + (1-\alpha_h) J T^h x_h\|^2 \\
&\leq \|w\|^2 - 2\alpha_h \langle w, J x_h\rangle - 2(1-\alpha_h)\langle w, J T^h x_h\rangle \\
&\quad + \alpha_h \|x_h\|^2 + (1-\alpha_h)\|T^h x_h\|^2 \\
&= \alpha_h \phi(w, x_h) + (1-\alpha_h)\phi(w, T^h x_h) \\
&\leq \alpha_h \phi(w, x_h) + (1-\alpha_h) k_h^{(t)} \phi(w, x_h) \\
&\leq \alpha_n \phi(w, x_h) + (1-\alpha_h) k_h \phi(w, x_h) \\
&\leq \phi(w, x_h) + (k_h - 1)\phi(w, x_h). \qquad (21)
\end{aligned}$$

It follows that

$$\begin{aligned}
\phi(w, u_h) &= \phi(w, K_{r_h} y_h) \leq \phi(w, y_h) \\
&\leq \phi(w, J^{-1}(\beta_h J x_h + (1-\beta_h) J S^h z_h)) \\
&= \|w\|^2 - 2\langle w, \beta_h J x_h + (1-\beta_h) J S^h z_h\rangle + \|\beta_h J x_h + (1-\beta_h) J S^h z_h\|^2 \\
&\leq \|w\|^2 - 2\beta_h \langle w, J x_h\rangle - 2(1-\beta_h)\langle w, J S^h z_h\rangle + \beta_h \|x_h\|^2 + (1-\beta_h)\|S^h z_h\|^2 \\
&= \beta_h \phi(w, x_h) + (1-\beta_h)\phi(w, S^h z_h) \\
&\leq \beta_h \phi(w, x_h) + (1-\beta_h) k_h^{(s)} \phi(w, z_h) \\
&\leq \beta_h \phi(w, x_h) + (1-\beta_h) k_h \phi(w, z_h) \\
&= (1-(1-\beta_n))\phi(w, x_h) + (1-\beta_h) k_h \phi(w, z_h) \\
&= \phi(w, x_h) + (1-\beta_h)[k_h \phi(w, z_h) - \phi(w, x_h)] \\
&\leq \phi(w, x_h) + (1-\beta_h)[k_h(\phi(w, x_h) + (k_h - 1)\phi(w, x_h)) - \phi(w, x_h)] \\
&= \phi(w, x_h) + (1-\beta_h)[k_h \phi(w, x_h) + (k_h^2 - k_h)\phi(w, x_h) - \phi(w, x_h))] \\
&= \phi(w, x_h) + (1-\beta_h)(k_h^2 - 1)\phi(w, x_h) \\
&\leq \phi(w, x_h) + (1-\beta_h)(k_h^2 - 1)M_h \\
&= \phi(w, x_h) + \theta_n. \qquad (22)
\end{aligned}$$

This shows that $w \in C_{h+1}$. This implies that $\Omega \subset C_n$ for each $n \geq 1$.

From $x_n = \Pi_{C_n} x_1$, we see that

$$\langle x_n - w, J x_1 - J x_n\rangle \geq 0, \ \forall w \in C_n.$$

Since $\Omega \subset C_n$ for each $n \geq 1$, we arrive at

$$\langle x_n - w, J x_1 - J x_n\rangle \geq 0, \ \forall w \in \Omega. \qquad (23)$$

(III) Now we prove that $\{x_n\}$ is bounded.

In view of Lemma 2.2, we see that

$$\phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1) \leq \phi(w, x_1) - \phi(w, x_n) \leq \phi(w, x_1),$$

for each $w \in C_n$. Therefore, we obtain that the sequence $\phi(x_n, x_1)$ is bounded, so are $\{x_n\}$, $\{y_n\}$, $\{T^n x_n\}$, $\{S^n x_n\}$ and $\{z_n\}$.

(IV) Now we prove that $\|x_n - T^n x_n\| \to 0$ and $\|z_n - S^n z_n\| \to 0$.
Since $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_1) \le \phi(x_{n+1}, x_1),\ \forall n \ge 1.$$

This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing, and so the limit $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. By the construction of C_n, we have

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_1) \le \phi(x_m, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\ &= \phi(x_m, x_1) - \phi(x_n, x_1). \end{aligned} \tag{24}$$

Letting $m, n \to \infty$ in (24), we see that $\phi(x_m, x_n) \to 0$. It follows from Lemma 2.6 that $x_m - x_n \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since E a Banach space and C is closed and convex, we can assume that

$$\lim_{n\to\infty} x_n = p \in C. \tag{25}$$

Now, we are in a position to state that $p \in \Omega = F(T) \cap F(S) \cap EP$. By taking $m = n+1$ in (24), we obtain that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{26}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1}$, from definition of C_{n+1} we have

$$\phi(x_{n+1}, u_n) \le \phi(x_{n+1}, x_n) + \theta_n,\ \forall n \ge 1, \tag{27}$$

and

$$k_n \phi(x_{n+1}, z_n) \le \phi(x_{n+1}, x_n) + (k_n^2 - 1) M_n,\ \forall n \ge 1. \tag{28}$$

Since E is uniformly smooth and uniformly convex, from (26)-(28), $\theta_n \to 0$ as $n \to \infty$ and Lemma 2.6, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - u_n\| = \lim_{n\to\infty} \|x_{n+1} - z_n\|, \tag{29}$$

and so

$$\lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|x_n - z_n\| = \lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{30}$$

Since $u_n = K_{r_n} y_n$, from (22) we have

$$\phi(u, y_n) \le \phi(u, x_n) + \theta_n,\ \forall u \in \Omega. \tag{31}$$

Since $\|x_n - u_n\| \to 0$ and J is uniformly continuous, we have

$$\begin{aligned} \phi(u_n, y_n) &= \phi(K_{r_n} y_n, y_n) \\ &\le \phi(u, y_n) - \phi(u, K_{r_n} y_n) \\ &\le \phi(u, x_n) - \phi(u, K_{r_n} y_n) + \theta_n \\ &= \phi(u, x_n) - \phi(u, u_n) + \theta_n \\ &= \|x_n\|^2 - \|u_n\|^2 - 2\langle u, Jx_n - Ju_n\rangle + \theta_n \\ &\le \|x_n - u_n\| \times (\|x_n\| + \|u_n\|) - 2\langle u, Jx_n - Ju_n\rangle + \theta_n \to 0. \end{aligned}$$

This implies that $\phi(y_n, u_n) \to 0$. Since E is smooth and uniformly convex, from Lemma 2.6, we have

$$\|y_n - u_n\| \to 0, \text{ and so } \|y_n - x_n\| \to 0. \tag{32}$$

From (17), we have

$$\|Jy_n - Jx_n\| = (1 - \beta_n)\|JS^n z_n - Jx_n\| \to 0, \tag{33}$$

and so $\|S^n z_n - x_n\| \to 0$. This together with $\|x_n - z_n\| \to 0$ yields

$$\|z_n - S^n z_n\| \to 0. \tag{34}$$

Again from (30) and (17) we have

$$\|Jz_n - Jx_n\| = (1 - \alpha_n)\|JT^n x_n - Jx_n\| \to 0. \tag{35}$$

This implies that $\|JT^nx_n - Jx_n\| \to 0$, and so

$$\|T^nx_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{36}$$

(V) Now we prove that $p \in F(T) \cap F(S) \cap EP = \Omega$.
From (36) and (30), we have

$$\lim_{n\to\infty} \|T^nx_n - z_n\| = 0. \tag{37}$$

Note that

$$\|T^nx_n - p\| \le \|T^nx_n - z_n\| + \|z_n - x_n\| + \|x_n - p\|. \tag{38}$$

It follows from (37), (30) and (25) that

$$\lim_{n\to\infty} \|T^nx_n - p\| = 0. \tag{39}$$

On other hand, we have

$$\|T^nx_n - p\| \le \|T^{n+1}x_n - T^nx_n\| + \|T^nx_n - p\|.$$

Since T is uniformly asymptotically regular and (39), we obtain that

$$\|T^{n+1}x_n - p\| = 0. \tag{40}$$

that is, $TT^nx_n \to p$ as $n \to \infty$. From the closedness of T, we see that $p \in F(T)$.
From (34) and (30), we have

$$\lim_{n\to\infty} \|S^nx_n - z_n\| = 0. \tag{41}$$

Note that

$$\|S^nx_n - p\| \le \|S^nx_n - z_n\| + \|z_n - x_n\| + \|x_n - p\|. \tag{42}$$

It follows from (41), (30) and (25) that

$$\lim_{n\to\infty} \|S^nx_n - p\| = 0. \tag{43}$$

On other hand, we have

$$\|S^nx_n - p\| \le \|S^{n+1}x_n - S^nx_n\| + \|S^nx_n - p\|.$$

Since S is uniformly asymptotically regular and (43), we obtain that

$$\|S^{n+1}x_n - p\| = 0. \tag{44}$$

that is, $SS^nx_n \to p$ as $n \to \infty$. From the closedness of S, we see that $p \in F(S)$.

Next we prove that $p \in EP$. Since $\lim_{n\to\infty} \|u_n - y_n\| = 0$ and J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0. \tag{45}$$

From the assumption $r_n > a$, we obtain

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0. \tag{46}$$

Noticing that $u_n = K_{r_n}y_n$, we have

$$H(u_n, y) + \frac{1}{r_n}\langle y - u_n, ju_n - Jy_n\rangle \ge 0, \ \forall y \in C. \tag{47}$$

From $(A2)$, we note that

$$\|y - u_n\|\frac{\|Ju_n - Jy_n\|}{r_n} \ge \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge -H(u_n, y) \ge H(y, u_n), \ \forall y \in C. \tag{48}$$

Taking the limit as $n \to \infty$ in the above inequality and from $(A4)$ and $u_n \to p$, we have $H(y, p) \le 0, \forall y \in C$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)p$. Noticing that $y, p \in C$, we obtain $y_t \in C$, which yields that $H(y_t, p) \le 0$. It follows from $(A1)$ that

$$0 = H(y_t, y_t) \le tH(y_t, y) + (1-t)H(y_t, p) \le tH(y_t, y),$$

that is, $H(y_t, y) \geq 0$.

Let $t \downarrow 0$; from $(A3)$, we obtain $H(p, y) \geq 0$, $\forall y \in C$. Therefore $p \in EP$, and so $p \in \Omega$.

(VI) Finally, we prove that $p = \Pi_\Omega x_1$. Taking the limit as $n \to \infty$ in (23), we obtain that

$$\langle p - z, Jx_1 - Jp \rangle \geq 0, \ \forall z \in \Omega$$

and hence $p = \Pi_\Omega x_1$ by Lemma 2.3. This complete the proof. □

The following Theorems can be obtained from Theorem 3.1 immediately.

Corollary 3.2. *Let E be a uniformly smooth and uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A1)$ - $(A4)$, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(t)}\} \subset [1, \infty)$ such that $k_n^{(t)} \to 1$ as $n \to \infty$ and $S : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(s)}\} \subset [1, \infty)$ such that $k_n^{(s)} \to 1$ as $n \to \infty$. Assume that T and S are uniformly asymptotically regular on C and $\Omega = F(T) \cap F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$(49) \quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JT^n x_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n) JS^n z_n), \\ f(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, Ju_n - Jx \rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1-\beta_n) k_n \phi(v, z_n) \leq \phi(v, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ \forall n \geq 1, \end{cases}$$

where $\theta_n = (1-\beta_n)(k_n^2 - 1)M_n \to 0$ as $n \to \infty$, $k_n = \max\{k_n^{(t)}, k_n^{(s)}\}$ for each $n \geq 1$, $M_n = \sup\{\phi(v, x_n) : v \in \Omega\}$ for each $n \geq 1$, $J : E \to E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If the following conditions are satisfied:

(i) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(ii) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(iii) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_1$, where Π_Ω is generalized projection of E onto Ω.

Proof. Putting $A = 0$ in Theorem 3.1, the conclusion of Theorem 3.2 can be obtained. □

Corollary 3.3. *Let E be a uniformly smooth and uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a continuous and monotone mapping and $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(t)}\} \subset [1, \infty)$ such that $k_n^{(t)} \to 1$ as $n \to \infty$ and $S : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(s)}\} \subset [1, \infty)$ such that $k_n^{(s)} \to 1$ as $n \to \infty$. Assume that T and S are uniformly asymptotically regular on C and $\Omega = F(T) \cap F(S) \cap VI(C, A, \varphi) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$(50) \quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JT^n x_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n) JS^n z_n), \\ \langle Au_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, Ju_n - Jx \rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1-\beta_n) k_n \phi(v, z_n) \leq \phi(v, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ \forall n \geq 1, \end{cases}$$

where $\theta_n = (1-\beta_n)(k_n^2-1)M_n \to 0$ as $n \to \infty$, $k_n = \max\{k_n^{(t)}, k_n^{(s)}\}$ for each $n \geq 1$, $M_n = \sup\{\phi(v, x_n) : v \in \Omega\}$ for each $n \geq 1$, $J : E \to E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If the following conditions are satisfied:

(i) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(ii) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(iii) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_1$, where Π_Ω is generalized projection of E onto Ω.

Proof. Putting $f = 0$ in Theorem 3.1, the conclusion of Theorem 3.3 can be obtained. □

Corollary 3.4. *Let E be a uniformly smooth and uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a continuous and monotone mapping and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A1)$ - $(A4)$, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $S : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(s)}\} \subset [1, \infty)$ such that $k_n^{(s)} \to 1$ as $n \to \infty$. Assume that S is uniformly asymptotically regular on C and $\Omega = F(T) \cap F(S) \cap EP \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$(51)\quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JS^n z_n), \\ f(u_n, y) + \langle Au_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, Ju_n - Jx\rangle \geq 0,\ \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n\phi(v, x_n) + (1-\beta_n)k_n\phi(v, z_n) \leq \phi(v, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1,\ \forall n \geq 1, \end{cases}$$

where $\theta_n = (1-\beta_n)(k_n^2-1)M_n \to 0$ as $n \to \infty$, $k_n = \max\{k_n^{(s)}\}$ for each $n \geq 1$, $M_n = \sup\{\phi(v, x_n) : v \in \Omega\}$ for each $n \geq 1$, $J : E \to E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If the following conditions are satisfied:

(i) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(ii) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(iii) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_1$, where Π_Ω is generalized projection of E onto Ω.

Proof. Taking $T = I$ in Theorem 3.1, then we have $z_n = x_n$, $\forall n \geq 1$. Hence the conclusion of Theorem 3.4 is obtained. □

Corollary 3.5. *Let E be a uniformly smooth and uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a continuous and monotone mapping and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A1)$ - $(A4)$, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(t)}\} \subset [1, \infty)$ such that $k_n^{(t)} \to 1$ as $n \to \infty$ and $S : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(s)}\} \subset [1, \infty)$ such that $k_n^{(s)} \to 1$ as $n \to \infty$. Assume that T and S are closed relatively nonexpansive mappings such that $\Omega = F(T) \cap F(S) \cap EP \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$(52)\quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n x_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JS^n z_n), \\ u_n = K_{r_n} y_n, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n\phi(v, x_n) + (1-\beta_n)k_n\phi(v, z_n) \leq \phi(v, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1,\ \forall n \geq 1, \end{cases}$$

where $\theta_n = (1-\beta_n)(k_n^2 - 1)M_n \to 0$ as $n \to \infty$, $k_n = \max\{k_n^{(t)}, k_n^{(s)}\}$ for each $n \geq 1$, $M_n = \sup\{\phi(v, x_n) : v \in \Omega\}$ for each $n \geq 1$, $J : E \to E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If the following conditions are satisfied:

(i) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(ii) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(iii) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_1$, where Π_Ω is generalized projection of E onto Ω.

Proof. Since every closed relatively nonexpansive mapping is quasi-ϕ-nonexpansive, the result is implied by Theorem 3.1. □

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