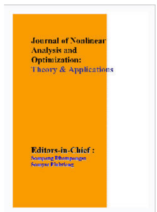
Journal of Nonlinear Analysis and Optimization
Vol. 1 No. 1 (2010), 45-53
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



Fixed point theorems for nonexpansive mappings with applications to generalized equilibrium and system of nonlinear variational inequalities problems*

Yeol Je Cho[1], Narin Petrot[2] and Suthep Suantai[3]

ABSTRACT: In this paper, a method for finding common element of two nonexpansive mappings are provided. Consequently, we provide some results for the fixed points of an infinite family of nonexpansive mappings and of an infinite family of strict pseudo-contraction mappings. Furthermore, we apply our main result to the problems of finding solution of generalized equilibrium and system of nonlinear variational inequalities problems. Some interesting remarks will be also pointed out and discussed.

KEYWORDS: Generalized equilibrium problem; system of nonlinear variational inequalities; nonexpansive mapping; strict pseudo-contraction mappings.

1. **Introduction and Preliminaries**

Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a nonempty closed convex subset of $\mathcal{H}$. A mapping $D : K \to K$ is said to be *nonexpansive mapping* if

$$\|Dx - Dy\| \leq \|x - y\|, \quad \forall x, y \in K.$$

If $D : K \to K$ is a nonexpansive mapping, we denote the set of fixed points of D by $F(D)$, that is, $F(D) = \{x \in K : Dx = x\}$. By assuming that D is a nonexpansive mapping such that its fixed points set is not empty, a classical iterative method to find the fixed point of D of minimal norm was firstly studied by Halpern [7]. He introduced the following explicit iteration scheme($u = 0$): for fixed $u, x_0 \in K, a_n \in (0,1)$

$$x_{n+1} = a_n u + (1 - a_n) D x_n, \quad n = 0, 1, 2, \ldots \tag{1}$$

and pointed out that the control conditions (C_1) $\lim_{n\to\infty} a_n = 0$ and (C_2) $\sum_{n=1}^{\infty} a_n = \infty$ are necessary for the convergence of the iteration scheme (1) to a fixed point of D. Since then, various extensions of Halpern result have been proposed. For examples, Lions [8] and Wittmann [21] established the strong convergence of the iteration scheme (1) under the control conditions

Corresponding author: Narin Petrot (narinp@nu.ac.th)
Manuscript received July 11, 2010. Accepted July 31, 2010.
*Supported by Faculty of Science, Naresuan University and the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

(C_1), (C_2) together with some additional control conditions. In 2007, Y. Yao et al.[22] introduced and studied the following implicit iterative scheme $\{x_n\}$:

(2) $$x_0 \in K, \ \ x_n = a_n u + b_n x_{n-1} + c_n D x_n, \ \ n \geq 1,$$

where u is an anchor and $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three real sequences in $(0,1)$. They showed that under some suitable control conditions, the sequence $\{x_n\}$ converges to a fixed point D.

Until now, the problem of finding the fixed points of the nonlinear mappings is the one subject of current interest in functional analysis. Motivated by Y. Yao et al.[22], in this paper we introduce the following an explicit iterative scheme $\{x_n\}$:

***The Algorithm*:** Let $G, D : K \to K$ be two mappings. For any $u, x_1 \in K$, we define the sequence $\{x_n\}$ in K as following:

(3) $$x_{n+1} = a_n u + b_n x_n + c_n \left(\gamma G(x_n) + (1-\gamma) D(x_n)\right), \quad \forall n \geq 1,$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are real sequences in $[0,1]$ such that $a_n + b_n + c_n = 1$ for all $n \geq 1$ and $\gamma \in (0,1)$. One main goal of us is to provide the suitable control conditions for the convergence of $\{x_n\}$.

Related to the fixed point problems, we also have the equilibrium problems which have had a great impact and influence in the development of several branches of pure and applied sciences. It is natural to construct a unified approach for these problems. In this direction, several authors have introduced some iterative schemes for finding a common element of the set of solutions of the equilibrium problems and the set of fixed points of nonlinear mappings (for examples, see [3, 12, 13, 15, 16, 23] and the references therein).

On the other hand, system of nonlinear variational inequalities problems were introduced by Verma [20]. Recently, Ceng et. al. [5] considered an iterative methods for a system of variational inequalities and obtained a strong convergence theorem for the such problem and a fixed point problem for a single nonexpansive mapping. For more examples, see [17, 18, 19] and the references therein.

All of above motivate us to apply the fixed point theory, our main result, to the problems of finding solution of generalized equilibrium and system of nonlinear variational inequalities problems. Also, some interesting remarks are also discussed. We would like to notice that, the results appeared in this paper can be viewed as an important improvement and extension of the previously known results.

Now we recall some well-known concepts and results.

Let K be a nonempty closed convex subset of $\mathcal{H}$. It is well known that, for any $z \in \mathcal{H}$, there exists a unique nearest point in K, denoted by $P_K z$, such that

$$\|z - P_K z\| \leq \|z - y\|, \quad \forall y \in K.$$

Such a mapping P_K is called the *metric projection* of $\mathcal{H}$ on to K. We know that P_K is nonexpansive. Furthermore, for any $z \in \mathcal{H}$ and $u \in K$,

(4) $$u = P_K z \Longleftrightarrow \langle u - z, w - u \rangle \geq 0, \quad \forall w \in K.$$

Lemma 1.1. [2] *Let K be a nonempty closed convex subset of a strictly convex Banach space E. If, for each $n \geq 1$, $S_n : K \to E$ is a nonexpansive mapping, then there exists a nonexpansive mapping $S : K \to E$ such that*

$$F(S) = \bigcap_{n=1}^{\infty} F(S_n).$$

In particular, if $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$, then the mapping $S = \sum_{n=1}^{\infty} \mu_n S_n$ satisfies the above requirement, where $\{\mu_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \mu_n = 1$.

Lemma 1.2. [1] *Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E and $S : K \to K$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero, i.e., if $\{x_n\}$ converges weakly to a point $x \in K$ and $\{x_n - Sx_n\}$ converges to zero, then $x = Sx$.*

Lemma 1.3. [14] *Let $\{x_n\}$ and $\{l_n\}$ be bounded sequences in a Banach space E and b_n be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1.$$

Suppose that $x_{n+1} = (1-b_n)l_n + b_n x_n$ for all $n \ge 1$ and

$$\limsup_{n\to\infty} (\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|l_n - x_n\| = 0$.

Lemma 1.4. [24] *Assume that $\{\theta_n\}$ is a sequence of nonnegative real numbers such that*

$$\theta_{n+1} \le (1-a_n)\theta_n + \delta_n, \quad \forall n \ge 1,$$

where $\{a_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} a_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{a_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \theta_n = 0$.

2. Main Results

Of course, we will use the sequence $\{x_n\}$, generated by (3), to obtain our main results in this paper. To do so, the behaviors of the sequences $\{a_n\}, \{b_n\}$ or $\{c_n\}$ should be controlled. Here, we assume the following control condition:

Condition ($\mathcal{C}$): Let $\{a_n\}$ and $\{b_n\}$ be defined as in (3). We say that the condition ($\mathcal{C}$) is satisfied if

(i) $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$,

are hold true.

Our main result is as following.

Theorem 2.1. *Let K be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $G, D : K \to K$ be two nonexpansive mappings such that $\Omega = F(G) \cap F(D) \neq \emptyset$. Let $u \in K$ be fixed and $\{x_n\}$ be a sequence in K generated by (3). If the condition ($\mathcal{C}$) is satisfied then the sequence $\{x_n\}$ converges strongly to a point $\widetilde{x} = P_\Omega u$.*

Proof. Firstly, we must assert that the mapping P_Ω is well defined by showing that Ω is a closed convex set. Indeed, since G, D are nonexpansive mappings, we have the set $F(G)$ and $F(D)$ are closed convex subsets of $\mathcal{H}$. Therefore, it follows that $\Omega = F(G) \cap F(D)$ is a closed convex subset of $\mathcal{H}$.

Now the proof is divided into the five steps as following:

Step 1: The sequence $\{x_n\}$ is bounded.

Write $e_n = \gamma G(x_n) + (1-\gamma)D(x_n)$ for all $n \ge 1$. Let $x^* \in \Omega$. Let us consider the following computation:

$$\begin{aligned}
\|e_n - x^*\| &= \|\gamma G(x_n) + (1-\gamma)D(x_n) - x^*\| \\
&\le \gamma\|G(x_n) - x^*\| + (1-\gamma)\|D(x_n) - x^*\| \\
&\le \gamma\|x_n - x^*\| + (1-\gamma)\|x_n - x^*\| \\
&= \|x_n - x^*\|, \quad \forall n \ge 1.
\end{aligned}$$

Consequently,

$$
\begin{aligned}
\|x_2 - x^*\| &= \|a_1 u + b_1 x_1 + c_1 e_1 - x^*\| \\
&\le a_1\|u - x^*\| + b_1\|x_1 - x^*\| + c_1\|e_1 - x^*\| \\
&\le a_1\|u - x^*\| + b_1\|x_1 - x^*\| + c_1\|x_1 - x^*\| \\
&\le a_1\|u - x^*\| + (1 - a_1)\|x_1 - x^*\| \\
&\le \max\{\|u - x^*\|, \|x_1 - x^*\|\}. \qquad (5)
\end{aligned}
$$

From (5) and induction, we know that the sequence $\{x_n\}$ is bounded, as required.

Step 2: $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
To do this, let us put

$$l_n = \frac{x_{n+1} - b_n x_n}{1 - b_n}, \quad \forall n \ge 1, \qquad (6)$$

which implies that

$$x_{n+1} - x_n = (1 - b_n)(l_n - x_n), \quad \forall n \ge 1. \qquad (7)$$

Now, by (6), (7), Lemma 1.3 and the condition (ii), we show that

$$\limsup_{n\to\infty} (\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \le 0. \qquad (8)$$

Consider the following computation:

$$
\begin{aligned}
l_{n+1} - l_n =& \frac{a_{n+1}u + c_{n+1}e_{n+1}}{1 - b_{n+1}} - \frac{a_n u + c_n e_n}{1 - b_n} \\
=& \frac{a_{n+1}}{1 - b_{n+1}} u + \frac{1 - b_{n+1} - a_{n+1}}{1 - b_{n+1}} e_{n+1} - \frac{a_n}{1 - b_n} u - \frac{1 - b_n - a_n}{1 - b_n} e_n \\
=& \frac{a_{n+1}}{1 - b_{n+1}} (u - e_{n+1}) + \frac{a_n}{1 - b_n} (e_n - u) + e_{n+1} - e_n, \quad \forall n \ge 1, \qquad (9)
\end{aligned}
$$

and,

$$
\begin{aligned}
\|e_{n+1} - e_n\| =& \|\gamma G(x_{n+1}) + (1 - \gamma)D(x_{n+1}) - (\gamma G(x_n) + (1 - \gamma)D(x_n)\| \\
\le& \gamma\|G(x_{n+1}) - G(x_n)\| + (1 - \gamma)\|D(x_{n+1}) - D(x_n)\| \\
\le& \gamma\|x_{n+1} - x_n\| + (1 - \gamma)\|x_{n+1} - x_n\| \\
=& \|x_{n+1} - x_n\|, \quad \forall n \ge 1. \qquad (10)
\end{aligned}
$$

Using (9) and (10), we have

$$\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \le \frac{a_{n+1}}{1 - b_{n+1}} \|u - e_{n+1}\| + \frac{a_n}{1 - b_n} \|e_n - u\|, \quad \forall n \ge 1.$$

Thus it follows from the conditions (i) and (ii) that

$$\limsup_{n\to\infty} (\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \le 0,$$

that is, (8) is satisfied.

Step 3: $x_n - e_n \to 0$ as $n \to \infty$.
From (3), we have

$$c_n(e_n - x_n) = x_{n+1} - x_n + a_n(x_n - u),$$

which implies that

$$c_n\|e_n - x_n\| \le \|x_{n+1} - x_n\| + a_n\|(x_n - u)\|$$

and so, from the conditions (i) and (ii), it follows that

$$\lim_{n\to\infty} \|e_n - x_n\| = 0. \qquad (11)$$

Step 4: $\limsup_{n\to\infty}\langle u - P_\Omega u, x_n - P_\Omega u\rangle \le 0.$

Write $\widetilde{x} = P_\Omega u$. Since $\{x_n\}$ is a bounded sequence, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $z \in K$ such that $\{x_{n_j}\}$ converges weakly to a point z as $j \to \infty$ and

$$\limsup_{n\to\infty}\langle u-\widetilde{x}, x_n-\widetilde{x}\rangle = \limsup_{j\to\infty}\langle u-\widetilde{x}, x_{n_j}-\widetilde{x}\rangle. \tag{12}$$

Now, we show that $z \in \Omega = F(G)\cap F(D)$. To show this, we define a mapping $S : K \to K$ by

$$Sx = \gamma G(x) + (1-\gamma)D(x), \quad \forall x \in K.$$

From Lemma 1.1, it follows that S is a nonexpansive mapping such that

$$F(S) = F(G)\cap F(D).$$

Furthermore, from (11), we obtain

$$\lim_{j\to\infty}\|Sx_{n_j} - x_{n_j}\| = 0.$$

Thus, by Lemma 1.2, we have $z \in F(S) = \Omega$. Consequently, from (4) and (12), it follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle u-\widetilde{x}, x_n-\widetilde{x}\rangle &= \limsup_{j\to\infty}\langle u-\widetilde{x}, x_{n_j}-\widetilde{x}\rangle\\ &= \langle u-\widetilde{x}, p-\widetilde{x}\rangle\\ &\le 0.\end{aligned}$$

Step 5: $x_n \to \widetilde{x}$ as $n \to \infty$, where $\widetilde{x} = P_\Omega u$.

Notice that

$$\begin{aligned}\|x_{n+1}-\widetilde{x}\|^2 &= \|a_n u + b_n x_n + c_n e_n - \widetilde{x}\|^2\\ &= \langle a_n(u-\widetilde{x}) + b_n(x_n-\widetilde{x}) + c_n(e_n-\widetilde{x}), x_{n+1}-\widetilde{x}\rangle\\ &\le a_n\langle u-\widetilde{x}, x_{n+1}-\widetilde{x}\rangle + b_n\|x_n-\widetilde{x}\|\|x_{n+1}-\widetilde{x}\| + c_n\|e_n-\widetilde{x}\|\|x_{n+1}-\widetilde{x}\|\\ &\le a_n\langle u-\widetilde{x}, x_{n+1}-\widetilde{x}\rangle + b_n\|x_n-\widetilde{x}\|\|x_{n+1}-\widetilde{x}\| + c_n\|x_n-\widetilde{x}\|\|x_{n+1}-\widetilde{x}\|\\ &= a_n\langle u-\widetilde{x}, x_{n+1}-\widetilde{x}\rangle + (1-a_n)\|x_n-\widetilde{x}\|\|x_{n+1}-\widetilde{x}\|\\ &\le a_n\langle u-\widetilde{x}, x_{n+1}-\widetilde{x}\rangle + \frac{(1-a_n)}{2}\left(\|x_n-\widetilde{x}\|^2 + \|x_{n+1}-\widetilde{x}\|^2\right).\end{aligned}$$

This implies that

$$\|x_{n+1}-\widetilde{x}\|^2 \le (1-a_n)\|x_n-\widetilde{x}\| + 2a_n\langle u-\widetilde{x}, x_{n+1}-\widetilde{x}\rangle. \tag{13}$$

Therefore, using (12) together with the conditions (i) and (ii), (13) and Lemma 1.4, it follows that $x_n \to \widetilde{x}$ as $n \to \infty$. This completes the proof. □

From Theorem 2.1, obviously, we can obtain the following result.

Corollary 2.2. *Let K be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $D : K \to K$ be a nonexpansive mapping such that $F(D) \neq \emptyset$. Let $u \in K$ be fixed and $\{x_n\}$ be a sequence in K generated by*

$$x_{n+1} = a_n u + b_n x_n + c_n D(x_n), \quad \forall n \ge 1. \tag{14}$$

If the condition (C) is satisfied then the sequence $\{x_n\}$ converges strongly to a point $\widetilde{x} = P_{F(D)}u$.

Using Lemma 1.1, as an application of Corollary 2.2, we also have the following results:

Corollary 2.3. *Let K be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $\{S_n\}$ be a family of nonexpansive mappings from K into itself such that $\Theta =: \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. Let $u \in C$ be fixed and $\{x_n\}$ be a sequence in K generated by (14) with $D = \sum_{n=1}^{\infty}\mu_n S_n$, where $\{\mu_n\}$ is a sequence of positive numbers with $\sum_{n=1}^{\infty}\mu_n = 1$. If the condition (C) is satisfied then the sequence $\{x_n\}$ converges strongly to a point $\widetilde{x} = P_\Theta u$.*

Remark 2.4. Recall that a mapping $W : C \to C$ is called a *τ-strict pseudo-contraction* with the coefficient $\tau \in [0,1)$ if

$$\|Wx - Wy\|^2 \leq \|x-y\|^2 + \tau\|(I-W)x - (I-W)y\|^2, \quad \forall x, y \in C.$$

It is obvious that every nonexpansive mapping is a 0-strict pseudo-contraction. Furthermore, if a mapping $W^{(\zeta)} : C \to C$ is defined by $W^{(\zeta)}x = \zeta x + (1-\zeta)Wx$ for all $x \in C$, where $\zeta \in [\tau, 1)$ is a fixed constant. Then $W^{(\zeta)}$ is a nonexpansive mapping such that $F(W^{(\zeta)}) = F(W)$, see [24]. Using this observation, in stead of the assumption that A and B are nonexpansive mappings, which were proposed in Theorem 2.1, we can further assume that the mappings A and B are strict pseudo-contractions.

Remark 2.5. If $f : C \to C$ is a contractive mapping and we replace u by $f(x_n)$ in the (3), then we can obtain the so-called viscosity iteration method (see [15] for more details).

3. applications

In this section, we will apply the Theorem 2.1 to some interesting problems as following:

3.1. Generalized equilibrium problem

Let $\varphi : K \to \mathbb{R}$ be a real-valued function, $Q : K \to \mathcal{H}$ be a mapping and $\Phi : \mathcal{H} \times K \times K \to \mathbb{R}$ be an equilibrium-like function. Let r be a positive number. For any $x \in K$, we consider the following problem:

$$(15) \qquad \begin{cases} \text{Find } y \in K \text{ such that} \\ \Phi(Qx, y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y\rangle \geq 0, \quad \forall z \in K, \end{cases}$$

which is known as the *auxiliary generalized equilibrium problem*. We denote the set of solution of problem (15) by $GEP(K, Q, \Phi, \varphi)$. In order to studying the problem (15), we related to the following concept:

Let $T^{(r)} : K \to K$ be the mapping such that, for each $x \in K$, $T^{(r)}(x)$ is the solution set of the auxiliary problem (15), i.e.,

$$T^{(r)}(x) = \{y \in K : \Phi(Qx, y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y\rangle \geq 0, \ \forall z \in K\}, \quad \forall x \in K.$$

We will assume the following **Condition (Δ)**:

(a) $T^{(r)}$ is single-valued;
(b) $T^{(r)}$ is nonexpansive;
(c) $F(T^{(r)}) = GEP(K, Q, \Phi, \varphi)$;

Notice that the examples of showing the sufficient conditions for the existence of the Condition (Δ) can be found in [4].

Assuming that the Condition (Δ) is satisfied, then we can introduce the following algorithm: Let r be a fixed positive number and $D : K \to K$ be a mapping. For any $u, x_1 \in K$, there exist sequences $\{u_n\}$, and $\{x_n\}$ in K such that

$$(16) \qquad \begin{cases} \Phi(Qx_n, u_n, v) + \varphi(v) - \varphi(u_n) + \frac{1}{r}\langle u_n - x_n, v - u_n\rangle \geq 0, \quad \forall v \in K, \\ x_{n+1} = a_n u + b_n x_n + c_n \left[\gamma u_n + (1-\gamma)D(x_n)\right], \quad \forall n \geq 1, \end{cases}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are real sequences in $[0,1]$ such that $a_n + b_n + c_n = 1$ for all $n \geq 1$ and $\gamma \in (0,1)$.

Theorem 3.1. *Let K be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $D : K \to K$ be a nonexpansive mapping. Assume that the Condition (Δ) is satisfied and*

$$\Omega = GEP(K, Q, \Phi, \varphi) \cap F(D) \neq \emptyset.$$

Let $u \in K$ be fixed and $\{x_n\}$ is defined by (16). If the condition ($\mathcal{C}$) is satisfied then the sequence $\{x_n\}$ converges strongly to a point $\widetilde{x} = P_\Omega u$.

Proof. Notice that, for each $n \geq 1$, we have $u_n = T^{(r)}(x_n)$. Hence, by setting $T^{(r)} =: G$, we see that (3) and (16) are the same. Therefore, thanks to the condition (Δ), the result is followed from Theorem 2.1.

□

Remark 3.2. It is worth to mention that for appropriate and suitable choice of the mapping Q, the functions Φ, φ and the convex set K, one can obtain a number of the various classes of equilibrium problems as special cases. This means, evidently, the Theorem 3.1 is very useful. For further applications of the problem (15), interested readers may refer to [6, 10, 11], and the references therein.

3.2. **System of nonlinear variational inequalities problems**

For two nonlinear mappings $A, B : K \to \mathcal{H}$, we consider the following *system of nonlinear variational inequalities problems*: Find $(x^*, y^*) \in K \times K$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in K, \\ \langle \rho B x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in K, \end{cases} \tag{17}$$

where λ and ρ are fixed positive numbers. In recent years, the problem of type (17) and its applications have been studied and investigated by many authors, see [5, 17, 18, 19, 20] for examples.

Now we have the following result, as the technical lemma:

Lemma 3.3. *Let K be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $G : K \to K$ be a nonexpansive self-mapping and $T, V : K \to \mathcal{H}$ be two nonexpansive non-self mappings. Assume that*

$$\Omega = F(G) \cap F(D) \neq \emptyset,$$

where $D : K \to K$ is defined by

$$D(x) = P_K[T \circ (P_K \circ V)](x), \quad \forall x \in K. \tag{18}$$

Let $u \in K$ be fixed and $\{x_n\}$ be a sequences in K generated by (3). If the condition ($\mathcal{C}$) is satisfied then the sequence $\{x_n\}$ converges strongly to a point $\widetilde{x} = P_\Omega u$.

Proof. Since $T, V : K \to \mathcal{H}$ are nonexpansive non-self mappings, it is obvious that $D := P_K[T \circ (P_K \circ V)]$ is a nonexpansive mapping. Therefore, the conclusion is followed immediately from Theorem 2.1.

□

We also need the following well-known lemma:

Lemma 3.4. [5] *Let ρ and λ be positive numbers. For any $x^*, y^* \in C$ with $y^* = P_C(x^* - \rho B x^*)$, (x^*, y^*) is a solution of the problem (1.3) if and only if x^* is a fixed point of the mapping $D : C \to C$ defined by*

$$D(x) = P_K[P_K(x - \rho B x) - \lambda A P_K(x - \rho B x)], \quad \forall x \in K.$$

Now we give the purposed result.

Theorem 3.5. *Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $A, B : K \to \mathcal{H}$ and $G : K \to K$ be a nonexpansive mapping. Assume that*

$$\Omega := F(G) \cap F(D) \neq \emptyset,$$

where the mapping $D : K \to K$ is defined by

$$D(x) = [P_K(I - \lambda A) \circ P_K(I - \rho B)](x), \quad \forall x \in K, \tag{19}$$

when ρ and λ are positive constants appeared in the problem (17). Assume that

(i) $(I - \lambda A)$ *and* $(I - \rho B)$ *are nonexpansive mappings;*
(ii) *the condition (C) is satisfied.*

If $\{x_n\}$ *is a sequence in* K *generated by (3) then* $\{x_n\}$ *converges strongly to a point* $\widetilde{x} = P_\Omega u$. *Moreover, if* $\widetilde{y} = P_K(\widetilde{x} - \rho B\widetilde{x})$, *then* $(\widetilde{x}, \widetilde{y})$ *is a solution to the problem (17).*

Proof. Firstly, since $I - \lambda A$ and $I - \rho B$ are nonexpansive mappings, we know that D is a nonexpansive mapping. Thus, thanks to Lemma 3.3, we know that $\{x_n\}$ converges strongly to $\widetilde{x} := P_\Omega u$. Moreover, by the definition of the mapping D, we observe that

$$D = P_K(I - \lambda A) \circ P_K(I - \rho B) = P_K\left[P_K(I - \rho B) - \lambda A P_K(I - \rho B)\right].$$

Consequently, in view of Lemma 3.4, the second part of the required result is followed immediately. □

Remark 3.6. Recall that a nonlinear mapping $A : K \to \mathcal{H}$ is said to be:

(1) *α-cocoercive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in K;$$

(2) *β-strongly monotone* if there exists a constant $\beta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in K.$$

Notice that if either

(A1) A is α-cocoercive mapping and $\lambda \in (0, 2\alpha]$ or,

(A2) A is β-strongly monotone and L-Lipschitz continuous mapping and $\lambda \in \left(0, \frac{2\beta}{L}\right]$;

is satisfied, then $I - \lambda A$ is a nonexpansive mapping. This means that the results obtained in the Theorem 3.5 can be viewed as an important extension of the previously known results.

Remark 3.7. In view of Corollary 2.3, one can apply Theorems 3.1 and 3.5 from a nonexpansive mapping to a family of nonexpansive mappings (or even a family of strict pseudo-contraction mappings).

References

1. F.E. Browder, Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces, Arch. Ration. Mech. Anal. 24 (1967) 82-90.
2. R.E. Bruck, Properties of fixed point sets of nonexpansive mappings in Banach spaces, Trans. Amer. Math. Soc. 179 (1973) 251-262.
3. S.S. Chang, H.W.J. Lee, C.K. Chan, A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization, Nonlinear Anal., 70 (2009), 3307-3319.
4. L.C. Ceng, Q. H. Ansari, J.C. Yao, Viscosity approximation methods for generalized equilibrium problems and fixed point problems, J. Glob. Optim. 43 (2009) 487-502.
5. L.C. Ceng, C.Y. Wang, J.C. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, Math. Meth. Oper. Res. 67 (2008) 375-390.
6. F. Flores-Bazan, Existence theorems for generalized noncoercive equilibrium problems: the quasi-convex case, SIAM J. Optim. 11 (2000) 675-690.
7. B. Halpern, Fixed points of nonexpanding maps, Bull. Amer. Math. Soc. 73 (1967) 957-961.
8. P.L. Lions, Approximation de points fixes de contraction, C.R.Acad. Sci. Ser. A-B Pairs (284) (1977) 1357-1359.
9. A. Moudafi, Viscosity approximation methods for fixed-points problems, J. Math. Anal. Appl. 241 (2000) 46-55.
10. M.A. Noor, Variational-like inequalities, Optim. 30 (1994) 323-330.
11. M.A. Noor, W. Oettli, On general nonlinear complementarity problems and quasi equilibria, Matematiche (Catania) 49 (1994) 313-331.
12. X. Qin, Y.J. Cho, S.M. Kang, Convergence theorems of common elements for equilib- rium problems and fixed point problems in Banach spaces, J. Comput. Appl. Math. 225(1) (2009), 20-30.
13. X. Qin, M. Shang, Y. Su, Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems, Math. Comput. Model. 48 (2008) 1033-1046.
14. T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one- parameter nonexpansive semigroups without Bochne integrals, J. Math. Anal. Appl. 305 (2005) 227-239.
15. A. Tada, W. Takahashi, Strong convergence theorem for an equilibrium problem and a nonexpansive mapping, J. Optim. Theory Appl. 133 (2007) 359-370.

16. S. Takahashi, W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Anal. 69 (2008) 1025-1033.
17. R.U. Verma, Generalized system for relaxed cocoercive variational inequalities and its projection methods, J. Optim. Theory Appl. 121 (2004) 203-210.
18. R.U. Verma, Generalized class of partial relaxed monotonicity and its connections, Adv. Nonlinear Var. Inequal. 7 (2004) 155-164.
19. R.U. Verma, General convergence analysis for two-step projection methods and applications to variational problems, Appl. Math. Lett. 18 (2005) 1286-1292.
20. R.U. Verma, On a new system of nonlinear variational inequalities and associated iterative algorithms, Math. Sci. Res. Hot-Line 3 (1999) 65-68.
21. R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math. 59 (1992) 486-491.
22. Y. Yao, Y.C. Liou, R. Chen, Strong convergence of an iterative algorithm for pseudocontractive mapping in Banach spaces, Nonlinear Analysis 67 (2007) 33113317.
23. Y. Yao, M.A. Noor, S. Zainab, Y.C. Liou, Mixed equilibrium problems and optimization problems, J. Math. Anal. Appl. 354 (2009) 319-329.
24. H.Y. Zhou, Convergence theorems of fixed points for k-strict pseudo-contractions in Hilbert spaces, Nonlinear Anal. 69 (2008) 456-462.

[1]Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju 660-701, Korea.
Email address: yjcho@gsnu.ac.kr

[2]Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, 65000, Thailand.
Email address: narinp@nu.ac.th

[3]Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand.
Email address: scmti005@chiangmai.ac.th