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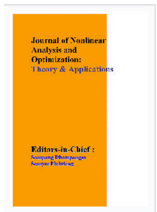

Existence and approximation of solution of the variational inequality problem with a skew monotone operator defined on the dual spaces of Banach spaces

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ABSTRACT: In this paper, we first study an existence theorem of the variational inequality problem for a skew monotone operator defined on the dual space of a smooth Banach space. Secondary, we prove a weak convergence theorem for finding a solution of the variational inequality problem by using projection algorithm method with a new projection which was introduce by Ibaraki and Takahashi [T. Ibaraki, W. Takahashi, A new projection and convence theorems for the projections in Banach spaces, Journal of Approximation Theory 149 (2007), 1-14]. Further, we apply our convergence theorem to the convex minimization problem and the problem of finding a zero point of maximal skew monotone operator.

KEYWORDS: Generalized nonexpansive retraction; Inverse-strongly-skew-monotone operator; Variational inequality; p-uniformly smooth.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed and convex subset of H. Let A be a monotone operator of C into H. The variational inequality problem [10, 16] is to find a point $u \in C$ such that

(1) $$\langle Au, v - u\rangle \geq 0, \quad \text{for all } \ v \in C.$$

Such a point $u \in C$ is called a solution of the problem and the set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

Variational inequality theory has played a fundamental and powerful role in the study of a wide range of problems arising in differential equations, mechanics, contact problems in elasticity, optimization and control problems, management science, operations research, general equilibrium problems in economics and transportation, etc. We now have a variety of techniques to suggest and analyze various iterative algorithms for solving variational inequalities and the related optimization problems. The projection operator technique, one usually establishes an equivalence between the variational inequalities and the fixed-point problem. This alternative equivalent formulation was used by Lions and Stampacchia [16] to study the existence of solutions of the variational inequalities. Projection method and its variant forms

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represent important tools for finding the approximate solution of variational inequalities. This method starts with any $x_1 = x \in C$ and updates iteratively x_{n+1} according to the formula

$$x_{n+1} = P_C(x_n - \lambda_n A x_n) \tag{2}$$

for every $n = 1, 2, ...$, where A is a monotone operator of C into H, P_C, is the metric projection of H onto C and $\{\lambda_n\}$ is a sequence of positive numbers. An operator A of C into E^* is said to be *inverse-strongly-monotone* [2, 8, 12] if there exists a positive real number α such that $\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2$ for all $x, y \in C$. In such a case A is said to be *α-inverse-strongly-monotone*. In the case where A is inverse-strongly-monotone, Iiduka, Takahashi and Toyoda [8] proved that the sequence $\{x_n\}$ generated by (2) converges weakly to some element of $VI(C, A)$.

Recently, Iiduka and Takahashi [7] introduced the following iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly-monotone operator A in a Banach space: $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n) \tag{3}$$

for every $n = 1, 2, ...$, where where A is an inverse-strongly-monotone operator of C into E^*, Π_C is the generalized metric projection from E onto C, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive numbers. They proved that the sequence $\{x_n\}$ generated by (3) converges weakly to some element of $VI(C, A)$. On the other hand, Ibaraki and Takahashi [5] introduced a new resolvent of a maximal monotone operator in a Banach space and the concept of a generalized nonexpansive mapping in a Banach space. Kohsaka and Takahashi [11], and Ibaraki and Takahashi [6] also studied some properties for generalized nonexpansive retractions in Banach spaces.

In this paper, motivated by Ibaraki and Takahashi [5] and Iiduka and Takahashi [7], we consider the following variational inequality problem: Let E be a smooth Banach space, let E^* be the dual space of E and let C be a nonempty and closed subset of E such that JC is closed and convex subset of E^*, where J is the duality mapping on E. Let A be a skew monotone operator of JC into E. Then, the variational inequality problem is to find

$$u \in C \text{ such that } \langle AJu, Jv - Ju\rangle \geq 0, \ \forall\, v \in C. \tag{4}$$

We denoted the set of solution of the variational inequality problem (4) by $VI(JC, A)$. If $E = H$ is Hilbert space and C is nonempty closed convex subset of H, then the variational inequality problem (4) is equivalent to the variational inequality problem (1). In this paper, we first prove existence theorem of the variational inequality problem for skew monotone operators defined on the dual space of E. Using the projection algorithm method with a new projection which was introduced by Ibaraki and Takahashi [5], we prove weak convergence theorem for finding a solution of the variational inequality problem (4) for an inverse-strongly-skew-monotone operator defined on the dual space of a uniformly convex and 2−uniformly smooth Banach space. Further, using this result we consider the convex minimization problem and the problem of finding a zero point of maximal skew monotone operator.

2. **Preliminaries**

Let E be a real Banach space. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. An operator $T \subset E \times E^*$ is said to be *monotone* if $\langle x - y, x^* - y^*\rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in T$. We denote the set $\{x \in E : 0 \in Tx\}$ by $T^{-1}0$. A monotone T is said to be *maximal* if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator. It is known that a monotone operator T is maximal if and only if for $(x, x^*) \in E \times E^*$, $\langle x - y, x^* - y^*\rangle \geq 0$ for every $(y, y^*) \in G(T)$ implies $x^* \in T(x)$. If T is maximal monotone, then the solution set $T^{-1}0$ is closed and convex.

The normalized duality mapping J from E into E^* is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

We recall (see [17]) that E is reflexive if and only if J is surjective; E is smooth if and only if J is single-valued; E is strictly convex if and only if J is one-to-one; if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. We note that in a Hilbert space, H, J is the identity operator. The definitions of the strict (uniform) convexity, (uniformly) smoothness of Banach spaces and related properties can be found in [17].

The duality J from a smooth Banach space E into E^* is said to be *weakly sequentially continuous* [4] if $x_n \rightharpoonup x$ implies $Jx_n \rightharpoonup^* Jx$, where $\rightharpoonup^*$ implies the weak* convergence.

Let E be a norm linear space with $\dim E \geq 2$. The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) = \sup\left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}.$$

The space E is said to be *smooth* if $\rho_E(\tau) > 0$, $\forall \tau > 0$. E is called *uniformly smooth* if and only if $\lim_{t\to 0^+} \frac{\rho_E(\tau)}{t} = 0$. Let $p > 1$. E is said to be $p-$*uniformly smooth* (or to have a modulus of smoothness of power type p) if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^p$, $t > 0$. It is well known (see, for example, [18]) that

$$L_p\ (l_p) \text{ or } (W_m)^p \text{ is } \begin{cases} \text{2-uniformly smooth if } p \geq 2 \\ \text{p-uniformly smooth if } 1 < p \leq 2. \end{cases}$$

We observe that every p-uniformly smooth Banach space is uniformly smooth. Furthermore, from the proof of [18, Remark 5, p.208], we have the following lemma

Lemma 2.1. [18] *Let E be a 2-uniformly smooth Banach space. Then, for all $x, y \in E$, there exists a constant $c > 0$ such that*

$$\|Jx - Jy\| \leq c\|x - y\|, \tag{5}$$

where J is the normalized duality mapping of E.

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for all } x, y \in E,$$

is studied by Alber [1], Kamimura and Takahashi [9] and Reich [13]. It is obvious from the definition of ϕ that $(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2$ for all $x, y \in E$.

Lemma 2.2. *(see [9]) Let E be a uniformly convex, smooth Banach space, and let $\{x_n\}$ and $\{y_n\}$ be sequences in E. If $\{x_n\}$ or $\{y_n\}$ is bounded and $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let E be a reflexive, strictly convex, smooth Banach space and J the duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one, surjective, and it is the duality mapping from E^* into E. We make use of the following mapping V studied in Alber [1]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{6}$$

for all $x \in E$ and $x^* \in E^*$. In other words, $V(x,x^*) = \phi(x, J^{-1}(x))$ for all $x \in E$ and $x^* \in E^*$.

Lemma 2.3. *(see [6]) Let E be a reflexive, strictly convex, smooth Banach space and let V be as in (6). Then*

$$V(x, x^*) + 2\langle y, Jx - x^*\rangle \leq V(x + y, x^*)$$

for all $x, y \in E$ and $x^* \in E^*$.

Let E be a smooth Banach space and let D be a nonempty closed subset of E. A mapping $R : D \to D$ is called generalized nonexpansive if $F(R) \neq \emptyset$ and $\phi(Rx, y) \leq \phi(x, y)$ for each $x \in D$ and $y \in F(R)$, where $F(R)$ is the set of fixed points of R. Let C be a nonempty closed subset of E. A mapping $R : E \to C$ is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx, \quad \forall x \in E,\ \forall\, t \geq 0.$$

A mapping $R : E \to C$ is said to be a retraction if $Rx = x$, $\forall x \in C$. If E is smooth and strictly convex, then a sunny generalized nonexpansive retraction of E onto C is uniquely determined if it exists (see [5]). We also know that if E is reflexive, smooth, and strictly convex and C is a nonempty closed subset of E, then there exists a sunny generalized nonexpansive retraction R_C of E onto C if and only if $J(C)$ is closed and convex. In this case R_C is given by $R_C = J^{-1}\Pi_{J(C)}J$

see [11]. Let C be a nonempty closed subset of a Banach space E. Then C is said to be a sunny generalized nonexpansive retract (resp. a generalized nonexpansive retract) of E if there exists a sunny generalized nonexpansive retraction (resp. a generalized nonexpansive retraction) of E onto C (see [5] for more detials). The set of fixed points of such a generalized nonexpansive retraction is C. The following Lemma was obtained in [5].

Lemma 2.4. *([5]) Let C be a nonempty and closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C, let $x \in E$ and let $z \in C$. Then the following hold:*
(a) $z = R_C x$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$
(b) $\phi(x, R_C x) + \phi(R_C x, z) \leq \phi(x, z)$.

3. Variational inequalities for skew monotone operators defined on the dual space of a Banach space.

In this section, we consider the variational inequalities for skew monotone operators defined on the dual space of a Banach space. Let E be a smooth Banach space and let C^* be the closed closed and convex subset of E^*. An operator $A : C^* \to E$ is said to be *skew monotone if $\langle Ax^* - Ay^*, x^* - y^* \rangle \geq 0$ for all $x^*, y^* \in C^*$. Let E^* be the dual space of E and let C be a nonempty and closed subset of E such that JC is closed and convex subset of E^*, where J is the duality mapping on E. Let A be a skew monotone operator of JC into E. Then, the variational inequality problem is to find*

(7) $$u \in C \text{ such that } \langle AJu, Jv - Ju \rangle \geq 0, \ \forall\, v \in C.$$

Such a point u is called a solution of the problem. We denote the set of solution of the variational inequality problem (7) by $VI(JC, A)$. i.e.,

$$VI(JC, A) = \{u \in C : \langle AJu, Jv - Ju \rangle \geq 0, \ \forall\, v \in C\}$$

.

Lemma 3.1. *Let E be a Banach space with the dual space E^*. Let C^* be a nonempty, compact and convex subset of E^* and A is a skew monotone operator of C^* into E. Then there exists $x_0^* \in C^*$ such that*

$$\langle Ax^*, x^* - x_0^* \rangle \geq 0, \ \forall x^* \in C^*.$$

Proof. For any $y^* \in C^*$, we assume that the set $\{x^* \in C^* : \langle Ax^*, x^* - y^* \rangle < 0\}$ is nonempty. We also define two multi-valued mappings T and B of C^* into itself by

$$Tx^* = \{y^* \in C^* : \langle Ay^*, x^* - y^* \rangle < 0\} \text{ and } Bx^* = \{y^* \in C^* : \langle Ax^*, x^* - y^* \rangle < 0\}.$$

Then, for any $y^* \in C^*$, the set $T^{-1}y^* = \{x^* \in C^* : \langle Ay^*, x^* - y^* \rangle < 0\}$ is convex. Also, for any $y^* \in C^*$, the set $B^{-1}y^* = \{x^* \in C^* : \langle Ax^*, x^* - y^* \rangle < 0\}$ is nonempty. Since A is skew monotone, we have that $\langle Ax^*, x^* - y^* \rangle \geq \langle Ay^*, x^* - y^* \rangle$, for all $x^*, y^* \in C^*$. So, we have that $Bx^* \subset Tx^*$ for all $x^* \in C^*$. Since Bx^* is open for all $x^* \in C^*$, it follows by [17, Theorem 6.1.5] that there exists a point $x_0^* \in C^*$ such that $x_0^* \in Tx_0^*$. Thus, we have

$$0 = \langle Ax_0^*, x_0^* - x_0^* \rangle < 0$$

This is a contradiction. □

An operator $A : D(A) \subset E^* \to E$ is said to be hemicontinuous if for all $x^*, y^* \in D(A)$, the mapping f of $[0,1]$ into E defined by $f(t) = A(tx^* + (1-t)y^*)$ is continuous.

Lemma 3.2. *Let E be a Banach space with the dual space E^*. Let C^* be a nonempty and convex subset of E^* and let A be a skew monotone and hemicontinuous operator of C^* into E. Let $x_0^* \in C^*$. Then*

(8) $$\langle Ax_0^*, x^* - x_0^* \rangle \geq 0, \ \forall x^* \in C^*$$

if and only if

(9) $$\langle Ax^*, x^* - x_0^* \rangle \geq 0, \ \forall x^* \in C^*.$$

Proof. Suppose that $\langle Ax_0^*, x^* - x_0^*\rangle \geq 0$, for all $x^* \in C^*$. By the skew monotonicity of A, we have

$$\langle Ax^*, x^* - x_0^*\rangle \geq \langle Ax_0^*, x^* - x_0^*\rangle \geq 0, \text{ for all } x^* \in C^*.$$

Conversely, suppose that $\langle Ax^*, x^* - x_0^*\rangle \geq 0$, for all $x^* \in C^*$. Let $y^* \in C^*$ and $0 < t < 1$. Since C^* is convex, we have $y_t^* = (1-t)x_0^* + ty^* \in C^*$. This implies that

$$0 \leq \langle Ay_t^*, y_t^* - x_0^*\rangle = t\langle Ay_t^*, y^* - x_0^*\rangle.$$

Since $t > 0$, it follow that $0 \leq \langle Ay_t^*, y^* - x_0^*\rangle$. Thus by the hemicontinuity of A , we have

$$0 \leq \langle Ax_0^*, y^* - x_0^*\rangle \text{ as } t \to 0.$$

This completes the proof. □

Using Lemma 3.1 and Lemma 3.2, we obtained the following Theorem

Theorem 3.3. *Let E be a Banach space with the dual space E^*. Let C^* be a nonempty, compact and convex subset of E^* and let A be a skew monotone and hemicontinuous operator of C^* into E. Then there exists $x_0^* \in C^*$ such that*

$$\langle Ax_0^*, x^* - x_0^*\rangle \geq 0, \ \forall x^* \in C^*.$$

We note from Theorem 3.3 that if JC is compact and convex and A is a skew monotone and hemicontinuous operator of JC into E, then $VI(JC, A)$ is nonempty.

Lemma 3.4. *Let C be a nonempty and closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex set. Let A be a skew monotone operator of JC into E. Then*

$$u \in VI(JC, A) \text{ if and only if } u = R_C(u - \lambda AJu), \ \forall \lambda > 0,$$

where R_C is sunny generalized nonexpansive retraction of E onto C.

Proof. From the definition of $VI(JC, A)$ and Lemma 2.4, we have

$$\begin{aligned} u \in VI(JC, A) &\Leftrightarrow \langle AJu, Jy - Ju\rangle \geq 0 \ \forall y \in C \\ &\Leftrightarrow \langle -\lambda AJu, Jy - Ju\rangle \leq 0 \ \forall y \in C, \ \forall \lambda > 0 \\ &\Leftrightarrow \langle u - \lambda AJu - u, Jy - Ju\rangle \leq 0 \ \forall y \in C, \ \forall \lambda > 0 \\ &\Leftrightarrow u = R_C(u - \lambda AJu), \ \forall \lambda > 0. \end{aligned}$$

□

Let E be a Banach space with the dual space E^* and let C be a nonempty and closed subset of E such that JC is closed and convex. Let i_{JC} be indicator function of JC. Since $i_{JC} : E^* \to (-\infty, \infty]$ is proper, lower semicontonuous and convex, the subdifferential ∂i_{JC} of i_{JC} defined by

$$\partial i_{JC}(x^*) = \{x \in E : i_{JC}(y^*) \geq i_{JC}(x^*) + \langle x, y^* - x^*\rangle \ (\forall \ y^* \in E^*) \ \}, \ (\forall \ x^* \in E^*)$$

is a maximal skew monotone operator by [14, 15]. *Next, let C be a nonempty and closed subset of E such that JC is closed and convex set and $x^* \in JC$. Then we denote by $N_{JC}(x^*)$ the skew normal cone of JC at a point $x^* \in JC$, that is,*

$$N_{JC}(x^*) = \{x \in E : \langle x, x^* - y^*\rangle \geq 0 \ \textit{for all } y^* \in JC\}.$$

We note from [14, 15] *that*

$$\partial i_{JC}(x^*) = \begin{cases} N_{JC}(x^*), & \textit{if } x^* \in JC, \\ \emptyset, & \textit{if } x^* \notin JC. \end{cases}$$

Then we obtain the following theorem:

Theorem 3.5. *Let C be a nonempty and closed subset of a smooth Banach space E such that JC is closed and convex set and let A be a skew monotone and hemicontinuous operator of JC into E. Let $B \subset E^* \times E$ be an operator defined as follows:*

$$Bv^* = \begin{cases} Av^* + N_{JC}(v^*), & v^* \in JC, \\ \emptyset, & v^* \notin JC. \end{cases}$$

Then B is maximal skew monotone and $(BJ)^{-1}0 = VI(JC, A)$.

Proof. We first show that B is skew monotone. Let $y_1 \in Ax_1^* + N_{JC}(x_1^*)$ and $y_2 \in Ax_2^* + N_{JC}(x_2^*)$. Then we can write them by $y_1 = Ax_1^* + z_1$ for some $z_1 \in N_{JC}(x_1^*)$ and $y_2 = Ax_2^* + z_2$ for some $z_2 \in N_{JC}(x_2^*)$. Since A is skew monotone, $z_1 \in N_{JC}(x_1^*)$ and $z_2 \in N_{JC}(x_2^*)$ it follows that

$$\begin{aligned}\langle y_1 - y_2, x_1^* - x_2^* \rangle &= \langle Ax_1^* + z_1 - (Ax_2^* + z_2), x_1^* - x_2^* \rangle \\ &= \langle Ax_1^* - Ax_2^*, x_1^* - x_2^* \rangle + \langle z_1 - z_2, x_1^* - x_2^* \rangle \\ &= \langle Ax_1^* - Ax_2^*, x_1^* - x_2^* \rangle - \langle z_1, x_2^* - x_1^* \rangle - \langle z_2, x_1^* - x_2^* \rangle \\ &\geq 0.\end{aligned}$$

This implies that B is skew monotone. Next, we shall show that B is maximal. Let $(x^*, x) \in E^* \times E$ such that $\langle y - x, y^* - x^* \rangle \geq 0$ for every $(y^*, y) \in G(B)$. Note that, for any $(y^*, y) \in G(B)$, we have $y \in By^* = Ay^* + N_{JC}(y^*)$. This implies that $y = Ay^* + z$ for some $z \in N_{JC}(y^*)$. So, the above inequality means that

$$\langle z, y^* - x^* \rangle + \langle Ay^* - x, y^* - x^* \rangle \geq 0. \tag{10}$$

for all $y^* \in JC$ and $z \in N_{JC}(y^*)$. It is clear from the definition of $N_{JC}(y^*)$ that if $z \in N_{JC}(y^*)$ and $\lambda \geq 0$, then $\lambda z \in N_{JC}(y^*)$. So from (10), we note that

$$\langle z, y^* - x^* \rangle \geq 0 \; \forall \, z \in N_{JC}(y^*). \tag{11}$$

In fact, if not, there exists $z \in N_{JC}(y^*)$ such that $\langle z, y^* - x^* \rangle < 0$. So, we have $\lambda \langle z, y^* - x^* \rangle \to -\infty$ as $\lambda \to \infty$. This is a contradiction. Then we got that (11) holds. Since $z \in N_{JC}(y^*) \Leftrightarrow z \in \partial i_{JC}(y^*)$, it follows from (11) that

$$\langle z - 0, y^* - x^* \rangle \geq 0 \; \forall \; (y^*, z) \in G(\partial i_{JC}).$$

Since ∂i_{JC} is maximal skew monotone, we have $0 \in \partial i_{JC}(x^*) = N_{JC}(x^*)$ and hence $x^* \in JC$. Define $x_t^* = tu^* + (1-t)x^*$, where $u^* \in JC$ and $t \in (0, 1)$. From the convexity of JC , we get $x_t^* \in JC$. By (10), we have from $0 \in N_{JC}(x_t^*)$ that

$$\langle 0, x_t^* - x^* \rangle + \langle Ax_t^* - x, x_t^* - x^* \rangle \geq 0,$$

and hence $\langle Ax_t^* - x, x_t^* - x^* \rangle \geq 0$. Since $x_t^* = tu^* + (1-t)x^*$, it follows that $t\langle Ax_t^* - x, u^* - x^* \rangle \geq 0$. Dividing this inequality by $t > 0$, we obtain

$$\langle Ax_t^* - x, u^* - x^* \rangle \geq 0.$$

So, letting $t \to 0$, we get

$$\langle x - Ax^*, x^* - u^* \rangle \geq 0 \; (\forall \, u^* \in JC)$$

and hence $x - Ax^* \in N_{JC}(x^*)$. This implies that $x \in Ax^* + N_{JC}(x^*) = Bx^*$. Therefore B is a maximal skew monotone operator. Finally, we will show that $(BJ)^{-1}0 = VI(JC, A)$.
We note that $(BJ)^{-1}0 = \{z \in C : 0 \in BJ(z)\}$. Thus, we have

$$\begin{aligned} z \in (BJ)^{-1}0 &\Leftrightarrow 0 \in AJz + N_{JC}(Jz) \\ &\Leftrightarrow -AJz \in N_{JC}(Jz) \\ &\Leftrightarrow \langle -AJz, Jz - y^* \rangle \geq 0 \; \forall \, y^* \in JC \\ &\Leftrightarrow \langle AJz, y^* - Jz \rangle \geq 0 \; \forall \, y^* \in JC \\ &\Leftrightarrow \langle AJz, Jy - Jz \rangle \geq 0 \; \forall \, y \in C \\ &\Leftrightarrow z \in VI(JC, A). \end{aligned}$$

□

Corollary 3.6. *Let E be a reflexive, strictly convex and smooth Banach space with a Fréchet differentiable norm and let C be a nonempty and closed subset of E such that JC is closed and convex and let A be a skew monotone and hemicontinuous operator of JC into E such that $VI(JC, A) \neq \emptyset$. Then $VI(JC, A)$ is closed and $JVI(JC, A)$ is closed and convex.*

Proof. Let $B \subset E^* \times E$ be an operator defined as follows:

$$Bv^* = \begin{cases} Av^* + N_{JC}(v^*), & v^* \in JC, \\ \emptyset, & v^* \notin JC. \end{cases}$$

By Theorem 3.5, we have that B is maximal skew monotone operator and $(BJ)^{-1}0 = VI(JC, A)$. Since E is reflexive and strictly convex, it follows that J is bijective. Thus, we have $JVI(JC, A) = JJ^{-1}B^{-1}0 = B^{-1}0$. Since B is maximal skew monotone, it follows that $B^{-1}0$ is closed and convex and hence $JVI(JC, A)$ is closed and convex. Next, let $\{x_n\} \subset (BJ)^{-1}0$ with $x_n \to x$. From $x_n \in (BJ)^{-1}0$, we have $J(x_n) \in B^{-1}0$. Since J is norm to norm continuous and $B^{-1}0$ is closed, we have $J(x_n) \to J(x) \in B^{-1}0$. This implies that $x \in (BJ)^{-1}0$. Hence $(BJ)^{-1}0$ is closed and therefore $VI(JC, A)$ is closed. □

4. **Weak convergence theorem**

In this section, using the projection algorithm method we prove weak convergence theorem for finding a solution of the variational inequality for an inverse-strongly-skew-monotone operator defined on the dual space of a uniformly convex and 2-uniformly smooth Banach space.

Let E be a real Banach space with the dual space E^*. An operator $A : D(A) \subset E^* :\to E$ is said to be inverse-strongly-skew-monotone if there exists a positive real number α such that $\langle Ax^* - Ay^*, x^* - y^* \rangle \geq \alpha \|Ax^* - Ay^*\|^2$ for all $x^*, y^* \in D(A)$. In such a case A is said to be α-inverse-strongly-skew-monotone. An operator $A : D(A) \subset E^* :\to E$ is said to be Lipscitz continuous if there exists $L \geq 0$ such that $\|Ax^* - Ay^*\| \leq L\|x^* - y^*\|$, for all $x^*, y^* \in D(A)$. If A is α-inverse-strongly-skew-monotone, then A is Lipscitz continuous, that is, $\|Ax^* - Ay^*\| \leq (\frac{1}{\alpha})\|x^* - y^*\|$, for all $x^*, y^* \in D(A)$.

Before proving our theorem we need the following Lemma.

Lemma 4.1. *Let C be a nonempty and closed subset of a uniformly convex and smooth Banach space E such that JC is closed and convex. Let $\{x_n\}$ be a sequence in E such that, for all $u \in C$,*

$$\phi(x_{n+1}, u) \leq \phi(x_n, u) \tag{12}$$

for every $n = 1, 2, ...$ Then $\{R_C(x_n)\}$ is a Cauchy sequence, where R_C is sunny generalized nonexpansive retraction of E onto C.

Proof. Put $u_n = R_C(x_n)$ for all $n \in \mathbb{N}$. From (12), we note that

$$\begin{aligned} \phi(x_{n+m}, u) &\leq \phi(x_{n+m-1}, u) \\ &\leq \phi(x_{n+m-2}, u) \leq ... \leq \phi(x_n, u) \end{aligned}$$

for every $n = 1, 2, ...$ Thus, we have

$$\phi(x_{n+m}, u_n) \leq \phi(x_n, u_n) \tag{13}$$

Since $u_{n+m} = R_C(x_{n+m})$, it follows from Lemma 2.4 (b) and (13) that

$$\begin{aligned} \phi(u_{n+m}, u_n) &= \phi(R_C(x_{n+m}), u_n) \\ &\leq \phi(x_{n+m}, u_n) - \phi(x_{n+m}, u_{n+m}) \\ &\leq \phi(x_n, u_n) - \phi(x_{n+m}, u_{n+m}). \end{aligned} \tag{14}$$

Consequently, we have $\limsup_{l\to\infty} \phi(x_l, u_l) \leq \phi(x_n, u_n)$, which implies that $\{\phi(x_n, u_n)\}$ converges. By Lemma 2.2 and (14), we note that $\{u_n\}$ is a Cauchy sequence. □

Now, we can prove the following weak convergence theorem.

Theorem 4.2. *Let E be a uniformly convex and 2-uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous. Let C be a nonempty and closed subset of E such that JC is closed and convex and let A be an α-inverse-strongly-skew-monotone operator of JC into E such that*

$VI(JC,A) \neq \emptyset$ and $\|AJy\| \leq \|AJy - AJu\|$ for all $y \in C$ and $u \in VI(JC,A)$. Let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and

$$x_{n+1} = R_C(x_n - \lambda_n AJx_n), \tag{15}$$

for every $n = 1,2,...$, where R_C is sunny generalized nonexpansive retraction of E onto C, $\{\lambda_n\} \subset [a,b]$ for some a,b with $0 < a < b < \frac{\alpha}{c}$, where c is a constant in (5). Then the sequence $\{x_n\}$ converges weakly to some element $z \in VI(JC,A)$. Further $z = \lim_{n\to\infty} R_{VI(JC,A)}(x_n)$.

Proof. Put $y_n = x_n - \lambda_n AJx_n$ for all $n = 1,2,....$ Let $u \in VI(JC,A)$. We first prove that $\{x_n\}$ is bounded. By Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}
\phi(x_{n+1},u) &= \phi(R_C y_n,u) \leq \phi(y_n,u)\\
&= V(y_n,Ju)\\
&= V(x_n - \lambda_n AJx_n, Ju)\\
&\leq V((x_n - \lambda_n AJx_n) + \lambda_n AJx_n, Ju) - 2\langle \lambda_n AJx_n, J(x_n - \lambda_n AJx_n) - Ju\rangle\\
&= V(x_n,Ju) - 2\lambda_n\langle AJx_n, Jy_n - Ju\rangle\\
&= \phi(x_n,u) - 2\lambda_n\langle AJx_n, Jx_n - Ju\rangle + 2\langle -\lambda_n AJx_n, Jy_n - Jx_n\rangle
\end{aligned} \tag{16}$$

for all $n \in \mathbb{N}$. Since A is α-inverse-strongly-skew-monotone and $u \in VI(JC,A)$, it follows that

$$\begin{aligned}
-2\lambda_n\langle AJx_n, Jx_n - Ju\rangle &= -2\lambda_n\langle AJx_n - AJu, Jx_n - Ju\rangle - 2\lambda_n\langle AJu, Jx_n - Ju\rangle\\
&\leq -2\alpha\lambda_n\|AJx_n - AJu\|^2
\end{aligned} \tag{17}$$

for all $n \in \mathbb{N}$. By Lemma 2.1 and our assumption, we obtain

$$\begin{aligned}
2\langle -\lambda_n AJx_n, Jy_n - Jx_n\rangle &= 2\langle -\lambda_n AJx_n, J(x_n - \lambda_n AJx_n) - Jx_n\rangle\\
&\leq 2\|\lambda_n AJx_n\|\|J(x_n - \lambda_n AJx_n) - Jx_n\|\\
&\leq 2c\|\lambda_n AJx_n\|\|(x_n - \lambda_n AJx_n) - x_n\|\\
&= 2c\lambda_n^2\|AJx_n\|^2 \leq 2c\lambda_n^2\|AJx_n - Ju\|^2
\end{aligned} \tag{18}$$

for all $n \in \mathbb{N}$. From (16), (17) and (18), we get

$$\begin{aligned}
\phi(x_{n+1},u) &\leq \phi(x_n,u) + 2\lambda_n(\lambda_n c - \alpha)\|AJx_n - AJu\|^2\\
&\leq \phi(x_n,u) + 2a(bc - \alpha)\|AJx_n - AJu\|^2\\
&\leq \phi(x_n,u)
\end{aligned} \tag{19}$$

for all $n \in \mathbb{N}$. Thus $\lim_{n\to\infty}\phi(x_n,u)$ exists and hence, $\{\phi(x_n,u)\}$ is bounded. It implies that $\{x_n\}$ is bounded. By (19), we note that

$$-2a(bc - \alpha)\|AJx_n - AJu\|^2 \leq \phi(x_n,u) - \phi(x_{n+1},u) \tag{20}$$

for all $n \in \mathbb{N}$ and hence $\lim_{n\to\infty}\|AJx_n - AJu\|^2 = 0$. From Lemma 2.3 , Lemma 2.4 and (18), we have

$$\begin{aligned}
\phi(x_{n+1},x_n) &= \phi(R_C y_n, x_n)\\
&\leq \phi(y_n,x_n) = \phi(x_n - \lambda_n AJx_n, x_n)\\
&= V(x_n - \lambda_n AJx_n, Jx_n)\\
&\leq V((x_n - \lambda_n AJx_n) + \lambda_n AJx_n, Jx_n) - 2\langle \lambda_n AJx_n, J(x_n - \lambda_n AJx_n) - Jx_n\rangle\\
&= \phi(x_n,x_n) + 2\langle -\lambda_n AJx_n, Jy_n - Jx_n\rangle\\
&\leq 2c\lambda_n^2\|AJx_n - Ju\|^2 \leq 2cb^2\|AJx_n - Ju\|^2
\end{aligned}$$

for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty}\|AJx_n - AJu\|^2 = 0$, we have $\lim_{n\to\infty}\phi(x_{n+1},x_n) = 0$. Applying Lemma 2.2, we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{21}$$

From the uniform smoothness of E, we have

(22) $$\lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in E$. It follows that $x_{n_i+1} \rightharpoonup z$ as $i \rightharpoonup \infty$. We shall show that $z \in VI(JC, A)$. Let $B \subset E^* \times E$ be an operator as follows:

$$Bv^* = \begin{cases} Av^* + N_{JC}(v^*), & v^* \in JC, \\ \emptyset, & v^* \notin JC. \end{cases}$$

By Theorem 3.5, B is maximal skew monotone and $(BJ)^{-1}0 = VI(JC, A)$. Let $(v^*, w) \in G(B)$. Since $w \in Bv^* = Av^* + N_{JC}(v^*)$, it follows that $w - Av^* \in N_{JC}(v^*)$. From $Jx_{n+1} \in JC$, we have

(23) $$\langle w - Av^*, v^* - Jx_{n+1}\rangle \geq 0.$$

Since $w \in Bv^*$, we get $v^* \in JC$. This implies that there is $v \in C$ such that $Jv = v^*$. Thus it follow from (23) that

(24) $$\langle w - AJv, Jv - Jx_{n+1}\rangle \geq 0.$$

On the other hand, from $x_{n+1} = R_C(x_n - \lambda_n AJx_n)$ and Lemma 2.4, we have $\langle (x_n - \lambda_n AJx_n) - x_{n+1}, Jx_{n+1} - Jv\rangle \geq 0$ and hence

(25) $$\left\langle \frac{x_n - x_{n+1}}{\lambda_n} - AJx_n, Jv - Jx_{n+1} \right\rangle \leq 0.$$

From (24) and (25), we have

$$\begin{aligned}
\langle w, Jv - Jx_{n+1}\rangle &\geq \langle AJv, Jv - Jx_{n+1}\rangle \\
&\geq \langle AJv, Jv - Jx_{n+1}\rangle + \left\langle \frac{x_n - x_{n+1}}{\lambda_n} - AJx_n, Jv - Jx_{n+1} \right\rangle \\
&= \langle AJv - AJx_n, Jv - Jx_{n+1}\rangle + \left\langle \frac{x_n - x_{n+1}}{\lambda_n}, Jv - Jx_{n+1} \right\rangle \\
&= \langle AJv - AJx_{n+1}, Jv - Jx_{n+1}\rangle + \langle AJx_{n+1} - AJx_n, Jv - Jx_{n+1}\rangle \\
&\quad + \left\langle \frac{x_n - x_{n+1}}{\lambda_n}, Jv - Jx_{n+1} \right\rangle \\
&\geq -\frac{\|Jx_{n+1} - Jx_n\|}{\alpha}\|Jv - Jx_{n+1}\| - \frac{\|x_n - x_{n+1}\|}{a}\|Jv - Jx_{n+1}\| \\
&\geq -M\left(\frac{\|Jx_{n+1} - Jx_n\|}{\alpha} + \frac{\|x_n - x_{n+1}\|}{a}\right)
\end{aligned}$$

for all $n \in \mathbb{N}$, where $M = \sup\{\|Jv - Jx_{n+1}\| : n \in \mathbb{N}\}$. Taking $n = n_i$, from (21), (22) and the weakly sequential continuity of J, we obtain $\langle w, Jv - Jz\rangle \geq 0$ as $i \to \infty$. Hence, by the skew maximality of B, we obtain $Jz \in B^{-1}0$. That is $z \in (BJ)^{-1}0 = VI(JC, A)$.

From Corollary 3.6, we note that $VI(JC, A)$ is closed and $JVI(JC, A)$ is closed and convex. Put $u_n = R_{VI(JC,A)}(x_n)$ for all $n \in \mathbb{N}$. It holds from (19) and Lemma 4.1 that $\{u_n\}$ is Cauchy sequence. Since $VI(JC, A)$ is closed, $\{u_n\}$ converges strongly to $w \in VI(JC, A)$. By the uniform smoothness of E, we also have $\lim_{n\to\infty} \|Ju_n - Jw\| = 0$. Finally, we prove that $z = w$. From Lemma 2.4 and $z \in VI(JC, A)$, we have

(26) $$\langle x_n - u_n, Jz - Ju_n\rangle \leq 0$$

for all $n \in \mathbb{N}$. So, we get

$$\begin{aligned}
\langle x_n - u_n, Jz - Jw\rangle &= \langle x_n - u_n, Jz - Ju_n\rangle + \langle x_n - u_n, Ju_n - Jw\rangle \\
&\leq \|x_n - u_n\|\|Ju_n - Jw\| \\
&\leq K\|Ju_n - Jw\|
\end{aligned}$$

for all $n \in \mathbb{N}$, where $K = \sup\{\|x_n - u_n\| : n = 1, 2, ...\}$. Taking $n = n_i$, from $\lim_{n\to\infty} \|u_n - w\| = 0$ and $\lim_{n\to\infty} \|Ju_n - Jw\| = 0$, we obtain

$$\langle z - w, Jz - Jw\rangle \leq 0 \text{ as } i \to \infty.$$

This implies that $\langle z - w, Jz - Jw\rangle = 0$. Since E is strictly convex, it follows that $z = w$. Therefore the sequence $\{x_n\}$ converges weakly to $z = \lim_{n\to\infty} R_{VI(JC,A)}(x_n)$. This completes the proof. □

5. Application

In this section, we study the problem of finding a zero point of a maximal skew monotone operator of E^* into E and a minimizer of a continuously Fréchet differentiable and convex functional in a Banach space. To prove this, we need the following lemma:

Lemma 5.1. *(see* [3]*.) Let E be a Banach space, f a continuously Fréchet differentiable and convex function on E^* and ∇f the gradient of f. If ∇f is $1/\alpha$-Lipschitz continuous, then ∇f is α-inverse-strongly-skew-monotone.*

Now, we consider the problem of finding a zero point of a maximal skew monotone operator of E^* into E and a zero point of an inverse-strongly-skew-monotone operator of E^* into E. In the case where $JC = E^*$.

Theorem 5.2. *Let E be a uniformly convex and $2-$uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous. Let A be an α-inverse-strongly-skew-monotone of E^* into E with $A^{-1}0 \neq \emptyset$. Let $x_1 = x \in E$ and $\{x_n\}$ is given by*

$$x_{n+1} = x_n - \lambda_n AJx_n,$$

for every $n = 1, 2, ...$, where $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < \frac{\alpha}{c}$, where c is a constant in (2.1). Then the sequence $\{x_n\}$ converges weakly to some element z in $(AJ)^{-1}0$. Further $z = \lim_{n\to\infty} R_{(AJ)^{-1}(0)}(x_n)$.

Proof. From $R_E = I$, $VI(JE, A) = (AJ)^{-1}0$ and $\|AJy\| = \|AJy - 0\| = \|AJy - AJu\|$ for all $y \in E$ and $u \in (AJ)^{-1}0$, by using Theorem 4.2, $\{x_n\}$ converges weakly to some element z in $(AJ)^{-1}0$. □

Corollary 5.3. *Let E be a uniformly convex and $2-$uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous. Assume that f is a function on E^* such that f is a continuously Fréchet differentiable and convex function on E^*, ∇f is $1/\alpha$-Lipschitz continuous and $(\nabla f)^{-1}0 = \{z^* \in E^* : f(z^*) = \min_{y^*\in E^*} f(y^*)\} \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$ and*

$$x_{n+1} = x_n - \lambda_n(\nabla f)Jx_n,$$

for every $n = 1, 2, ...$, where $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < \frac{\alpha}{c}$, where c is a constant in (2.1). Then the sequence $\{x_n\}$ converges weakly to some element z in $((\nabla f)J)^{-1}0$. Further $z = \lim_{n\to\infty} R_{((\nabla)fJ)^{-1}0}(x_n)$.

Proof. By Lemma 5.1, we have ∇f is an α-inverse-strongly-skew-monotone operator of E^* into E. Hence, by Theorem 5.2, $\{x_n\}$ converges weakly to some element z in $((\nabla f)J)^{-1}0$. □

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