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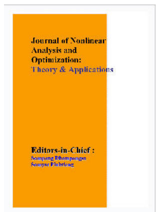

Asymptotic stability of stochastic pantograph differential equations with Markovian switching

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ABSTRACT: In this paper, we investigate the almost surely asymptotic stability of the nonlinear stochastic pantograph differential equations (SPDEs) with Markovian switching under the weakened linear growth condition. Linear SPDEs with Markovian switching and nonlinear examples with Markovian switching will be discussed to illustrate the theory.

KEYWORDS: Stochastic pantograph differential equations; Asymptotic stability; Generalized Itô formula; Markov chain.

1. Introduction

Recently, the study of stochastic pantograph differential equations (SPDEs) has received a great deal of attention. For example, Baker and Buckwar [1] gave the necessary analytical theory for existence and uniqueness of a strong solution of the linear stochastic pantograph equation, and of strong approximations to the solution obtained by a continuous extension of the θ-Euler scheme. They also proved that the numerical solution produced by the continuous θ-method converges to the true solution with order $1/2$. Appleby and Buckwar [2] studied the asymptotic growth and delay properties of solutions of the linear stochastic pantograph equation. They give sufficient conditions on the parameters for solutions to grow at a polynomial rate on pth mean and in the almost sure sense. Fan et al. [3] investigated the existence and uniqueness of the solutions and convergence of semi-implicit Euler methods for stochastic pantograph equations under the local Lipschitz condition and the linear growth condition. Fan et al. [4] investigated the αth moment asymptotical stability of the analytic solution and the numerical methods for the stochastic pantograph equation by using the Razumikhin technique. Li et al. [5] investigated the convergence of the Euler method of the stochastic pantograph differential equations with Markovian switching under the weaker conditions.

The classical stochastic stability theory deals with not only moment stability but also almost sure stability [6–10]. However, to the best of our knowledge, most of the existing results on the linear stochastic pantograph differential equations[1,2,4,11] are about the moment stability, while little is known on the almost surely asymptotic stability for SPDEs with Markovian switching under the non-linear growth condition which is the main topic of the present paper.

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The paper is organised as follows. In Section 2, we introduce the SPDEs with Markovian switching. We investigate the almost surely asymptotic stability for the stochastic pantograph differential equations with Markovian switching under the non-linear growth condition in Section 3. In Section 4, Some examples are discussed to illustrate the theory.

2. SPDEs with Markovian switching

Throughout this paper,we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right-continuous while $\mathcal{F}_0$ contains all P-null sets). Moreover, $|\cdot|$ is the Euclidean norm in $\mathcal{R}^n$. Let x_0 be an $\mathcal{F}_0-$measurable R^n-valued random variable such that $E|x_0|^2 < \infty$. Let $w(t) = (w_t^1, \cdots, w_t^m)^{\mathrm{T}}, t \geq 0$, be a m-dimensional Brownian motion defined on the probability space.

Let $r(t), t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta) & \text{if } i \neq j, \\ 1 + \gamma_{ij}\delta + o(\delta) & \text{if } i = j, \end{cases}$$

where $\delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{i\neq j}\gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is useful to recall that a continuous-time Markov chain $r(t)$ with generator $\Gamma = \{\gamma_{ij}\}_{N\times N}$ can be represented as a stochastic integral with respect to a Poisson random measure ([12, 13]).

$$dr(t) = \int_R \overline{h}(r(t-), y)v(dt, dy),\ t \geq 0 \tag{1}$$

with initial value $r(0) = i_0 \in S$, where $v(dt, dy)$ is a Poisson random measure with intensity $dt \times m(dy)$ in which m is the Lebesgue measure on R while the explicit definition of $\overline{h} : S \times R \to R$ can be found in ([12, 13]) but we will not need it in this paper.

Consider an n-dimensional stochastic pantograph differential equations with Markovian switching

$$dx(t) = f(t, x(t), x(qt))dt + f(t, x(t), x(qt))dw(t). \tag{2}$$

on $t \geq 0$ with initial data $x(0) = x_0, 0 < q < 1$ and $r(0) = i_0 \in S$, where $f : R_+ \times R^n \times R^n \times S \to R^n$ and $g : R_+ \times R^n \times R^n \times S \to R^{n\times m}$.

In this paper, the following hypothesis is imposed on the coefficients f and g.

Assumption H. Both f and g satisfy the local Lipschitz condition. For each integer $h \geq 1$ and $i \in S$, there exists a positive constant L_h such that

$$|f(t, x_1, x_2, i) - f(t, y_1, y_2, i)| \vee |g(t, x_1, x_2, i) - g(t, y_1, y_2, i)| \leq L_h(|x_1 - y_1| + |x_2 - y_2|)$$

for $x_1, x_2, y_1, y_2 \in R^n$ with $|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \leq h$. Moreover,

$$\sup\{|f(t, 0, 0, i)| \vee |g(t, 0, 0, i)| : t \geq 0, i \in S\} < \infty.$$

In general, this hypothesis will only guarantee a unique maximal local solution to Eq. (2) for any given initial value x_0 and i_0. However, the additional conditions imposed in our main result, Theorem 3.1, will guarantee that this maximal local solution is in fact a unique global solution (see Lemma 3.2), which is denoted by $x(t; x_0; i_0)$ in this paper. The main purpose of this paper is to discuss the almost surely asymptotic stability of the solution([6, 14]).

To state our main result, we will need a few more notations. Let $C(R^n; R_+)$ and $C(R_+ \times R^n; R_+)$ denote the families of all continuous nonnegative functions defined on R^n and $R_+ \times R^n$, respectively. Moreover, let $\mathcal{K}$ denote the class of continuous increasing functions μ from R_+ to R_+ with $\mu(0) = 0$. Let $\mathcal{K}_\infty$ denote the class of functions μ with $\mu(s) \to \infty$ as $s \to \infty$. Functions in $\mathcal{K}$ and $\mathcal{K}_\infty$ are called class $\mathcal{K}$ and $\mathcal{K}_\infty$ functions, respectively. If $\mu \in \mathcal{K}$, its iverse function is denoted by μ^{-1} with domain $[0, \mu(\infty))$. We also denote by $L^1(R_+; R_+)$ the family of all functions $\gamma : R_+ \to R_+$ such that $\int_0^\infty \gamma(t)dt < \infty$. If E is a subset of R^n, denote by $d(x, E)$ the Haussdorff semi-distance between $x \in R^n$ and the set E,namely $d(x, E) = \inf_{y\in E}|x - y|$.

If W is a real-valued function defined on R^n, then its kernel is denoted by $Ker(W)$, namely $Ker(W) = \{x \in R^n : W(x) = 0\}$. Let $C^{1,2}(R_+ \times R^n \times S; R_+)$ denote the family of all non-negative functions $V(t,x,i)$ on $R_+ \times R^n \times S$ which are continuously twice differentiable in x and once differentiable in t. If $V \in C^{1,2}(R_+ \times R^n \times S; R_+)$, define an operator LV from $R_+ \times R^n \times R^n \times S$ to R by

$$\begin{aligned} LV(t,x,y,i) &= V_t(t,x,i) + V_x(t,x,i)f(t,x,y,i) + \frac{1}{2}trace[g^T(t,x,y,i)V_{xx}(t,x,i)g(t,x,y,i)] \\ &\quad + \sum_{j=1}^{N} \gamma_{ij} V(t,x,j), \end{aligned} \tag{3}$$

where
$V_t(t,x,i) = \frac{\partial V(t,x,i)}{\partial t}$, $V_x(t,x,i) = (\frac{\partial V(t,x,i)}{\partial x_1}, ..., \frac{\partial V(t,x,i)}{\partial x_n})$, $V_{xx}(t,x,i) = (\frac{\partial^2 V(t,x,i)}{\partial x_i \partial x_j})_{n\times n}$.
For the convenience of the reader we cite the generalized Itô's formula ([14]): If $V \in C^{1,2}(R_+ \times R^n \times S)$, then for any $t \geq 0$

$$\begin{aligned} V(t,x(t),r(t)) &= V(0,x(0),r(0)) + \int_0^t LV(s,x(s),x(qs),r(s))ds + \int_0^t V_x(s,x(s),r(s))dw(s) \\ &\quad + \int_0^t \int_R (V(s,x(s),i_0 + \overline{h}(r(s-),l)) - V(s,x(s),r(s)))u(ds,dl), \end{aligned} \tag{4}$$

where $u(ds,dl) = v(ds,dl) - m(dl)ds$ is a martingale measure.

To establish our main result for locating limit sets of the solutions of the stochastic pantograph equations with Markovian switching, let us cite the useful convergence theorem of nonnegative semi-martingale ([15])as a lemma.

Lemma 2.1 *Let $A(t)$ and $U(t)$ be two continuous adapted increasing processes on $t \geq 0$ with $A(0) = U(0) = 0$ a.s. Let $M(t)$ be a real-valued continuous local martingale with $M(0) = 0$ a.s. Let ξ be a nonnegative $\mathcal{F}_0-$measurable random variable. Define*

$$X(t) = \xi + A(t) - U(t) + M(t) \text{ for } t \geq 0.$$

If $X(t)$ is nonnegative, then

$$\{\lim_{t\to\infty} A(t) < \infty\} \subset \{\lim_{t\to\infty} X(t) < \infty\} \cap \{\lim_{t\to\infty} U(t) < \infty\} \text{ a.s.},$$

where $B \subset D$ a.s. means $P(B \cap D_c) = 0$. In particular, if $\lim_{t\to\infty} A(t) < \infty$ a.s., then, with probability one,

$$\lim_{t\to\infty} X(t) < \infty, \ \lim_{t\to\infty} U(t) < \infty \text{ and } -\infty < \lim_{t\to\infty} M(t) < \infty.$$

That is , all of the three processes $X(t), U(t)$ and $M(t)$ converge to finite random variable.

3. **Asymptotic stability**

With the above notations, we can now state our main result.

Theorem 3.1 *Let (H) hold. Assume that there are functions $V \in C^{1,2}(R_+ \times R^n \times S; R_+), \gamma \in L^1(R_+; R_+)$ and $w_1, w_2 \in C(R^n; R_+)$ such that*

$$LV(t,x,y,i) \leq \gamma(t) - w_1(x) + qw_2(y) \tag{5}$$

for all $(t,x,y,i) \in R_+ \times R^n \times R^n \times S$ and

$$w_1(0) = w_2(0) = 0, \ w_1(x) > w_2(x) \text{ for all } x \neq 0. \tag{6}$$

and

$$\lim_{|x|\to\infty} \Big[\inf_{(t,i)\in R_+\times S} V(t,x,i) \Big] = \infty. \tag{7}$$

Then for any initial value x_0,

$$\lim_{t\to\infty} x(t;x_0,i_0) = 0 \text{ a.s.} \tag{8}$$

That is, the solution of Eq. (2) is almost surely asymptotically stable.

To prove this theorem, we can also give the following lemma by the standard truncated technique (see e.g. [14]).

Lemma 3.2 *Under the conditions of Theorem 3.1, for any initial vulue x_0 and $r(0) = i_0 \in S$, Eq. (2) has a unique global solution.*

Let us now begin to prove our main result.

Proof of Theorem 3.1 We divide the proof into three steps.

Step 1. Fix any x_0 and i_0 and write $x(t; x_0, i_0) = x(t)$ for simplicity. By the generalized Itô's formula, (5) and (6) we derive that

$$\begin{aligned} V(t,x(t),r(t)) &\leq V(0,x(0),r(0)) + M(t) + \int_0^t [\gamma(s) - w_1(x(s)) + qw_2(x(qs))]ds \\ &\leq V(0,x(0),r(0)) + \int_0^t \gamma(s)ds - \int_0^t [w_1(x(s)) - w_2(x(s))]ds + M(t) \end{aligned} \tag{9}$$

where

$$M(t) = \int_0^t V_x(s,x(s),r(s))dw(s) + \int_0^t \int_R (V(s,x(s),i_0 + \bar{h}(r(s-),l)) - V(s,x(s),r(s)))u(ds,dl),$$

which is a continuous local martingale with $M(0) = 0$ a.s. Applying Lemma 2.1 we immediately obtain

$$\limsup_{t\to\infty} V(t,x(t),r(t)) < \infty \text{ a.s.} \tag{10}$$

Moreover, taking the expectations on both sides of (9) and letting $t \to \infty$, we obtain that

$$E\int_0^\infty [w_1(x(s)) - w_2(x(s))]ds < \infty \text{ a.s.} \tag{11}$$

This implies

$$\int_0^\infty [w_1(x(s)) - w_2(x(s))]ds < \infty \text{ a.s.} \tag{12}$$

Step 2. Set $\omega = w_1 - w_2$. Clearly, $\omega \in C(R^n; R_+)$. It is straightforward to see from (12) that

$$\liminf_{t\to\infty} \omega(x(t)) = 0 \text{ a.s.} \tag{13}$$

We now claim that

$$\lim_{t\to\infty} \omega(x(t)) = 0 \text{ a.s.} \tag{14}$$

If this is false, then

$$P\{\limsup_{t\to\infty} \omega(x(t)) > 0\} > 0.$$

Hence, there is a number $\varepsilon > 0$ such that

$$P(\Omega_1) \geq 3\varepsilon, \tag{15}$$

where

$$\Omega_1 = \{\limsup_{t\to\infty} \omega(x(t)) > 2\varepsilon\}.$$

It is easy to observe from (10) and continuity of both the solution $x(t)$ and the function $V(t,x,i)$ that

$$\sup_{0\leq t<\infty} V(t,x(t),r(t)) < \infty \text{ a.s.}$$

Define $\rho : R_+ \to R_+$ by

$$\rho(r) = \inf_{|x|\geq r, 0\leq t<\infty} V(t,x(t),i) \text{ for } r \geq 0.$$

Obviously,

$$\sup_{0\le t<\infty} \rho(|x(t)|) \le \sup_{0\le t<\infty} V(t,x(t),r(t)) < \infty \text{ a.s.}$$

On the other hand, by (7) we have

$$\lim_{r\to\infty} \rho(r) = \infty.$$

Therefore

$$\sup_{0\le t<\infty} |x(t) < \infty \text{ a.s.} \tag{16}$$

Recalling the boundedness of the initial value we can then find a positive number h, which depends on ε, sufficiently large for $|x_0| < h$, while

$$P(\Omega_2) \ge 1-\varepsilon, \tag{17}$$

where

$$\Omega_2 = \{ \sup_{0\le t<\infty} |x(t)| < h\}.$$

It is easy to see from (15)and (17) that

$$P(\Omega_1 \cap \Omega_2) \ge 2\varepsilon. \tag{18}$$

We now define a sequence of stopping times,

$$\begin{aligned}
\tau_h &= \inf\{t \ge 0 : |x(t)| \ge h\},\\
\sigma_1 &= \inf\{t \ge 0 : \omega(x(t)) \ge 2\varepsilon\},\\
\sigma_{2k} &= \inf\{t \ge \sigma_{2k-1} : \omega(x(t)) \le \varepsilon\},\ k = 1,2,...,\\
\sigma_{2k+1} &= \inf\{t \ge \sigma_{2k} : \omega(x(t)) \ge \varepsilon\},\ k = 1,2,...,
\end{aligned}$$

where throughout this paper e set $\inf \phi = \infty$. Note from (13) and the definition of Ω_1 and Ω_2 that if $\varpi \in \Omega_1 \cap \Omega_2$, then

$$\tau_h = \infty \text{ and} \sigma_k < \infty\ \forall k \ge 1. \tag{19}$$

Let I_A is denote the indicator function of set A, using the fact $\sigma_{2k} < \infty$ whenever $\sigma_{2k-1} < \infty$ and (13), by (11) we can compute

$$\begin{aligned}
\infty &> E\int_0^\infty \omega(x(t))dt\\
&\ge \sum_{k=1}^\infty E\Big[I_{\{}\sigma_{2k-1} < \infty, \sigma_{2k} < \infty, \tau_h = \infty\} \int_{\sigma_{2k-1}}^{\sigma_{2k}} \omega(x(t))dt\Big]\\
&\ge \varepsilon \sum_{k=1}^\infty E\Big[I_{\{}\sigma_{2k-1} < \infty, \tau_h = \infty\}(\sigma_{2k} - \sigma_{2k-1})\Big].
\end{aligned} \tag{20}$$

On the other hand, by hypothesis (H),there exists a constant $K_h > 0$ such that $|f(t,x,y,i)|^2 \vee |g(t,x,y,i)|^2 \le K_h$ whenever $|x| \vee |y| \le h$. By Hölder's inequality and Doob's martingale inequality, we easily compute

$$E\Big[I_{\{}\sigma_{2k-1} \wedge \tau_h < \infty\} \sup_{0\le t\le T} |x(\tau_h \wedge (\sigma_{2k-1}+t)) - x(\tau_h \wedge \sigma_{2k-1})|^2\Big] \le 2K_h(T+4)T. \tag{21}$$

Since $\omega(\cdot)$ is continuous in R^n, it must be uniformly continuous in the closed ball $\overline{S}_h = \{x \in R^n : |x| \le h\}$. We can therefore choose $\delta = \delta(\varepsilon) > 0$ so small such that

$$|\omega(x) - \omega(x)| < \varepsilon \text{ whenever } x, y \in \overline{S}_h, |x-y| < \delta. \tag{22}$$

We furthermore chose $T = T(\varepsilon,\delta,h) > 0$ sufficiently small for $2K_h(T+4)T/\delta^2 < \varepsilon$. It then follows from (21) that

$$P\Big(\{\sigma_{2k-1} \wedge \tau_h < \infty\} \cap \Big\{ \sup_{0\le t\le T} |x(\tau_h \wedge (\sigma_{2k-1}+t)) - x(\tau_h \wedge \sigma_{2k-1})| \ge \delta\Big\}\Big) \le \frac{2K_h(T+4)T}{\delta^2} < \varepsilon.$$

This together with (18) and (19) yields

$$P\Big(\{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \Big\{ \sup_{0\le t\le T} |x(\sigma_{2k-1}+t) - x(\sigma_{2k-1})| \ge \delta\Big\}\Big) \le \varepsilon.$$

and

$$P\Big(\{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \Big\{ \sup_{0\le t\le T} |x(\sigma_{2k-1}+t) - x(\sigma_{2k-1})| < \delta\Big\}\Big) \ge \varepsilon.$$

Using (22), we derive that

$$(23)\quad P\Big(\{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \Big\{ \sup_{0\le t\le T} |\omega(x(\sigma_{2k-1}+t)) - \omega(x(\sigma_{2k-1}))| < \varepsilon Big\}\Big) \ge \varepsilon.$$

Set

$$\overline{\Omega}_k = \Big\{ \sup_{1\le t\le T} |\omega(x(\sigma_{2k-1}+t)) - \omega(x(\sigma_{2k-1}))| < \varepsilon\Big\}.$$

Noting that

$$\sigma_{2k}(\varpi) - \sigma_{2k-1}(\varpi) \ge T \text{ if } \varpi \in \{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \overline{\Omega}_k,$$

we derive from (20) and (23) that

$$\begin{aligned} \infty &> \varepsilon \sum_{k=1}^{\infty} E[I_{\{}\sigma_{2k-1} < \infty, \tau_h = \infty\}(\sigma_{2k} - \sigma_{2k-1})] \\ &\ge \varepsilon \sum_{k=1}^{\infty} E[I_{\{\sigma_{2k-1}<\infty,\tau_h=\infty\}\cap\overline{\Omega}_k}(\sigma_{2k} - \sigma_{2k-1})] \\ &\ge \varepsilon T \sum_{k=1}^{\infty} P(\{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \overline{\Omega}_k) \\ &\ge \varepsilon T \sum_{k=1}^{\infty} \varepsilon = \infty, \qquad (24) \end{aligned}$$

which is a contradiction. So (14) must hold.

Step 3. We observe from (14) and (16) there is an $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that

$$(25)\qquad \lim_{t\to\infty} \omega(x(t,\varpi)) = 0 \text{ and } \sup_{0\le t<\infty} |x(t,\varpi)| < \infty \text{ for all } \varpi \in \Omega_0.$$

We shall now show that

$$(26)\qquad \lim_{t\to\infty} x(t,\varpi) = 0\ \forall \varpi \in \Omega_0.$$

If this is false, then there is some $\hat{\sigma} \in \Omega_0$ such that

$$\limsup_{t\to\infty} |x(t,\hat{\sigma})| > 0,$$

whence there is a subsequence $\{x(t_k,\hat{\sigma})\}_{k\ge1}$ of $\{x(t,\hat{\sigma})\}_{t\ge0}$ such that

$$|x(t_k,\hat{\sigma})| \ge \alpha \forall\ k \ge 1$$

for some $\alpha > 0$. Since $\{x(t_k,\hat{\sigma})\}_{k\ge1}$ is bounded so there must be an increasing subsequence $\{\bar{t}_k\}_{k\ge1}$ such that $\{x(\bar{t}_k,\varpi)\}_{k\ge1}$ converges to some $z \in R^n$ with $|z| \ge \alpha$. Hence

$$\omega(z) = \lim_{k\to\infty} \omega(x(t_k,\varpi)) > 0.$$

However, by (25), $\omega(z) = 0$. This is a contradiction and hence (26) must hold. This implies that the solution of Eq. (2) is almost surely asymptotically stable and the proof is therefore complete.

It is not difficult to observe from the proof of Theorem 3.1 that the following more general result holds.

Theorem 3.3 *Assume that all the conditions of Theorem 3.1 hold except Condition (6) is replaced by*

$$w_1(x) \ge w_2(x),\ x \in R^n.$$

Then

$$Ker(w_1 - w_2) \neq \phi \text{ and } \lim_{t\to\infty} d(x(t;x_0,i_0), Ker(w_1 - w_2)) = 0 \text{ a.s.}$$

4. Examples

In this section we discuss a linear example and a nonlinear example to illustrate our theory. In the following examples we let $w(t)$ be a scalar Brownian motion.

Example 4.1 Let $r(t)$ be a right-continuous Markov chain. Assume that $w(t)$ and $r(t)$ are independent. Consider a one-dimensional linear autonomous stochastic pantograph differential with Markovian switching of the form

$$d(x(t)) = [A(r(t))x(t) + B(r(t))x(qt)]dt + [C(r(t))x(t) + D(r(t))x(qt)]dw(t) \tag{27}$$

on $t \geq 0$. For $i \in S$, we will write $A(r(t)) = A_i, B(r(t)) = B_i, C(r(t)) = C_i, D(r(t)) = D_i$, for simplicity. Let $V(t,x,i) = |x|^2$. Then

$$\begin{aligned} LV(t,x,y,i) &= 2x(A_ix + B_iy) + (C_ix + D_iy)^2 \\ &\leq (2A_i + |B_i| + |C_iD_i| + C_i^2)x^2 + (|B_i| + |C_iD_i| + D_i^2)y^2 \end{aligned}$$

By Theorem 4.1, if $1 + 2A_i + |B_i| + |C_iD_i| + C_i^2 = 0$ and $|B_i| + |C_iD_i| + D_i^2 < q$, we can conclude that the solution of Eq. (27) is almost surely asymptotically stable.

Example 4.2 Let $r(t)$ be a right-continuous Markov chain taking values in $S \in \{1,2\}$ with generator

$$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Assume that $w(t)$ and $r(t)$ are independent. Assume that $B(t)$ and $r(t)$ are independent. Consider a one-dimensional stochastic differential pantograph equation with Markov switching of the form

$$d(x(t)) = f(t,x(t),r(t))dt + g(t,x(qt),r(t))dw(t) \tag{28}$$

on $t \geq 0$ as $1/2 < q < 1$, where

$$f(t,x,1) = \frac{1}{4}x\sin t, f(t,x,2) = e^{-t} - 4x - 3x^3, g(t,x,1) = \frac{1}{8}x\cos t, g(t,x,2) = \frac{1}{\sqrt{2}}x\sin t,$$

Clearly

$$xf(t,x,1) \leq \frac{1}{4}|x|^2, xf(t,x,2) \leq |x|e^{-t} - 4x^2, g^2(t,x,1) \leq \frac{1}{64}|x|^2, g^2(t,x,2) = \frac{1}{2}|x|^2$$

for all $(t,x) \in (R_+, R)$. To examine the asymptotic stability, we construct a function $V : R \times S \to R_+$ by $V(x,i) = \beta_i|x|^2$ with $\beta_2 = 1$ and $\beta_1 = \beta$ a constant to be determined. It is easy to show that the operator LV from $R_+ \times R \times R \times S$ to R has the form

$$LV(t,x,y,i) = 2\beta_ixf(t,x,i) + \beta_i|g(t,y,i)|^2 + (\gamma_{i1}\beta + \gamma_{i2})|x|^2.$$

By the conditions, we then have

$$LV(t,x,y,1) \leq -(\frac{\beta}{2} - 1)x^2 + \frac{\beta}{64}y^2,$$

and

$$LV(t,x,y,2) \leq 2|x|e^{-t} + (2\beta - 10)x^2 + \frac{1}{2}y^2$$

Setting $\beta = 4$, and noting that $2|x|e^{-t} \leq x^2 + e^{-2t}$, we then have

$$LV(t,x,y,i) \leq e^{-2t} - x^2 + \frac{1}{2}y^2.$$

Although f does not satisfy the linear growth condition, by Theorem 3.1, we can also conclude that the solution of Eq. (28) is almost surely asymptotically stable.

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