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CONVERGENCE THEOREMS FOR OPERATORS WITH PROPERTY (E) IN $CAT(0)$ SPACES

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ABSTRACT. In this paper, we study the convergence of the SP*-iteration process to fixed point for operators with property (E) in $CAT(0)$ spaces. We also prove the stability of the SP*-iteration process in $CAT(0)$ spaces. Our results improve and extend some recently results in the literature of fixed point theory in $CAT(0)$ spaces.

KEYWORDS:Fixed point, iteration process, stability, Δ-convergence, $CAT(0)$ space, Garcia-Falset mapping.
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1. Introduction

It is essential for many fields of study, including mathematics, to have fixed points. The conditions under which maps have solutions are given by fixed point results. In particular, fixed point methods have been applied in many fields, such as informatics, biology, chemistry, economics, and engineering. Determining the precise value of the intended fixed point is a crucial and ultimately the last step in solving the problem, but determining its existence is a crucial initial step. Using an iterative procedure is one of the best ways to obtain the intended fixed point. A number of researchers have recently shown interest in these areas and have developed iterative procedures that have been investigated to estimate fixed points for a larger class of nonexpansive mappings as well as for nonexpansive mappings. The existence of a fixed point is very important in several areas of mathematics and other sciences. The numerous numbers of researchers attracted in these direction and developed iterative process has been investigated to approximate fixed point for not only nonexpansive mapping, but also for some wider class of nonexpansive mappings. This is an active area of research, several well known scientists in the world paid and still pay attention to the qualitative study of iteration methods. The well-known

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Banach contraction theorem use Picard iteration process [28] for approximation of fixed point. Some of the well-known iterative processes are those of Mann [24], Ishikawa [17], Noor [25], SP-iteration [29], Picard Normal S-iteration [18] and so on. Let $\mathcal{X}$ be a real Banach space and $\mathcal{M}$ be a nonempty subset of $\mathcal{X}$, and $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ be a mapping. We have $\{\tau_n\}, \{\sigma_n\}$ and $\{\kappa_n\}$ real sequences in $[0, 1]$. Recently, Phuengrattana and Suantai ([29]) defined the SP-iteration as follows:

$$\begin{cases} z_n = (1-\kappa_n)u_n + \kappa_n \mathcal{G}u_n, \\ v_n = (1-\sigma_n)z_n + \sigma_n \mathcal{G}z_n, \\ u_{n+1} = (1-\tau_n)v_n + \tau_n \mathcal{G}v_n, \forall n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $u_1 \in \mathcal{M}$. They showed that the Mann, Ishikawa, Noor and SP-iterations are equivalent and the SP-iteration converges better than the others for the class of continuous and nondecreasing functions. In 2014, Kadıoglu and Yıldırım [18] introduced Picard Normal S-iteration process and they established that the rate of convergence of the Picard Normal S-iteration process is faster than other fixed point iteration process that was in existence then. The Picard Normal S-iteration [18] as follows:

$$\begin{cases} z_n = (1-\sigma_n)u_n + \sigma_n \mathcal{G}u_n, \\ v_n = (1-\tau_n)z_n + \tau_n \mathcal{G}z_n, \\ u_{n+1} = \mathcal{G}v_n, \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $u_1 \in \mathcal{M}$.

In 2021, Temir and Korkut [35] introduced SP*-iteration process and they established that the rate of convergence of the SP*-iteration scheme is faster than above iteration processes. Now we give SP*-iteration process:for arbitrary $u_1 \in \mathcal{M}$ construct a sequence $\{u_n\}$ by

$$\begin{cases} z_n = \mathcal{G}((1-\kappa_n)u_n + \kappa_n \mathcal{G}u_n), \\ v_n = \mathcal{G}((1-\sigma_n)z_n + \sigma_n \mathcal{G}z_n), \\ u_{n+1} = \mathcal{G}((1-\tau_n)v_n + \tau_n \mathcal{G}v_n), \forall n \in \mathbb{N}. \end{cases} \tag{1.3}$$

Some generalizations of nonexpansive mappings and the study of related fixed point theorems have been intensively carried out over past decades [1, 4, 14, 26, 27, 33, 34, 36, 37]. A class of generalized nonexpansive mappings (in short GNMs) on a nonempty subset $\mathcal{M}$ of a Banach space $\mathcal{X}$ has been defined by Suzuki [33]. Such mappings were referred to as belonging to the class of mappings satisfying condition (C) (also referred as Suzuki GNM), which properly includes the class of nonexpansive mappings. Every self-mapping $\mathcal{G}$ on $\mathcal{M}$ providing condition (C) has an almost fixed point sequence for a nonempty bounded and convex subset $\mathcal{M}$. Two new classes of GNMs that are wider than those providing the condition (C) were presented in 2011 by Garcia-Falset et al. [14], while retaining their fixed point properties. The resulting property was called condition (E) (in the sequel, the class of mappings satisfying condition (E) will be referred to as Garcia-Falset-generalized nonexpansive mappings or Garcia-Falset mappings).

In this paper, we apply SP*-iteration (1.3) for operators with property (E) in the context of $CAT(0)$ space as follows

$$\begin{cases} z_n = \mathcal{G}((1-\kappa_n)u_n \oplus \kappa_n \mathcal{G}u_n), \\ v_n = \mathcal{G}((1-\sigma_n)z_n \oplus \sigma_n \mathcal{G}z_n), \\ u_{n+1} = \mathcal{G}((1-\tau_n)v_n \oplus \tau_n \mathcal{G}v_n), \forall n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\mathcal{M}$ is a nonempty closed convex subset of a $CAT(0)$ space, $u_1 \in \mathcal{M}$, $\{\tau_n\}$, $\{\sigma_n\}$ and $\{\kappa_n\} \in [0, 1]$.

Inspired and motivated by these facts, in this paper, we prove some convergence theorems of SP*-iterative process generated by (1.4) to fixed point of operators with Property (E) in $CAT(0)$ spaces. In 2021, Temir and Korkut [35] introduced the iterative process generated by (1.4) (SP*-iteration process) and they established that the rate of convergence of the SP*-iteration process is faster than the SP-iteration process and the Picard Normal S-iteration process. Since only the convergence analysis of the SP*-iterative process was studied in [35], we also prove the stability of the SP*-iterative process in this study. In addition, we provide an example that satisfies condition (E) but the mapping is neither a generalized α-nonexpansive mapping nor does it satisfy condition (C).

2. Preliminaries

First we present some basic concepts and definitions.

Let $\mathcal{G}$ be a self-mapping defined on a nonempty subset of a $CAT(0)$ space. A point $u \in \mathcal{M}$ is called a fixed point of $\mathcal{G}$ if $\mathcal{G}u = u$ and we denote by $Fix(\mathcal{G})$ the set of fixed points of $\mathcal{G}$, that is, $Fix(\mathcal{G}) = \{u \in \mathcal{M} : \mathcal{G}u = u\}$. A mapping $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ is called contraction if there exists $\theta \in [0, 1)$ such that

$$d(\mathcal{G}u, \mathcal{G}v) \leq \theta d(u, v),$$

for all $u, v \in \mathcal{M}$. If $\theta = 1$ in inequality above, then $\mathcal{G}$ is said to be a nonexpansive mapping.

Definition 2.1. A mapping $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ satisfies condition (C) on $\mathcal{M}$ if for all $u, v \in \mathcal{M}$, $\frac{1}{2}d(u, \mathcal{G}u) \leq d(u, v) \Rightarrow d(\mathcal{G}u, \mathcal{G}v) \leq d(u, v)$.

Suzuki [33] showed that the mapping satisfying condition (C) is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. In 2017, Pant and Shukla [26] introduced a new type of nonexpansive mappings called generalized α-nonexpansive mappings and obtain a number of existence and convergence theorems. This new class of nonlinear mappings properly contains nonexpansive, Suzuki-type GNMs and partially extends firmly nonexpansive and α-nonexpansive mappings.

Definition 2.2. A mapping $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ is called a generalized α-nonexpansive mapping if there exists an $\alpha \in [0, 1)$ and for each $u, v \in \mathcal{M}$,

$$\frac{1}{2}d(u, \mathcal{G}u) \leq d(u, v) \text{ implies } d(\mathcal{G}u, \mathcal{G}v) \leq \alpha d(\mathcal{G}u, v) + \alpha d(\mathcal{G}v, u) + (1 - 2\alpha)d(u, v).$$

Recently, Garcia-Falset et al. [14] studied GNMs satisfying condition (E) that have a weaker property than Suzuki GNMs.

Definition 2.3. A mapping $\mathcal{G} : \mathcal{M} \to \mathcal{X}$ satisfies condition (E_μ) on $\mathcal{M}$, if there exists $\mu \geq 1$ such that

$$d(u, \mathcal{G}v) \leq \mu d(u, \mathcal{G}u) + d(u, v)$$

for all $u, v \in \mathcal{M}$.

Moreover, it is said that $\mathcal{G}$ satisfies condition (E) on $\mathcal{M}$, whenever $\mathcal{G}$ satisfies condition (E_μ), for some $\mu \geq 1$. It is clearly seen that if $\mathcal{G} : \mathcal{M} \to \mathcal{X}$ is nonexpansive, then it satisfies condition (E_1) and from Lemma 7 in [33] we know that if $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ satisfies condition (C) on $\mathcal{M}$, then $\mathcal{G}$ satisfies condition (E_3) (see [14]). By Lemma 5.2 in [26], if $\mathcal{G} : \mathcal{M} \to \mathcal{X}$ is a generalized α-nonexpansive mapping, then it satisfies condition (E) on $\mathcal{M}$; see [26] for a proof. Therefore, the class of generalized α-nonexpansive mappings is subordinated to the class of mappings satisfying condition (E). Proposition 1 in [14], we know also that if $\mathcal{G} : \mathcal{M} \to \mathcal{X}$ a mapping which satisfies condition (E) on $\mathcal{M}$ has some fixed point, then $\mathcal{G}$ is quasi-nonexpansive. Example 2 that is in [14] shows the converse is not true.

It is well-known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a $CAT(0)$ space. Other examples include Pre-Hilbert spaces, any convex subset of a Euclidian space $\mathbb{R}^n$ with the induced metric, the complex Hilbert ball with a hyperbolic metric and many others. For discussion of these spaces and of the fundamental role they play in geometry see Bridson and Haefliger [6]. Burago et al. [8] contains a somewhat more elementary treatment, and Gromov [15] a deeper study. Fixed point theory in $CAT(0)$ space has been first studied by Kirk (see [19],[20]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete $CAT(0)$ space always has a fixed point. On the other hand, we know that not every Banach space is a $CAT(0)$ space. Since then the fixed point theory in CAT(0) has been rapidly developed and much papers a appeared.(see [9],[10],[11],[12],[13],[19],[20],[21], [22]).

Recently, Kirk and Panyanak [22] used the concept of $\Delta-$convergence introduced by Lim [23] to prove on the $CAT(0)$ space analogs of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [9] obtained $\Delta-$convergence theorems for the Picard, Mann and Ishikawa iteration processes for nonexpansive mappings in the $CAT(0)$ space. In addition, the convergence results for generalized nonexpansive mappings are obtained by using different iteration processes in $CAT(0)$ spaces (see [2], [3], [30], [31]).

If u, v_1, v_2 are points of a $CAT(0)$ spaces, and and if v_0 is the midpoint of the segment $[y_1, y_2]$ then the $CAT(0)$ inequality implies

$$d^2(u, v_0) \leq \frac{1}{2}d^2(u, v_1) + \frac{1}{2}d^2(u, v_2) - \frac{1}{4}d^2(v_1, v_2).$$

This is the (CN) inequality of Bruhat and Tits [7]. In fact, a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality ([[6], p. 163]).

In the sequel, we need the following definitions and useful lemmas to prove our main results of this paper.

Lemma 2.4. *([9]) Let $\mathcal{X}$ be a $CAT(0)$ space.*

(i) For $u, v \in \mathcal{X}$ and $t \in [0, 1]$, there exists a unique point $z \in [u, v]$ such that $d(u, z) = td(u, v)$ and $d(v, z) = (1 - t)d(u, v)$.

(ii) For $u, v \in \mathcal{X}$ and $t \in [0, 1]$, we have $d((1-t)u \oplus tv, z) \leq (1-t)d(u, z)+td(v, z)$.

Let $\{u_n\}$ be a bounded sequence in a closed convex subset $\mathcal{M}$ of a $CAT(0)$ space $\mathcal{X}$. For $x \in \mathcal{X}$, set $r(u, \{u_n\}) = \limsup_{n\to\infty} d(u, u_n)$. The asymptotic radius $r(\{u_n\})$ of $\{u_n\}$ is given by $r(\mathcal{M}, \{u_n\}) = \inf_n \{r(u, \{u_n\}) : u \in \mathcal{M}\}$ and the asymptotic center of u_n relative to K is the set $A(\mathcal{M}, \{u_n\}) = \{u \in \mathcal{M} : r(x, \{u_n\}) = r(\mathcal{M}, \{u_n\})\}$. It is known that, in a $CAT(0)$ space, $A(\mathcal{M}, \{u_n\})$ consists of exactly one point; see [12], Proposition 7.

We now recall the definition of Δ-convergence and weak convergence in $CAT(0)$ space.

Definition 2.5. ([22],[23]) A sequence $\{u_n\}$ in a $CAT(0)$ space $\mathcal{X}$ is said to Δ-converge to $u \in \mathcal{X}$ if u is the unique asymptotic center of every subsequence $\{u_n\}$. In this case we write $\Delta - \lim_{n\to\infty} u_n = u$ and call u is the $\Delta-$limit of $\{u_n\}$.

Lemma 2.6. *([22]) Given $\{u_n\} \in X$ such that $\{u_n\}$, Δ-converges to u and given $v \in \mathcal{X}$ with $v \neq u$, then $\limsup_{n\to\infty} d(u_n, u) < \limsup_{n\to\infty} d(u_n, v)$.*

Lemma 2.7. *([22]) Every bounded sequence in a complete $CAT(0)$ space always has a Δ-convergent subsequence.*

Lemma 2.8. *([11]) Let $\mathcal{M}$ be closed convex subset of a complete $CAT(0)$ space and $\{u_n\}$ be a bounded sequence in $\mathcal{M}$. Then asymptotic center of $\{u_n\}$ is in $\mathcal{M}$.*

Next, Harder and Hicks [16] introduced the following definition of $\mathcal{G}$-stability :

Definition 2.9. ([16]) Let $\{t_n\}_{n=1}^{\infty}$ be an arbitrary sequence in $\mathcal{M}$. Then , an iteration process

$$t_{n+1} = f(\mathcal{G}, t_n),\ \ for\ n = 1, 2, ...$$

is said to be $\mathcal{G}$-stable or stable with respect to $\mathcal{G}$ for some function f, converging to fixed point p, if $\epsilon_n = d(t_{n+1}, f(\mathcal{G}, t_n))$ for $n = 1, 2, ...$, we have $\lim\limits_{n\longrightarrow\infty} \epsilon_n = 0 \Leftrightarrow \lim\limits_{n\longrightarrow\infty} t_n = p$.

In what follows, we shall make use of the following well-known lemma.

Lemma 2.10. *([5]) Let $\{\epsilon_n\}$ and $\{u_n\}$ be sequences of positive real numbers satisfying*

$$u_{n+1} \leq \delta u_n + \epsilon_n,$$

$n \in \mathbb{N}$ and $\delta \in [0, 1)$. If $\lim\limits_{n\to\infty} \epsilon_n = 0$ then $\lim\limits_{n\to\infty} u_n = 0$.

3. Stability of SP*-iteration process

In this section, we prove that the SP*-iteration process defined by (1.4) is stable. First, we prove the following strong convergence theorem.

Theorem 3.1. *Let $\mathcal{M}$ be a nonempty closed convex subset of a complete $CAT(0)$ space $\mathcal{X}$, $\mathcal{G}$ be a contraction mapping with $Fix(\mathcal{G}) \neq \emptyset$. For arbitrary chosen $u_1 \in \mathcal{M}$, $\{u_n\}$ be a sequence generated by (1.4) with real sequences $\{\tau_n\}, \{\sigma_n\}$ and $\{\kappa_n\} \in [0, 1]$ with $\sum_{n=1}^{\infty} \tau_n = \infty$. Then $\{u_n\}_{n=1}^{\infty}$ converges strongly to an unique fixed point of $\mathcal{G}$.*

Proof. We will prove that $u_n \to p$ as $n \to \infty$ from (1.4), we have,

$$\begin{aligned} d(z_n, p) &= d(\mathcal{G}((1-\kappa_n)u_n \oplus \kappa_n \mathcal{G}u_n), p) \\ &\leq \theta[(1-\kappa_n)d(u_n, p) + \kappa_n \theta d(u_n, p)] \\ &= \theta[1-\kappa_n(1-\theta)]d(u_n, p). \end{aligned} \tag{3.1}$$

Similarly, from (1.4) and (3.1), we get

$$\begin{aligned} d(v_n, p) &= d(\mathcal{G}((1-\sigma_n)z_n \oplus \sigma_n \mathcal{G}z_n), p) \\ &\leq \theta[(1-\sigma_n)d(z_n, p) + \sigma_n d(\mathcal{G}z_n, p)] \\ &\leq \theta[(1-\sigma_n)d(z_n, p) + \sigma_n \theta d(z_n, p)] \\ &= \theta[(1-\sigma_n(1-\theta))d(z_n, p)] \\ &\leq \theta^2[(1-\sigma_n(1-\theta))(1-\kappa_n(1-\theta))]d(u_n, p). \end{aligned} \tag{3.2}$$

From (1.4) and (3.2) , we get

$$\begin{aligned} d(u_{n+1}, p) &= d(\mathcal{G}((1-\tau_n)v_n \oplus \tau_n \mathcal{G}v_n), p) \\ &\leq \theta[(1-\tau_n)d(v_n, p) + \tau_n d(\mathcal{G}v_n, p)] \\ &\leq \theta[(1-\tau_n)d(v_n, p) + \tau_n \theta d(v_n, p)] \\ &= \theta[1-\tau_n(1-\theta)d(v_n, p)] \\ &\leq \theta^3[(1-\tau_n(1-\theta))(1-\sigma_n(1-\theta))(1-\kappa_n(1-\theta))]d(u_n, p) \end{aligned}$$

Considering that $\{\tau_n\}, \{\sigma_n\}$ and $\{\kappa_n\} \in [0,1]$, $\theta \in [0,1)$, and rearranging the above inequality, we get

$$d(u_{n+1}, p) \leq \theta^3[1-\tau_n(1-\theta)]d(u_n, p)$$

By induction, we get

$$\begin{aligned} d(u_n, p) &\leq \theta^3[1-\tau_{n-1}(1-\theta)]d(u_{n-1}, p) \\ &. \\ &. \\ &. \\ d(u_2, p) &\leq \theta^3[1-\tau_1(1-\theta)]d(u_1, p). \end{aligned}$$

Therefore, we obtain

$$d(u_{n+1}, p) \leq \theta^{3n}\prod_{k=1}^{n}[1-\tau_k(1-\theta)]d(u_1, p),$$

$\theta < 1$ and $\tau_k \in [0,1]$ for $k = 1, 2, ...$ Then we have $[1-\tau_k(1-\theta)] \leq 1$ for $k = 1, 2, ...$ So, we know that $1-u \leq e^{-u}$ for all $u \in [0,1]$. Hence we have

$$d(u_{n+1}, p) \leq \theta^{3n}e^{-(1-\theta)\sum_{k=1}^{n}\tau_k}d(u_1, p). \tag{3.3}$$

Taking the limit of both sides of the above inequality , $u_n \to p$ as $n \to \infty$. □

Now we prove that the iteration defined by (1.4) is stable with respect to $\mathcal{G}$.

Theorem 3.2. *Suppose that all conditions of Theorem 3.1 hold. Then the iteration process (1.4) is $\mathcal{G}$-stable.*

Proof. Let $\{t_n\}$ be any arbitrary sequence in $\mathcal{M}$. $t_{n+1} = f(\mathcal{G}, t_n)$ is the sequence generated by (1.4) and $\epsilon_n = d(t_{n+1}, f(\mathcal{G}, t_n))$ for $n = 1, 2, ...$, in which

$$\begin{cases} r_n = \mathcal{G}((1-\kappa_n)t_n \oplus \kappa_n\mathcal{G}t_n), \\ s_n = \mathcal{G}((1-\sigma_n)r_n \oplus \sigma_n\mathcal{G}r_n), \\ t_{n+1} = \mathcal{G}((1-\tau_n)s_n \oplus \tau_n\mathcal{G}s_n), \forall n \in \mathbb{N}. \end{cases}$$

We have to prove that $\lim_{n\longrightarrow\infty}\epsilon_n = 0 \Leftrightarrow \lim_{n\longrightarrow\infty}t_n = p$.

Suppose $\lim_{n\longrightarrow\infty}\epsilon_n = 0$. We prove that $\lim_{n\longrightarrow\infty}t_n = p$:

$$\begin{aligned} d(t_{n+1}, p) &\leq d(t_{n+1}, f(\mathcal{G}, t_n)) + d(f(\mathcal{G}, t_n), p) \\ &\leq \epsilon_n + \theta[1-\tau_n(1-\theta)d(s_n, p)] \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} d(s_n, p) &= d(\mathcal{G}((1-\sigma_n)r_n \oplus \sigma_n\mathcal{G}r_n), p) \\ &\leq \theta[(1-\sigma_n)d(r_n, p) + \sigma_n d(\mathcal{G}r_n, p)] \\ &\leq \theta[(1-\sigma_n)d(r_n, p) + \sigma_n\theta d(r_n, p)] \\ &= \theta[(1-\sigma_n(1-\theta))]d(r_n, p) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} d(r_n, p) &= d(\mathcal{G}((1-\kappa_n)t_n \oplus \kappa_n\mathcal{G}t_n), p) \\ &\leq \theta[(1-\kappa_n)d(t_n, p) + \kappa_n\theta d(t_n, p)] \\ &= \theta[1-\kappa_n(1-\theta)]d(t_n, p). \end{aligned} \tag{3.6}$$

Substituting (3.6) in (3.5), we obtain

$$d(s_n, p) \leq \theta^2[(1-\sigma_n(1-\theta))(1-\kappa_n(1-\theta))]d(t_n, p). \tag{3.7}$$

Substituting (3.7) in (3.4), we get

$$d(t_{n+1}, p) \leq \epsilon_n + \theta^3[(1-\tau_n(1-\theta))(1-\sigma_n(1-\theta))(1-\kappa_n(1-\theta))]d(t_n, p).$$

Since $\{\tau_n\}, \{\sigma_n\}$ and $\{\kappa_n\} \in [0,1]$, $\theta \in [0,1)$ and $\theta^3[(1-\tau_n(1-\theta))(1-\sigma_n(1-\theta))(1-\kappa_n(1-\theta))] < 1$, we can easily seen that all conditions of Lemma 2.10 are fulfilled by above inequality. Hence by Lemma 2.10 we get $\lim_{n\longrightarrow\infty} t_n = p$.

Conversely, let $\lim_{n\longrightarrow\infty} t_n = p$, we have

$$\begin{aligned} \epsilon_n &= d(t_{n+1}, f(\mathcal{G}, t_n)) \\ &\leq d(t_{n+1}, p) + d(f(\mathcal{G}, t_n), p) \\ &\leq d(t_{n+1}, p) + \theta^3[(1-\tau_n(1-\theta))(1-\sigma_n(1-\theta))(1-\kappa_n(1-\theta))]d(t_n, p). \end{aligned}$$

By taking the limit as $n \to \infty$ in the above inequality we have $\lim_{n\longrightarrow\infty} \epsilon_n = 0$. Hence (1.4) is stable with respect to $\mathcal{G}$. □

4. Convergence of SP*-iteration process for operators with property (E)

Lemma 4.1. *Let $\mathcal{M}$ be a nonempty closed convex subset of a complete $CAT(0)$ space $\mathcal{X}$, $\mathcal{G}$ be a mapping satisfying condition (E) with $Fix(\mathcal{G}) \neq \emptyset$. For arbitrary chosen $x_1 \in \mathcal{M}$, let $\{u_n\}$ be a sequence generated by (1.4) with $\{\tau_n\}$, $\{\sigma_n\}$ and $\{\kappa_n\}$ real sequences in $[0,1]$. Assume that $\liminf_{n\to\infty}(1-\kappa_n)\kappa_n > 0, \liminf_{n\to\infty}(1-\sigma_n)\sigma_n > 0$ and $\liminf_{n\to\infty}(1-\tau_n)\tau_n > 0$. Then $Fix(\mathcal{G}) \neq \emptyset$ if and only if $\{u_n\}$ is bounded and $\lim_{n\to\infty} d(u_n, \mathcal{G}u_n) = 0$.*

Proof. Assume that $Fix(\mathcal{G}) \neq \emptyset$. $\mathcal{G}$ is a quasi-nonexpansive because $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ is a Garcia-Falset GNM. Using (1.4), for any $p \in Fix(\mathcal{G})$, because of $\mathcal{G}$ quasi-nonexpansive mapping, then we have

$$\begin{aligned} d^2(z_n, p) &= d^2(\mathcal{G}((1-\kappa_n)u_n \oplus \kappa_n \mathcal{G}u_n), p) \\ &\leq d^2((1-\kappa_n)u_n \oplus \kappa_n \mathcal{G}u_n, p) \\ &\leq (1-\kappa_n)d^2(u_n, p) + \kappa_n d^2(\mathcal{G}u_n, p) - (1-\kappa_n)\kappa_n d^2(\mathcal{G}u_n, u_n) \\ &\leq d^2(u_n, p) - (1-\kappa_n)\kappa_n d^2(\mathcal{G}u_n, u_n) \leq d^2(u_n, p). \end{aligned} \tag{4.1}$$

Using (1.4) and (4.1), we get

$$\begin{aligned} d^2(v_n, p) &= d^2(\mathcal{G}((1-\sigma_n)z_n \oplus \sigma_n \mathcal{G}z_n), p) \\ &\leq d^2((1-\sigma_n)z_n \oplus \sigma_n \mathcal{G}z_n, p) \\ &\leq (1-\sigma_n)d^2(z_n, p) + \sigma_n d^2(\mathcal{G}z_n, p) - (1-\sigma_n)\sigma_n d^2(\mathcal{G}z_n, z_n) \\ &\leq d^2(z_n, p) - (1-\sigma_n)\sigma_n d^2(\mathcal{G}z_n, z_n) \\ &\leq d^2(z_n, p) \leq d^2(u_n, p). \end{aligned} \tag{4.2}$$

By using (1.4) and (4.2), we get

$$\begin{aligned} d^2(u_{n+1}, p) &= d^2(\mathcal{G}((1-\tau_n)v_n \oplus \tau_n \mathcal{G}v_n), p) \\ &\leq d^2((1-\tau_n)v_n \oplus \tau_n \mathcal{G}v_n, p) \\ &\leq (1-\tau_n)d^2(v_n, p) + \tau_n d^2(\mathcal{G}v_n, p) - (1-\tau_n)\tau_n d^2(\mathcal{G}v_n, v_n) \end{aligned} \tag{4.3}$$

$$\leq \quad d^2(v_n,p) - (1-\tau_n)\tau_n d^2(\mathcal{G}v_n, v_n)$$
$$\leq \quad d^2(v_n,p) \leq d^2(u_n,p).$$

This implies that $\{d(u_n,p)\}$ is bounded and non-increasing for all $p \in Fix(\mathcal{G})$. Put $\lim_{n\to\infty} d(u_n,p) = c$. From (4.1) and (4.2), we have

$$\limsup_{n\to\infty} d(z_n,p) \leq \limsup_{n\to\infty} d(u_n,p) = c$$

and

$$\limsup_{n\to\infty} d(v_n,p) \leq \limsup_{n\to\infty} d(z_n,p) \leq \limsup_{n\to\infty} d(u_n,p) = c.$$

From (4.3), we can get $d(u_{n+1},p) \leq d(v_n,p)$. Therefore $c \leq \liminf_{n\to\infty} d(v_n,p)$. Thus we have $c = \lim_{n\to\infty} d(v_n,p)$. Next

$$c = \lim_{n\to\infty} d(v_n,p) \leq \lim_{n\to\infty} d(z_n,p) \leq \lim_{n\to\infty} d(u_n,p) = c.$$

Now, using (4.1), we know that

$$d^2(z_n,p) \leq d^2(u_n,p) - (1-\kappa_n)\kappa_n d^2(\mathcal{G}u_n, u_n).$$

Thus

$$(1-\kappa_n)\kappa_n d^2(\mathcal{G}u_n,u_n) \leq d^2(u_n,p) - d^2(z_n,p)$$

so that

$$d^2(\mathcal{G}u_n,u_n) \leq \frac{1}{(1-\kappa_n)\kappa_n}[d^2(u_n,p) - d^2(z_n,p)].$$

We have

$$\lim_{n\to\infty} d^2(\mathcal{G}u_n,u_n) \leq 0.$$

Hence $\lim_{n\to\infty} d(\mathcal{G}u_n,u_n) = 0$.

Conversely, suppose that $\{u_n\}$ is bounded and $\lim_{n\to\infty} d(u_n,\mathcal{G}u_n) = 0$. Let $p \in A(\mathcal{M},\{u_n\})$. Then we have,

$$\begin{aligned} r(\mathcal{G}p,\{u_n\}) = \limsup_{n\to\infty} d(u_n,\mathcal{G}p) &\leq \limsup_{n\to\infty} \mu d(\mathcal{G}u_n,u_n) + \limsup_{n\to\infty} d(u_n,p) \\ &= \limsup_{n\to\infty} d(u_n,p) = r(p,\{u_n\}). \end{aligned}$$

This implies that for $\mathcal{G}p = p \in A(\mathcal{M},\{u_n\})$. Since $\mathcal{X}$ is complete $CAT(0)$ then $A(\mathcal{M},\{u_n\})$ is singleton, hence $\mathcal{G}p = p$. This completes the proof. □

Now , we prove the Δ-convergence theorem of a iterative process generated by (1.4) in $CAT(0)$ spaces.

Theorem 4.2. *Let $\mathcal{X},\mathcal{M},\mathcal{G}$ and $\{u_n\}$ be as in Lemma 4.1 with $Fix(\mathcal{G}) \neq \emptyset$. Then u_n, Δ-converges to a fixed point of $\mathcal{G}$.*

Proof. Lemma 4.1 guarantees that the sequence $\{u_n\}$ is bounded and $\lim_{n\to\infty} d(\mathcal{G}u_n,u_n) = 0$. Let $W_\Delta(u_n) = \bigcup A(\{\omega_n\})$; where the union is taken over all subsequences $\{\omega_n\}$ of $\{u_n\}$: We claim that $W_\Delta(u_n) \subseteq Fix(\mathcal{G})$. Let $\omega \in W_\Delta(u_n)$. Then, there exists a subsequence $\{\omega_n\}$ of $\{u_n\}$ such that $A(\{\omega_n\}) = \omega$. Since $\mathcal{G}$ is a mapping with condition (E), we obtain $d(\omega_n,\mathcal{G}\omega) \leq \mu d(\omega_n,\mathcal{G}\omega_n) + d(\omega_n,\omega)$. Using this last inequality and fact that $\lim_{n\to\infty} d(\omega_n,\mathcal{G}\omega_n) = 0$, taking limsup on both sides implies that $\limsup_{n\to\infty} d(\omega_n,\mathcal{G}\omega) \leq \limsup_{n\to\infty} d(\omega_n,\omega)$. Hence $r(\mathcal{G}\omega,\{\omega_n\}) \leq r(\omega,\{\omega_n\})$. However, ω is the unique asymptotic center of $\{\omega_n\}$, which implies that $\omega = \mathcal{G}\omega$, that is, $\omega \in Fix(\mathcal{G})$.

By Lemma 2.7 and Lemma 2.8, there exists a subsequence $\{\zeta_n\}$ of $\{\omega_n\}$ such that $\Delta - \lim_{n\to\infty} \zeta_n = \zeta \in \mathcal{G}$. Since $\lim_{n\to\infty} d(\zeta_n, \mathcal{G}\zeta_n) = 0$ and $\mathcal{G}$ is a Garcia-Falset mapping, then, we have

$$d(\zeta_n, \mathcal{G}\zeta) \leq \mu d(\mathcal{G}\zeta_n, \zeta_n) + d(\zeta_n, \zeta).$$

By taking limsup and using Opial property, we obtain $\zeta \in Fix(\mathcal{G})$. Now, we claim that $\omega = \zeta$. Assume on contrary, that $\omega \neq \zeta$. By Lemma 4.1, $\lim_{n\to\infty} d(u_n, \zeta)$ exists and by the uniqueness of asymptotic centers, then we have

$$\begin{aligned}
\lim_{n\to\infty} d(\zeta_n, \zeta) &< \lim_{n\to\infty} d(\zeta_n, \omega) \leq \lim_{n\to\infty} d(\omega_n, \omega) \\
&< \lim_{n\to\infty} d(\omega_n, \zeta) \;\; = \lim_{n\to\infty} d(\omega_n, \zeta) \\
&= \lim_{n\to\infty} d(\zeta_n, \zeta),
\end{aligned}$$

which is contradiction. Thus $\omega = \zeta \in Fix(\mathcal{G})$ and $W_\Delta(\omega_n) \subseteq Fix(\mathcal{G})$. To show that $\{\omega_n\}$, Δ-converges to a fixed point of $\mathcal{G}$, we show that $W\Delta(u_n)$ consists of exactly one point. By Lemma 2.7 and Lemma 2.8, there exists a subsequence $\{\zeta_n\}$ of ω_n such that $\Delta - \lim_{n\to\infty} \zeta_n = \zeta \in \mathcal{M}$. Let $A(\{\omega_n\}) = \{\omega\}$ and $A(\{\omega_n\}) = \{\rho\}$. We have already seen that $\omega = \zeta$ and $\zeta \in Fix(\mathcal{G})$. Finally, we claim that $\rho = \zeta$. If not, then existence $\lim_{n\to\infty} d(u_n, \zeta)$ and uniqueness of asymptotic centers imply that

$$\begin{aligned}
\lim_{n\to\infty} d(\zeta_n, \zeta) &< \lim_{n\to\infty} d(\zeta_n, \rho) \leq \lim_{n\to\infty} d(\omega_n, \rho) \\
&< \lim_{n\to\infty} d(\omega_n, \zeta) \;\; = \lim_{n\to\infty} d(\zeta_n, \zeta).
\end{aligned}$$

This is a contradiction and hence $\rho = \zeta \in Fix(\mathcal{G})$. Therefore, $W_\Delta(\omega_n) = \rho$. In conclusion $W_\Delta(\omega_n)$ is a singleton and unique element is a fixed point of $\mathcal{G}$. This proves Δ-convergence of u_n. □

In the next result, we prove the strong convergence theorem as follows.

Theorem 4.3. *Let $\mathcal{X}, \mathcal{M}, \mathcal{G}$ and $\{u_n\}$ be as in Lemma 4.1 with $Fix(\mathcal{G}) \neq \emptyset$ such that $\mathcal{M}$ is compact subset of $\mathcal{X}$. Then $\{u_n\}$ converges strongly to a fixed point of $\mathcal{G}$.*

Proof. By Lemma 4.1, we have $\lim_{n\to\infty} d(u_n, \mathcal{G}u_n) = 0$. Since $\mathcal{M}$ is compact, by Lemma 2.7, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $p \in \mathcal{M}$ such that $\{u_{n_k}\}$ converges p. Then we have $d(u_{n_k}, \mathcal{G}p) \leq \mu d(\mathcal{G}u_{n_k}, u_{n_k}) + d(u_{n_k}, p)$ for all $k \geq 1$. So $\{u_{n_k}\}$ converges $\mathcal{G}p$. This implies $\mathcal{G}p = p$. Since $\mathcal{G}$ is quasi-nonexpansive , we have $d(u_{n+1}, p) \leq d(u_n, p)$ for all $n \in \mathbb{N}$. Therefore $\{u_n\}$ converges strongly to p. □

Finally, we briefly discuss the strong convergence theorem using condition (I) introduced by Senter and Dotson[32] in $CAT(0)$ space $\mathcal{X}$ as follows.

Theorem 4.4. *Let $\mathcal{G}$ be a Garcia-Falset mapping on a nonempty closed convex subset $\mathcal{M}$ of a complete $CAT(0)$ space $\mathcal{X}$. $\{u_n\}$ be as in Lemma 4.1 with $Fix(\mathcal{G}) \neq \emptyset$. Also if, for $\mathcal{G}$ satisfies condition (I), then $\{u_n\}$ defined by (1.4) converges strongly to a fixed point of $(\mathcal{G})$.*

Proof. By Lemma 4.1, we have $\lim_{n\to\infty} d(u_n, p)$ exists and so $\lim_{n\to\infty} d(u_n, Fix(\mathcal{G}))$. Also by Lemma 4.1, $\lim_{n\to\infty} d(u_n, \mathcal{G}u_n) = 0$.

It follows from condition (I) that $\lim_{n\to\infty} f(d(u_n, Fix(\mathcal{G})) \leq \lim_{n\to\infty} d(u_n, \mathcal{G}u_n)$. That is, $\lim_{n\to\infty} f(d(u_n, Fix(\mathcal{G})) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$, we have $\lim_{n\to\infty} d(u_n, Fix(\mathcal{G}))=0$. Thus , we have a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $\{y_k\}\subset Fix(\mathcal{G})$ such that $d(x_{n_k}, y_k)<\frac{1}{2^k}$ for all $k\in\mathbb{N}$. We can easily show that $\{y_k\}$ is a Cauchy sequence in $Fix(\mathcal{G})$ and so it converges to a point p. Since $Fix(\mathcal{G})$ is closed, therefore $p\in Fix(\mathcal{G})$ and $\{u_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty} d(u_n,p)$ exists, we have that $u_n\to p$. Thus the proof is completed. □

Next, we give the following example satisfying condition (E), but it is neither a generalized α-nonexpansive mapping nor does it satisfy condition (C).

Example 4.5. Let $\mathcal{X}=\mathbb{R}$ be a $CAT(0)$ space and $\mathcal{M}=[0,1]$ be a closed convex subset of $\mathbb{R}$ endowed with the usual norm. Define a mapping $\mathcal{G}:\mathcal{M}\to\mathcal{M}$ by $\mathcal{G}u=\begin{cases} 0, & 0\le u<\frac{1}{100} \\ \frac{2u}{3}, & \frac{1}{100}\le u\le 1. \end{cases}$ In order to see that $\mathcal{G}$ satisfies condition (E_3) on $[0,1]$, we consider the following cases:

(i) $u\in[0,\frac{1}{100})$ and $v\in[0,\frac{1}{100})$. Then we have

$$d(u,\mathcal{G}v)=|u-0|=|u|=d(u,\mathcal{G}u)\le\mu d(u,\mathcal{G}u)+d(u,v).$$

So, $\mathcal{G}$ satisfies condition (E_1).

(ii) $u\in[\frac{1}{100},1]$ and $v\in[\frac{1}{100},1]$. Then we have

$$\begin{aligned} d(u,\mathcal{G}v) &= \left|u-\frac{2v}{3}\right| = \left|\frac{3u-2v}{3}\right| \\ &= \left|\frac{u}{3}+\frac{2u}{3}-\frac{2v}{3}\right| \\ &= \frac{u}{3}+\frac{2}{3}\left|u-v\right|. \end{aligned}$$

Turning to the right side of the inequality in Definition 2.3,

$$\mu d(u,\mathcal{G}u)+d(u,v)=\mu\left|u-\frac{2u}{3}\right|+\left|u-v\right|.$$

If we choose the admissible parameter $\mu=1$, the mapping will satisfy condition (E).

(iii) $u\in[\frac{1}{100},1]$ and $v\in[0,\frac{1}{100})$, which leads to $d(u,\mathcal{G}u)=\left|u-\frac{2u}{3}\right|$. Evaluating condition (E) for this case, we have

$$\begin{aligned} d(u,\mathcal{G}v) &= |u-0|\le\frac{3u}{3}+|u-v| \\ &= 3(\frac{u}{3})+d(u,v)=3d(u,\mathcal{G}u)+d(u,v). \end{aligned}$$

So, if we choose the admissible parameter $\mu=3$, then the mapping will prove to have condition (E). Taking the maximum value of μ, we conclude that $\mathcal{G}$ satisfies (E_3) with $\mathcal{G}(0)=0$ fixed point.

Now, let us prove that $\mathcal{G}$ is not a generalized α-nonexpansive mapping. We shall take $u=\frac{1}{150}$ and $v=\frac{1}{100}$. It follows that

$$\frac{1}{2}d(u,\mathcal{G}u)=\frac{1}{2}\left|\frac{1}{150}-0\right|=\frac{1}{300}=\left|\frac{1}{150}-\frac{1}{100}\right|=\left|u-v\right|.$$

If we consider the left side of the inequality in Definition 2.2,

$$d(\mathcal{G}u,\mathcal{G}v)=\left|0-\frac{2}{3}\frac{1}{100}\right|=\frac{1}{150}.$$

Turning to the right side of the inequality in Definition 2.2, for $\alpha \in [0,1)$,

$$\begin{aligned}
& \alpha d(\mathcal{G}u, v) + \alpha d(\mathcal{G}v, u) + (1-2\alpha) d(u,v) \\
= \; & \alpha \left| \frac{1}{150} - \frac{2}{3}\frac{1}{100} \right| + \alpha \left| 0 - \frac{1}{150} \right| + (1-2\alpha) \left| \frac{1}{150} - \frac{1}{100} \right| \\
= \; & 0 + \frac{\alpha}{100} + \frac{1}{300} - \frac{\alpha}{300} \\
= \; & \frac{3\alpha}{300} + \frac{1}{300} - \frac{2\alpha}{300} \\
= \; & \frac{\alpha}{300} + \frac{1}{300} = (\alpha+1)\frac{1}{300}.
\end{aligned}$$

So, for $\alpha \in [0,1)$, the implications fails to be satisfied, which leads to the conclusion that $\mathcal{G}$ is not a generalized α-nonexpansive mapping.

In order to we show that $\mathcal{G}$ does not satisfy condition (C), we take also $u = \frac{1}{150}$ and $v = \frac{1}{100}$. Then we have

$$\frac{1}{2} d(u, \mathcal{G}u) = \frac{1}{2}\left| \frac{1}{150} - 0 \right| = \frac{1}{300} = \left| \frac{1}{150} - \frac{1}{100} \right| = |u - v| = d(u,v).$$

If we apply the inequality in Definition 2.1, we get

$$d(\mathcal{G}u, \mathcal{G}v) = \left| 0 - \frac{2}{3}\frac{1}{100} \right| = \frac{1}{150} > \frac{1}{300} = d(u,v).$$

Thus $\mathcal{G}$ does not satisfy condition (C).

5. Conclusions

We get some results on the strong and Δ-convergence of SP*-iteration process (1.4) in given [35] for the mapping with Property (E) in nonlinear $CAT(0)$ spaces. The result herein complements the some results of [14, 36, 37] from linear setting to $CAT(0)$ spaces. We also prove the stability of SP*-iteration process generated by (1.4) in given [35] in this paper. In addition, we give an illustrative numerical example that satisfies condition (E). As seen in Example 4.5, the mapping is neither a generalized α-nonexpansive mapping nor does it satisfy condition (C). Further, in future studies, iteration process can be developed and iteration that converges faster than prominent iterations can be presented.

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